Aalto university
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Exercise sheet 8
Complex Analysis, MS-C1300.
Hand in exercise 1 and 2 for grading. Deadline Wednesday 18.11 at 23:59. The exercises should be uploaded to the correct folder on MyCourses as one pdf-file with name and student number in the file name. **Submission via MyCourses is the only accepted way.** Done during class Thursday 19.11 or Friday 20.11.

(1) Use the Cauchy integral formulas to evaluate the following contour integrals when the circles are positively oriented (that is, the winding numbers are 1 inside).

(a)
$$\int_{|z|=1} \frac{\operatorname{Log}(z+e)}{z}\, dz$$
(3p)

(b)
$$\int_{|z|=2} \frac{e^z}{(z+1)^2}\, dz$$
(3p)

(2) Let f be an entire function with the property that
$$|f(z)| \leq c|z|^\lambda + d$$
for all z, where λ, c, and d are positive constants. Prove that f is necessarily a polynomial function of z whose degree does not exceed λ. (6p)

(3) Let $u\colon \Delta(0,r) \to \mathbb{R}$ be a harmonic function (that is $u_{xx}+u_{yy}=0$ in $\Delta(0,r)$). Use Cauchys integral formula to show that
$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{it})\, dt$$
when $0 < \rho < r$. You may assume that $u = \operatorname{Re}(f)$ in $\Delta(0,r)$ for some analytic function $f\colon \Delta(0,r) \to \mathbb{C}$.

(4) If f is a non-constant entire function, prove that the range $f(\mathbb{C})$ of f must almost "fill up" the complex plane in the following sense: for every point $w_0 \in \mathbb{C}$ and every $r > 0$ we have $f(\mathbb{C}) \cap \Delta(w_0, r) \neq \emptyset$. (We say that $f(\mathbb{C})$ is dense in $\mathbb{C}$.) (*Hint*: Assume there is $w_0 \in \mathbb{C}$ and $r > 0$ such that $f(\mathbb{C}) \cap \Delta(w_0, r) = \emptyset$. Study $g(z) = 1/(f(z) - w_0)$.)