Formal "Loop shaping"

- Classical idea: use a compensator to get a desired loop gain
- New approach: minimization of H_∞, H_2 – norms
- Use of weights, sensitivity functions

System $\quad y = Gu + w$

Control law $\quad u = -F_y y$

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Sensitivity functions

$$S = \left(I + GF_y\right)^{-1}$$

$$T = I - S = \left(I + GF_y\right)^{-1} GF_y = GF_y \left(I + GF_y\right)^{-1}$$

$$G_{wu} = -\left(I + F_y G\right)^{-1} F_y = -F_y \left(I + GF_y\right)^{-1} = -F_y S$$
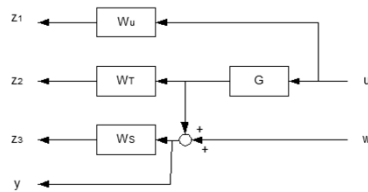
and weight matrices

$$W_S(i\omega)S(i\omega)$$
$$W_T(i\omega)T(i\omega)$$
$$W_u(i\omega)G_{wu}(i\omega)$$

which are used to "shape" the sensitivity functions. But the dimension of the system and the compensator grow.

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Generalized control configuration

$$z_1 = W_u u$$
$$z_2 = W_T Gu$$
$$z_3 = W_S(Gu + w)$$
$$y = Gu + w$$

"weights" are used to form an augmented system

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$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$
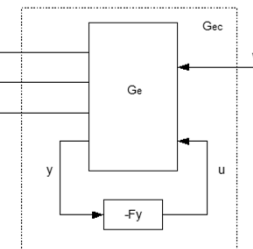
Generalized control configuration. When the control loop is closed, the transfer function from w to z is obtained.

$$z = \begin{bmatrix} W_u G_{wu} \\ -W_T T \\ W_S S \end{bmatrix} w = G_{ec} w$$

and the motivation of using the weights becomes obvious. Similarly, it is clear why the norms between w and z are minimized for performance.

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Control design can be carried out based on the generalized process model *Ge* e.g. by using the Matlab commands *mixsyn, hinfsyn* and *h2syn* (Robust Control Toolbox).

But look at the issue from theoretical viewpoint. Form a realization of the open loop system *Ge* with inputs *u,w* and outputs *z,y*

$$\dot{x} = Ax + Bu + Nw$$
$$z = Mx + Du$$
$$y = Cx + w$$

where certain assumptions have been made (*z* does not depend directly from *w*, *y* does not depend directly from *u*). Moreover, assume that $D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

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The assumptions are sometimes restrictive (they are needed for the mathematic solution to be appropriate), but often they can be relaxed, if needed . (In this case the solution gets more difficult and nasty to derive.)

To see how this can be done, let $D^T D$ be invertible. Change variables from *u* to

$$\tilde{u} = \left(D^T D\right)^{1/2} u + \left(D^T D\right)^{-1/2} D^T Mx$$

which gives

$$z = \tilde{M}x + \tilde{D}\tilde{u}, \quad \tilde{M} = \left(I - D\left(D^T D\right)^{-1} D^T\right)M, \quad \tilde{D} = D\left(D^T D\right)^{-1/2}$$

Now $\tilde{D}^T \begin{bmatrix} \tilde{M} & \tilde{D} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$ which is easy to verify.

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Example. DC motor

$$G(s) = \frac{20}{s(s+1)}$$

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 20 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Use simple weights $W_u = 1, W_T = 1, W_S = 1/s$

An augmented system model is obtained by taking the new state (see the figure)

$$z_3 = x_3, \quad x_3 = \frac{1}{p}\left(Gu + w\right)$$

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giving

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w$$

$$z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x + w$$

which also fulfils $D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

Control design is done from this model, the dimension of which has grown because of weights. In more complex cases it is more difficult to form the extended state representation. But see Matlab command *augw*.

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In Matlab, the command *mixsyn* turns out to be helpful here.

```
[K,CL,GAM,INFO]=mixsyn(G,W1,W2,W3)

mixsyn H-infinity mixed-sensitivity synthesis method for robust
control design. Controller K stabilizes plant G and minimizes
the H-infinity cost function

        ||  W1*S ||
        || W2*K*S ||
        ||  W3*T ||
where
  S := inv(I+G*K)        % sensitivity
  T := I-S = G*K/(I+G*K)  % complementary sensitivity
  W1, W2 and  W3 are stable LTI 'weights'

Inputs:
  G       LTI plant
  W1,W2,W3  LTI weights (either SISO or compatibly dimensioned MIMO)
        To omit weight, use empty matrix (e.g., W2=[] omits W2)
```

