PHYS-C0252 - Quantum Mechanics

Exercise set 4

Due date : November 20, 2020 before 10.00

1. (a) Prove the following commutation relations:

   i. $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$, where $\hat{A}$, $\hat{B}$ and $\hat{C}$ are arbitrary linear operators acting on $\mathcal{H}$.

   iii. $[f(\hat{x}), \hat{p}_x] = \mathrm{i}\hbar\frac{\partial f(\hat{x})}{\partial \hat{x}}$,where $\hat{x}$ and $\hat{p}_x$ are the position and the momentum operator, respectively.

   iii. $[\hat{x}^2, \hat{p}_x^2] = 2\mathrm{i}\hbar(\hat{x}\hat{p}_x + \hat{p}_x\hat{x})$.

   (b) Show that the commutator of two Hermitian operators is anti Hermitian.

2. Consider the matrix representation of the Hadamard gate

$$H_{\mathrm{g}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   (a) Show that the Hadamard gate is Hermitian and unitary.
   (b) Verify that $\hat{H}_{\mathrm{g}}\hat{\sigma}_x\hat{H}_{\mathrm{g}} = \hat{\sigma}_z$, where $\{\hat{\sigma}_{i=x,z}\}$ are the Pauli spin-1/2 operators.

3. Consider the classical one-dimensional harmonic oscillator described by a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2,$$

   where $m > 0$ is the mass of the point-like particle and $\omega > 0$ is a constant with units of angular frequency. The equations of motion can be obtained from the standard Hamilton equations as

$$\dot{p} = -\frac{\partial H}{\partial q} \quad \dot{q} = \frac{\partial H}{\partial p}.$$

   (a) Solve equations of motion and find a general solution for q using the initial conditions $q(t = 0) = q_0$ and $p(t = 0) = p_0$ and plot the solution of q in p-q space.

   (b) Use the recipe given on Lecture III to quantize the system with the exception that you pick the mechanical momentum of the particle as the free coordinate. Does the physics change, i.e., are the equations of motion for physical quantities different from what we had on the lectures or above? Hint: You may use symbols p and q to denote the mechanical momentum and coordinate, respectively, as above and use symbols p_{c} and q_{c} to denote the canonical momentum and coordinate, respectively. Thus you start by defining $q_{\mathrm{c}} \equiv p$ and by writing down the potential energy $V(q_{\mathrm{c}})$. Then just follow the recipe until you have written down the quantum-mechanical Hamiltonian.

4. Consider the raising and lowering operators of a one-dimensional harmonic oscillator, $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{\mathrm{i}}{m\omega}\hat{p}_x)$ and $\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{\mathrm{i}}{m\omega}\hat{p}_x)$. On the lectures we showed that

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle,$$
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

(a) Show that $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$, where $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator.
(b) Evaluate the matrix elements $\langle n|\hat{a}|n'\rangle$ and $\langle n|\hat{a}^\dagger|n'\rangle$.
(c) Show that $|\alpha\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}|0\rangle$ is an eigenstate of $\hat{a}$ with eigenvalue α.

5. Consider a particle of mass m in a one-dimensional potential

$$V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2 - \lambda\hat{x}$$

(a) Find the algebraic representation of the corresponding Hamiltonian $\hat{H}$ in terms of the operators $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{\mathrm{i}}{m\omega}\hat{p}_x)$ and $\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{\mathrm{i}}{m\omega}\hat{p}_x)$.

(b) Find the eigenenergies of $\hat{H}$. Hint: Try to simplify the Hamiltonian into the form $\hat{A}^\dagger\hat{A} + C\hat{I}$, where $\hat{A} \in \mathcal{L}(\mathcal{H})$ and $C \in \mathbb{R}$.

(d) Let the particle be at $t = 0$ in the ground state.

i. Find expectation value of $V(\hat{x})$ at time $t = 0$.
ii. Find $\Delta\hat{x} = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2}$ and $\Delta\hat{p}_x = \sqrt{\langle\hat{p}_x^2\rangle - \langle\hat{p}_x\rangle^2}$ at time $t = 0$.