ELEC-E8116 Model-based control systems
/exercises with solutions 9

Problem 1. Consider the system

$$\frac{Y(s)}{U(s)} = \frac{s+0.5}{s^2+2s+4} \text{ and the criterion to be minimized } J = \int_0^\infty (3y^2 + 0.5u^2)\, dt\,.$$

Write a *Matlab m-file* to do the following:

Solve the optimal control law by using the *lqr*-function in Matlab. Calculate the *damping ratio* of the closed loop system. Simulate the system by letting the reference signal be zero (regulator problem) and letting the initial states be non-zero. Then consider the tracking problem. Use a static pre-compensator to set the static gain of the closed-loop system to the value 1. Then simulate the system for a step change in the reference signal.

Solution. Note that Matlab's *lqr* function uses the cost $J = \int_0^\infty \left(x^T Q x + u^T R u \right) dt$

$$J = \int_0^\infty (3y^2 + 0.5u^2)dt = \int_0^\infty (3y^T y + 0.5u^2)dt = \int_0^\infty (3(Cx)^T Cx + 0.5u^2)dt$$

$$\int_0^\infty (x^T (3C^T C)x + 0.5u^2)dt$$

The Matlab m-code shows the solution and simulation.

The process model in state-space form is

```
A =

   -2   -2
    2    0

B =

    1
    0

C =

   1.0000    0.2500
```

D =

0

The process and optimal controller are

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u = -Lx$$

which leads to the closed loop representation

$$\dot{x} = (A - BL)x$$
$$y = Cx$$

The state feedback matrix, the Riccati equation solution and the closed loop poles are

L =

1.2197	0.0917

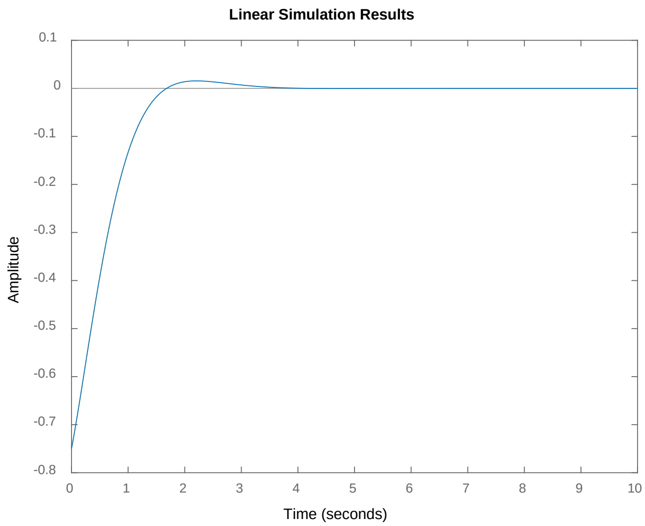
S =

0.6099	0.0458
0.0458	0.3086

E =

-1.6099 + 1.2616i
-1.6099 - 1.2616i

The damping ratio is 0.787. The simulation result using the initial condition [-1 1] is shown in the figure. Note that it shows the output signal *y*. If both states should be plotted, you should have defined the C-matrix as an identity matrix. The system would not change, but both states would be defined as outputs.



As for the servo problem, the controller is $u = -Lx + kr$ where k is a constant and r the reference. The closed loop is

$$\dot{x} = (A - BL)x + kBr$$
$$y = Cx$$

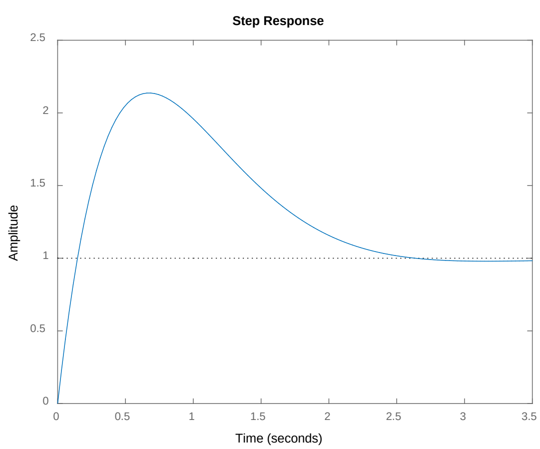
The corresponding transfer function from reference to output is

$$G_{cl}(s) = kC\left(sI - A + BL\right)^{-1} B$$

The static gain is $G_{cl}(0)$, and it must be set to 1 in order the output to follow the reference. Hence

