ELEC-E8116 Model-based control systems /exercises 10 Solutions

1. Consider a SISO system and a state feedback control

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$u(t) = -Lx(t)$$

where L is chosen as a solution to the infinite time optimal (LQ) horizon problem.

a. Prove that the loop gain is $H(s) = L(sI - A)^{-1}B$

b. Prove that $\left|1 + H(i\omega)\right| \geq 1$

c. Show that for the LQ controller

- phase margin is at least 60 degrees
- gain margin is infinite
- the magnitude of the sensitivity function is less than 1
- the magnitude of the complementary sensitivity function is less than 2.

Solution:

a. First solve for x: $px = Ax + Bu \Rightarrow x = \left[pI - A\right]^{-1} Bu$

Starting from the output of the controller u go around the loop and meet the signal u again. We get

$$u = -Lx = -L\left[pI - A\right]^{-1} Bu$$

The open loop transfer function is the forward loop transfer function multiplied by the feedback loop transfer function. The open loop is then

$$H(s) = L\left[sI - A\right]^{-1} B$$

as given in the problem. Note: no minus sign, because it is the feedback sign.

b. In the *LQ* problem

$H(s) = L\left[sI - A\right]^{-1} B$ Note that L is now the state feedback gain, H is the open loop transfer function.

The (stationary) Riccati equation: $A^T S + SA + Q - SBR^{-1}B^T S = 0$.

State feedback gain: $L = R^{-1}B^T S$.

In the exercise session the problem was solved in the simple case of assuming one-dimensional state variable *x*. Then all the matrices are scalars:

$$
\begin{aligned}
&\left|1+H(j\omega)\right|^2 = (1+H(j\omega))^*(1+H(j\omega)) = (1+H(-j\omega))(1+H(j\omega)) \\
&= \left(1+\frac{lb}{-j\omega-a}\right)\left(1+\frac{lb}{j\omega-a}\right) = \frac{-a+lb-j\omega}{-a-j\omega}\cdot\frac{-a+lb+j\omega}{-a+j\omega} \\
&= \frac{\left(-a+lb\right)^2+\omega^2}{a^2+\omega^2} = \frac{a^2-2abl+b^2l^2+\omega^2}{a^2+\omega^2} \\
&= \frac{a^2-2a\frac{b^2}{r}s+b^2\frac{b^2s^2}{r^2}+\omega^2}{a^2+\omega^2} = \frac{a^2+\frac{b^2}{r}(\frac{b^2s^2}{r}-2as)+\omega^2}{a^2+\omega^2} \\
&= \frac{a^2+\frac{b^2}{r}q+\omega^2}{a^2+\omega^2} \geq 1
\end{aligned}
$$

because $\frac{b^2}{r}q \geq 0$. Note how the Riccati equation was used in the last part of the derivation.

But the general inequality is

$$\left[I+H(-j\omega)\right]^T R\left[I+H(j\omega)\right] \geq R$$

which applies also to multivariable cases. In the case of single transfer functions the above trivially simplifies to

$$\left|1+H(i\omega)\right| \geq 1$$

The general proof (MIMO case) is however a bit more complicated.

$$
\begin{aligned}
&\left[I+H(-j\omega)\right]^{T} R\left[I+H(j\omega)\right]=\left[I+H(-j\omega)\right]^{T}\left[R+RH(j\omega)\right] \\
&=R+RH(j\omega)+H(-j\omega)^{T}R+H(-j\omega)^{T}RH(j\omega) \\
&=R+RL\left[j\omega I-A\right]^{-1}B+B^{T}\left[-j\omega I-A\right]^{-T}L^{T}R+B^{T}\left[-j\omega I-A\right]^{-T}L^{T}RL\left[j\omega I-A\right]^{-1}B \\
&=R+B^{T}S\left[j\omega I-A\right]^{-1}B+B^{T}\left[-j\omega I-A^{T}\right]^{-1}SB+B^{T}\left[-j\omega I-A^{T}\right]^{-1}SBR^{-1}B^{T}S\left[j\omega I-A\right]^{-1}B \\
&=R+B^{T}\left[-j\omega I-A^{T}\right]^{-1}\left\{\left[-j\omega I-A^{T}\right]S+S\left[j\omega I-A\right]+SBR^{-1}B^{T}S\right\}\left[j\omega I-A\right]^{-1}B \\
&=R+B^{T}\left[-j\omega I-A^{T}\right]^{-1}\left\{-A^{T}S-SA+A^{T}S+SA+Q\right\}\left[j\omega I-A\right]^{-1}B \\
&=R+B^{T}\left[-j\omega I-A^{T}\right]^{-1}Q\left[j\omega I-A\right]^{-1}B\geq R
\end{aligned}
$$