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```
Outputs:
  K       H-infinity Controller
  CL      CL=[W1*S; W2*K*S; W3*T]; weighted closed-loop system
  GAM     GAM=hinfnorm(CL), closed-loop H-infinity norm
  INFO    Information STRUCT, see HINFSYN documentation for details
```

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Example:

```
G=ss(-1,2,3,4);   % plant to be controlled
  w0=10;     % desired closed-loop bandwidth
  A=1/1000;  % desired disturbance attenuation inside bandwidth
  M=2 ;      % desired bound on hinfnorm(S) & hinfnorm(T)
  s=tf('s'); % Laplace transform variable 's'
  W1=(s/M+w0)/(s+w0*A); % Sensitivity weight
  W2=[];              % Empty control weight
  W3=(s+w0/M)/(A*s+w0); % Complementary sensitivity weight
  [K,CL,GAM,INFO]=mixsyn(G,W1,W2,W3);

Plot results of successful design:
  L=G*K;  % loop transfer function
  S=inv(1+L); % Sensitivity
  T=1-S;      % complementary sensitivity
```

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Mixsyn does the H infinity problem formulation automatically and solves the problem. If you use the command *hinfsyn*, you have to form the augmented plant yourself and pose the problem accordingly.

This is *Mixed Sensitivity Design*, an advanced form of *Loop Shaping Control*.

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Slide 1

Example of control design
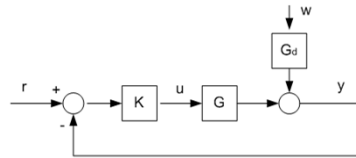
$$G(s) = \frac{200}{(10s+1)(0.05s+1)^2}$$

$$G_d(s) = \frac{100}{10s+1}$$

Command tracking + disturbance rejection problem

Both demands are difficult to meet simultaneously (trade-off in control design)

Let us try loop shaping by H_∞ control.

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Slide 2

Example of control design...