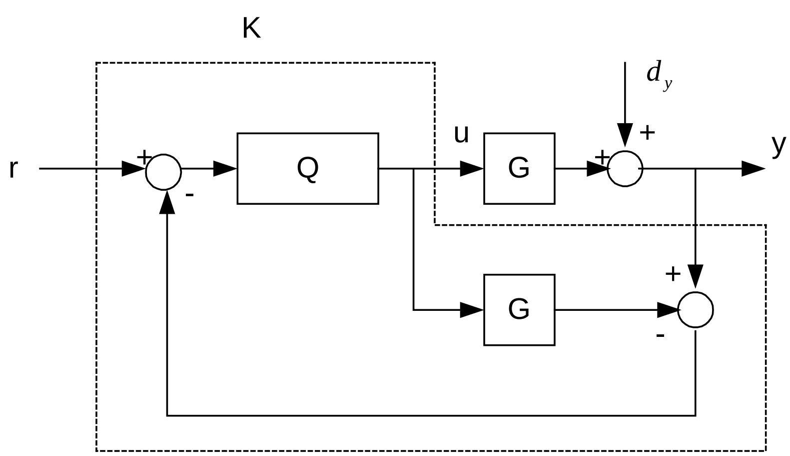
$$k = \frac{1}{C(-A + BL)^{-1} B}$$

The simulation result for a step change in r is shown below.



```
% Model-based control systems
% Exercise 9, Problem 1
%
%Model
s=tf('s');
G=(s+0.5)/(s^2+2*s+4);
Gss=ss(G);
[A,B,C,D]=ssdata(Gss);
%Cost function
Q=3*C'*C; R=0.5;
%Optimal control u=-Lx, regulator problem
[L,S,E]=lqr(A,B,Q,R);
damp(A-B*L);
%Closed loop
Gclss=ss(A-B*L,zeros(2,1),C,0);
%Simulation
x0=[-1,1]';
T=0:0.01:10;
U=zeros(size(T));
lsim(Gclss,U,T,x0)
%Optimal control u=-Lx+kr, servo problem
k=1/(C*inv(-A+B*L)*B);
G2clss=ss(A-B*L,k*B,C,0);
figure
step(G2clss)
```

Problem 2: Consider the following IMC-control configuration, in which the process G is assumed stable.



a. Prove that to study the internal stability, the stability of the transfer functions

$$K(I+GK)^{-1} = Q$$
$$(I+GK)^{-1} = I - GQ$$
$$(I+KG)^{-1} = I - QG$$
$$G(I+KG)^{-1} = G(I-QG)$$

must be investigated. Prove that the system is internally unstable, if either Q or G is unstable.

b. Let a stable controller K be given. How can you characterize those processes, which can be stabilized with this controller? (Hint: Change the roles of the controller and process.)

Solution:

a. For the control it holds

$$u = Q[r - (y - Gu)] = Q(r-y) + QGu$$

from which it follows easily

$$u = (I - QG)^{-1} Q(r-y)$$

But this has the form

$$u = K(r-y)$$

where $K = (I - QG)^{-1} Q$

and

$$Q = K(I+GK)^{-1}$$

By this controller the configuration is equivalent to the ”one-degree-of-freedom”-structure. Based on lectures (Chapter 3, Internal stability of closed-loop systems) it is known that the system is internally stable, if the transfer functions

$$K(I+GK)^{-1} = Q$$
$$(I+GK)^{-1} = I - GQ$$
$$(I+KG)^{-1} = I - QG$$
$$G(I+KG)^{-1} = G(I-QG)$$

are stable (the ”right sides” follow easily from the choice of Q).