To see the last inequality note that R is positive definite. The matrix

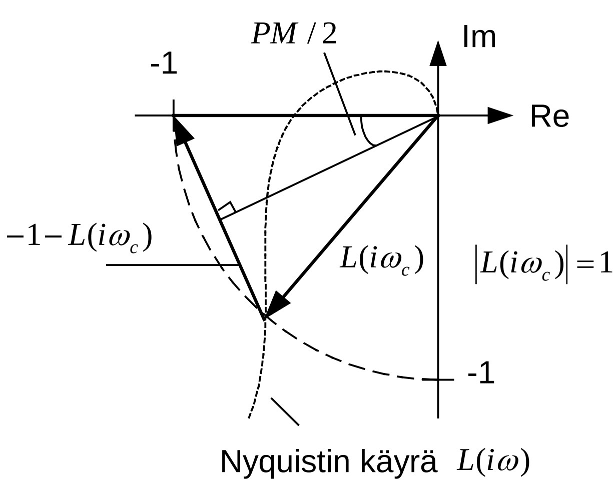
$$Z=B^{T}\left[-j\omega I-A^{T}\right]^{-1}Q\left[j\omega I-A\right]^{-1}B$$

is clearly real, because $Z^{*}=Z$ (the matrix is in fact Hermitian). But for any non-zero vector x with appropriate dimension

$$
\begin{aligned}
&x^{*}Zx=x^{*}B^{T}\left[-j\omega I-A^{T}\right]^{-1}Q\left[j\omega I-A\right]^{-1}Bx \\
&=\left[\left(j\omega I-A\right)^{-1}Bx\right]^{*}Q\left[\left(j\omega I-A\right)^{-1}Bx\right]=y^{*}Qy\geq 0
\end{aligned}
$$

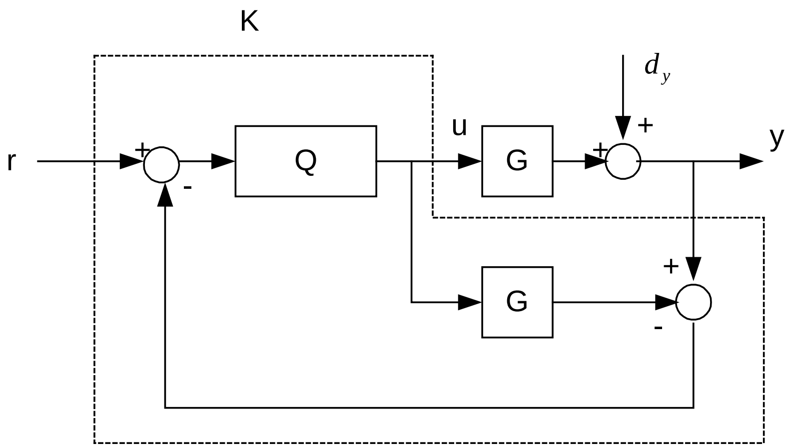
Hence Z is positive demidefinite. Note that Q and R are positive definite by definition.

c. Consider the following figure, where $L = H$ now is the loop transfer function.



Because $|1+H(i\omega)|\geq 1$ the Nyquist curve will never enter inside the circle centered at (-1,0) and with the radius 1. Therefore the gain margin is infinite and the sensitivity function is never larger than 1 in magnitude. The complementary sensitivity function cannot be larger than 2, because the two sentitivity functions can differ at most by 1 in magnitude. Now the Nyquist curve touches the dashed line at the gain crossover frequency ω_c and if $|1+L(i\omega)|=1$ (minimum) we have an equilateral triangle (see figure) so that each angle is 60 degrees. But generally $|1+L(i\omega)|\geq 1$ so that the phase margin is at least 60 degrees.

2. Consider the IMC control structure, which is used to control a stable and minimum phase SISO process *G*.



Note that in addition to the reference *r* a disturbance signal d_y is modelled to enter at the output of the process. By using the IMC design discussed in the lectures analyse the response to step inputs at *r* and d_y.

Solution:

The figure represents a two-degrees-of-freedom control configuration, where the inputs to the controller *K* are *r* and *y*. Again, it is easy to write

$$u = Q[r-(y-Gu] = Q(r-y)+QGu \Rightarrow u=(I-QG)^{-1}Q(r-y)$$

But that can be interpreted as a one-degree-of-freedom configuration with the controller

$$u = K_1(r-y), \quad K_1=(I-QG)^{-1}Q=\frac{Q}{1-QG} \qquad \text{(SISO!)}$$

Using the design (see lecture slides)

$Q = \frac{1}{(\lambda s + 1)^n} G^{-1}$ and writing equations from the topology in the figure

$$y = d_y + Gu = d_y + GK_1(r - y) \Rightarrow y = \frac{GK_1}{1 + GK_1} r + \frac{1}{1 + GK_1} d_y$$

Setting K_1 to this gives after simple calculations

$$y = \frac{\frac{GQ}{1 - QG}}{1 + \frac{GQ}{1 - QG}} r + \frac{1}{1 + \frac{GQ}{1 - QG}} d_y = GQr + (1 - QG) d_y = \frac{1}{(\lambda s + 1)^n} r + \left[1 - \frac{1}{(\lambda s + 1)^n}\right] d_y$$

Note that $GQ = QG$ for SISO systems. Also $y = GQr + (1 - QG) d_y$ could have been obtained directly from the figure (careful!).

Setting $s = 0$ we find that the static gain from r to y is 1 and from d_y to y 0, so that the output follows the reference and mitigates the disturbance asymptotically. Note that internal stability was guaranteed by the fact that G was stable and minimum phase (G^{-1} stable) and Q stable.