```
% Mixed sensitivity design
%
% Uses the Robust Control Toolbox
%
s=tf('s');
G=200/(10*s+1)/(0.05*s+1)^2;
Gd=100/(10*s+1);
M=1.5; wb=10; A=1e-4;
Ws=tf([1/M wb], [1 wb*A]); Wu=1;
[Fy,CL,gopt]=mixsyn(G,Ws,Wu,[]);
```
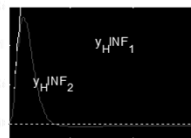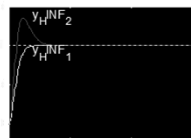
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Slide 3
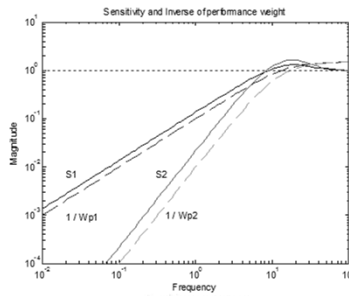
Note: $W_P = W_S$

$$W_{P1} = \frac{s/M + \omega_B^*}{s + \omega_B^* A}; \quad M = 1.5, \omega_B^* = 10, A = 10^{-4}$$

$$W_{P2} = \frac{(s/M^{1/2} + \omega_B^*)^2}{(s + \omega_B^* A^{1/2})^2}; \quad M = 1.5, \omega_B^* = 10, A = 10^{-4}$$

Because the load response is very poor in design 1, higher gains for the controller at low frequencies are needed (integral action).

To that end, use W_{P2}, and the result is clearly better.

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Slide 4

Some norm theory:

Measure the output z by using the 2-norm

$$\|z(t)\|_2 = \sqrt{\sum_i \int_{-\infty}^{\infty} |z_i(\tau)|^2 \, d\tau}$$

Note. In what follows the norm is denoted by "two bars" to make a distinction to the absolute value of a scalar value. (The textbook uses, for some obscure reason, two bars only in the case of a system norm).

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The system 2-norm (euclidian norm) is

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\left(G(i\omega)^* G(i\omega)\right) d\omega}$$

where

$$\mathrm{tr}\left|G(i\omega)^* G(i\omega)\right| = \sum_{i,j}\left|G_{ij}(i\omega)\right|^2 = \|G(i\omega)\|_F^2$$

is called the Frobenius norm. The system must be "strictly proper", $D=0$, in order the 2-norm to be finite. By using the Parseval theorem

$$\|G(s)\|_2 = \|g(t)\|_2 = \sqrt{\int_0^\infty \mathrm{tr}\left(g(\tau)^T g(\tau)\right)d\tau} = \sqrt{\int_0^\infty \sum_{i,j}\left|g_{ij}\right|^2 d\tau} = \sqrt{\sum_{i,j}\int_0^\infty\left|g_{ij}\right|^2 d\tau}$$

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and it is seen that the 2-norm can be interpreted as a size measure of the output, when impulses are fed at the input. That has a connection to the stochastic interpretation, because impulse inputs can be interpreted to be white noise.

H2 –norm is then:

$$\|G(s)\|_2 = \max \|z(t)\|_2 \qquad \text{when input } w \text{ is composed of unit impulses.}$$

Let the system be "proper" (not necessarily "strictly", D can be non-zero). Define the $H\infty$ - norm

$$\|G(s)\|_\infty = \max_\omega \bar{\sigma}\left(G(i\omega)\right) \qquad \text{the maximum of the largest singular value of the frequency function}$$

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It can be shown that

$$\|G(s)\|_\infty = \max_{\omega(t)\neq 0}\frac{\|z(t)\|_2}{\|w(t)\|_2}$$

is the largest gain to non-zero input signals.

Differences between H2 – and H∞- norms:

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{i,j}\left|G_{ij}(i\omega)\right|^2 d\omega} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_i \sigma_i^2(G(i\omega))d\omega}$$

(because it can be shown that the Frobenius norm can be written by means of singular values; not proved here)

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It is seen that the minimization of these norms means:
H∞ : minimize the maximum of the largest singular value
-H2:   minimize all singular values of all frequencies

But what are the consequences of all this? We considered the closed-loop system

$$z = \begin{bmatrix} -W_u G_{wu} \\ -W_T T \\ W_S S \end{bmatrix} w = G_{ec} w$$

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and now

$$\|G_{ec}(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{tr}\left(G_{ec}(i\omega)^* G_{ec}(i\omega)\right)d\omega}$$

so that

$$\|G_{ec}(s)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\|W_S(i\omega)S(i\omega)\|_2^2 + \|W_T(i\omega)T(i\omega)\|_2^2 + \|W_u(i\omega)G_{wu}(i\omega)\|_2^2\right]d\omega$$

These should be "pushed down" on the whole frequency range. But as that could be interpreted as the minimization of the impulse response, set the criterion

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$$V(F_y) = \|z\|_2^2 = \|Mx + Du\|_2^2 = \left[(Mx+Du)^T(Mx+Du)\right]^2$$
$$= \left[x^T M^T Mx + x^T M^T Du + u^T D^T Mx + u^T D^T Du\right]^2$$
$$= \|Mx\|_2^2 + \|u\|_2^2$$

where the assumption has been used

$$D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \Rightarrow D^T M = 0, D^T D = I$$

The familiar *LQ* (*LQG*) –criterion was obtained. So, H2 – minimization corresponds to *LQ(G)* –control. The difference is the more general formulation (generalized model, input and output variables,weights), when compared to the conventional *LQ*-theory. But note that it is easy to formulate this kind of a control problem, which does not have a solution; H2-norm is then not finite.

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The solution is a state feedback from reconstructed states (if the states cannot be measured, the Kalman filter must be used).

But what about *H∞*-control: The norm to be minimized is

$$\|G_{ec}\|_\infty = \max_\omega \ \bar\sigma\left(G_{ec}(i\omega)\right)$$

the largest singular value of the closed-loop system.

But that cannot be made analytically! Instead, try to find a controller, which fulfils

$$\|G_{ec}\|_\infty \le \gamma \qquad$$ find iteratively the smallest γ, for which a corresponding controller exists.

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Result: Consider the open loop

$$\dot{x} = Ax + Bu + Nw$$
$$z = Mx + Du$$
$$y = Cx + w$$

where $D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

If the Riccati equation

$$A^T S + SA + M^T M + S\left(\gamma^{-2}NN^T - BB^T\right)S = 0$$

has a positive semidefinite solution, then for the system controlled by

$$u = -B^T S\hat{x}$$

it holds that

$$\|z\|_2 \le \gamma \|w\|_2$$

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for all inputs w. But the 2-norm of the signals induces the system ∞-norm. Then the ∞-norm of the system is smaller than γ.

The design procedure:
1. Determine the generalized plant *G*.
2. Design weights W_u, W_s, W_τ .
3. Pick γ.
4. If the controller exists, make γ smaller, otherwise make it larger; iterate until the smallest γ has been found (so-called γ-iteration).
5. Investigate the properties of the closed loop; if not good enough, goto item 2.

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Note. 1. Because of the weights the controller usually has a high dimension. Use model reduction techniques to reduce the dimension without changing much the controller properties.

Note. 2. γ-iteration and the design is done automatically by the command *hinfsyn* in the Robust Control-toolbox of Matlab. (Corresponding to *h2syn*).

The iteration need not be programmed by the designer.

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