Taloustieteen matemaattiset menetelmät
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Problem Set 5: Solutions

1. **Solution**

By Weierstrass's Theorem we know that a global maximum exists if the objective function is continuous and its domain is compact (i.e. closed and bounded). These conditions hold in the constraint set, so a solution exists.

The Lagrangian is

$$L(x, y, \mu, \lambda_x, \lambda_y) = x^2 + y^2 - \mu(2x + y - 4) - \lambda_x(-x) - \lambda_y(-y)$$

The first order conditions are:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 2x - 2\mu + \lambda_x = 0 \\
\frac{\partial L}{\partial y} &= 2y - \mu + \lambda_y = 0 \\
\frac{\partial L}{\partial \mu} &= 2x + y - 4 = 0 \\
\lambda_x x &= 0 \\
\lambda_y y &= 0 \\
x, y, \lambda_x, \lambda_y &\geq 0
\end{aligned}$$

Consider the following four cases.

(a) $x = y = 0$. From the third condition we get $y = 4 - 2x$, which cannot hold as $0 \neq 4$. Thus, $(0, 0)$ cannot be a solution.

(b) $x > 0$ and $y > 0$. Now $\lambda_x = \lambda_y = 0$, which leads to $\mu = x = 2y$. Substituting $x = 2y$ to $y = 4 - 2x$ yields $y = \frac{4}{5}$, and therefore $x = \frac{8}{5}$ and $\mu = \frac{8}{5}$.

(c) $x > 0$ and $y = 0$. From $y = 4 - 2x$ we get $x = 2$, and therefore $\lambda_x = 0$. Thus, the first two conditions yield $\mu = \lambda_y = 2$.

(d) $x = 0$ and $y > 0$. Now $y = 4 - 2x = 4$, so $\lambda_y = 0$ and $\mu = \frac{\lambda_x}{2} = 2y = 8$, so $\lambda_x = 16$.

By substituting x and y to the objective function we can conclude that it is maximized when $(x, y) = (0, 4)$. Now we have to check that the NDCQ is satisfied. The Jacobian of the binding constraints is

$$J = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix},$$

which is of full rank, so NDCQ is satisfied.

By the proposition in the slide 14 of lecture 12, we know that if a solution exists, it must be a critical point of the Lagrangian. Since by Weierstrass's Theorem we know that a solution exists, we can conclude that $(x^*, y^*) = (0, 4)$ is a global constrained maximizer.

2. **Solution**

A solution exists by Weierstrass's Theorem. The Lagrangian is

$$L = ax_1 + bx_2 - \mu(p_1x_1 + p_2x_2 - w) + \lambda_1 x_1 + \lambda_2 x_2.$$

The first order conditions are:

$$a - \mu p_1 + \lambda_1 = 0 \quad (1)$$
$$b - \mu p_2 + \lambda_2 = 0 \quad (2)$$
$$\mu(p_1x_1 + p_2x_2 - w) = 0 \quad (3)$$
$$\lambda_1 x_1 = 0 \quad (4)$$
$$\lambda_2 x_2 = 0 \quad (5)$$
$$\mu, \lambda_1, \lambda_2 \geq 0 \quad (6)$$
$$p_1x_1 + p_2x_2 \leq w,\ x_1 \geq 0,\ x_2 \geq 0. \quad (7)$$

Consider the following four cases.

(a) $x_1 = x_2 = 0$. By (3), $\mu = 0$. By (1), $\lambda_1 = -a < 0$, which violates (6). We conclude that $(x_1, x_2) = (0, 0)$ cannot be a solution.

(b) $x_1 > 0$ and $x_2 = 0$. By (4), $\lambda_1 = 0$. By (1), $\mu = \frac{a}{p_1} > 0$. Hence the budget constraint is binding, and $x_1 = \frac{w}{p_1}$. Using $\mu = \frac{a}{p_1}$ in (2), we get $\lambda_2 = a\frac{p_2}{p_1} - b$, which is nonnegative provided that $\frac{a}{b} \geq \frac{p_1}{p_2}$.

(c) $x_1 = 0$ and $x_2 > 0$. By (5), $\lambda_2 = 0$. By (2), $\mu = \frac{b}{p_2} > 0$. Hence the budget constraint is binding, and $x_2 = \frac{w}{p_2}$. Using $\mu = \frac{b}{p_2}$ in (1), we get $\lambda_1 = b\frac{p_1}{p_2} - a$, which is nonnegative provided that $\frac{a}{b} \leq \frac{p_1}{p_2}$.