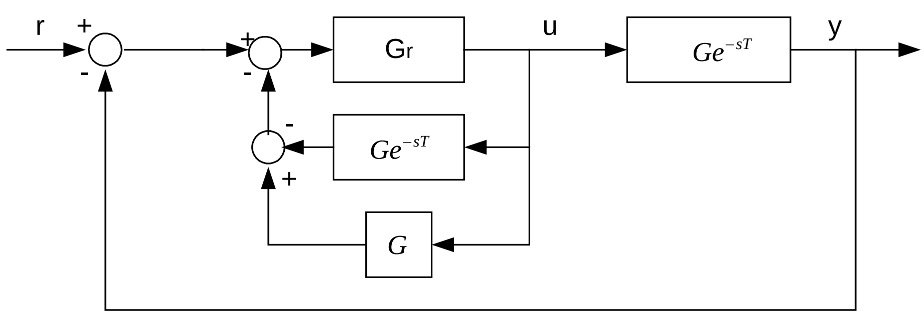
But the functions are clearly stable, if Q and G are stable. Correspondingly, if either one is unstable, the system is internally unstable.

b. These systems can be represented in the form (parameterization)

$$G = (I - QK)^{-1}Q = Q(I - KQ)^{-1}$$

where Q is any stable transfer function matrix.

Problem 3. Consider the control configuration shown in the figure (known as the *Smith-predictor*). Calculate the closed loop transfer function and verify the idea behind this controller. Compare to the *IMC*-controller and prove that the Smith predictor always leads to an internally unstable system, if the process is unstable.



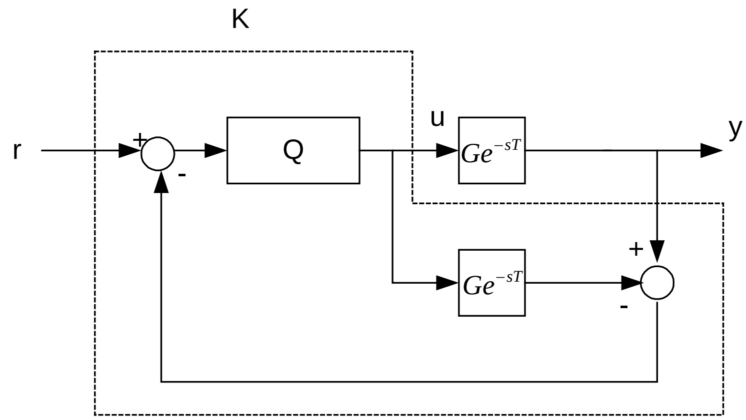
Solution: By using block diagram algebra the transfer function from r to y is easily calculated to be

$$Y(s) = \frac{G_r(s)G(s)}{1 + G_r(s)G(s)} e^{-sT} R(s)$$

which reveals the idea behind this control configuration: the basic controller G_r can be designed to give a good closed loop response without paying any attention to the process delay. The real response is then the same but added with a pure delay T. The term e^{-sT} is not shown in the characteristic equation (which would happen, if G_r would directly control the process with delay). But note that in this ideal case the process is exactly known and the intermediate block in the controller generates the predicted value of the output. In reality an inaccurate process model has to be used for this purpose.

But: moving the block Ge^{-sT} in the figure a bit (without changing the control signal *u*, of course), the configuration below is obtained. That is directly the *IMC*-structure. There

$$Q = \frac{G_r}{1 + G_r G}$$



But what if the process *G* is unstable? Look at the previous problem, in which it was shown that the closed loop system is internally unstable, if *Q* or *G* is unstable. Because the Smith predictor structure was above shown to be equivalent to the *IMC*-structure, the closed loop is inevitably (internally) unstable, if the process is unstable.