(d) $x_1 > 0$ and $x_2 > 0$. By (4) and (5), $\lambda_1 = 0$ and $\lambda_2 = 0$. By (1) and(2), $\mu = \frac{a}{p_1} = \frac{b}{p_2} > 0$, which holds provided that $\frac{a}{b} = \frac{p_1}{p_2}$. In addition, the budget constraint is binding.

Summing up, when $\frac{a}{b} > \frac{p_1}{p_2}$, the unique solution is $(x_1, x_2) = \left(\frac{w}{p_1}, 0\right)$. When $\frac{a}{b} < \frac{p_1}{p_2}$, the unique solution is $(x_1, x_2) = \left(0, \frac{w}{p_2}\right)$. When, $\frac{a}{b} = \frac{p_1}{p_2}$, we have infinitely many solutions: every (x_1, x_2) such that $p_1x_1 + p_2x_2 = w$ is a global constrained maximizer. You can easily check that the NDCQ is always satisfied.

3. **Solution**

The Lagrangian is

$$L = x^2 - x + \lambda x.$$

The first order conditions are

$$\begin{aligned} 2x - 1 + \lambda &= 0 \\ \lambda x &= 0 \\ x &\geq 0 \\ \lambda &\geq 0. \end{aligned}$$

The set of first order conditions admit two solutions: $(x, \lambda) = (0, 1)$ and $(x, \lambda) = (\frac{1}{2}, 0)$.

The NDCQ trivially holds. However, neither $(0, 1)$ nor $(\frac{1}{2}, 0)$ is a global maximizer. As a matter of fact, there are no global constrained maximizers in this problem. As $x \longrightarrow \infty$, $f(x)$ goes to infinity too. Given that a global maximizer does not exist, the Proposition mentioned allows us to find only potential *local* constrained maximizers. In other words, the Proposition rests on the hypothesis that local (and not necessarily global) maximizers exist. You can verify that $x = 0$ is a local maximizer, and $x = \frac{1}{2}$ is a local minimizer.

4. **Solution**

(a) The Lagrangian is

$$L = y - \mu(y^3 - x^2).$$

The first order conditions are

$$2\mu x = 0 \tag{8}$$

$$1 - 3\mu y^2 = 0 \tag{9}$$

$$y^3 - x^2 = 0. \tag{10}$$

From (8) we have either $\mu = 0$ or $x = 0$. If $\mu = 0$, (9) cannot hold. If $x = 0$, $y = 0$ by (10) and, consequently, (9) cannot hold. Thus the system (8)-(10) does not admit any solution.

(b) The NDCQ fails when $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$. That is, $3y^2 = 2x = 0$, which holds only at the point $(x, y) = (0, 0)$. Notice that $(0, 0)$ belongs to the constraint set.

(c) One can argue as follows. The constraint requires $y^3 = x^2$. Since $x^2 \geq 0$ for every x, this implies that $y \geq 0$. Since we want to minimize f, the lowest possible value that f can take on is when $y = 0$, which requires $x = 0$. Thus $(0, 0)$ is the unique global constrained minimizer.

5. **Solution**

The Lagrangian is

$$L = x^2 + y^2 + z^2 - \mu_1(x + 2y + z - 30) - \mu_2(2x - y - 3z - 10).$$

The first order conditions are

$$\begin{aligned} 2x - \mu_1 - 2\mu_2 &= 0 \\ 2y - 2\mu_1 + \mu_2 &= 0 \\ 2z - \mu_1 + 3\mu_2 &= 0 \\ x + 2y + z - 30 &= 0 \\ 2x - y - 3z - 10 &= 0. \end{aligned}$$

The above is a system of 5 linear equations in 5 unknowns. The unique solution is $(x, y, z, \mu_1, \mu_2) = (10, 10, 0, 12, 4)$.

The bordered Hessian is:

$$H = \begin{pmatrix} 0 & 0 & \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ 0 & 0 & \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\ \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} & L''_{xx} & L''_{xy} & L''_{xz} \\ \frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y} & L''_{yx} & L''_{yy} & L''_{yz} \\ \frac{\partial g_1}{\partial z} & \frac{\partial g_2}{\partial z} & L''_{zx} & L''_{zy} & L''_{zz} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 & -3 \\ 1 & 2 & 2 & 0 & 0 \\ 2 & -1 & 0 & 2 & 0 \\ 1 & -3 & 0 & 0 & 2 \end{pmatrix}.$$

In this problem, we have $n = 3$ variables and $m = 2$ constraints. We have to check the sign of the last $n - m$ leading principal minors. That is, we only need to check the sign of the determinant of the whole matrix H. This determinant is equal to 150. Since $(-1)^m = 1$ and $(-1)^n = -1$, and since $\det(H) > 0$, we conclude that H is positive definite on the constraint set. Therefore, $(10, 10, 0)$ is a strict local minimizer of f over the given constraint set.