Aalto university
Björn Ivarsson

Exercise sheet 9
Complex Analysis, MS-C1300.
Hand in exercise 1 and 2 for grading. Deadline Monday 23.11 at 23:59. The exercises should be uploaded to the correct folder on MyCourses as one pdf-file with name and student number in the file name. **Submission via MyCourses is the only accepted way.**
Done during class Tuesday 24.11 or Wednesday 25.11.

(1) Assume that a function f is analytic in an annulus D centered at a point z_0, that is
$$D = \{z \in \mathbb{C} \; ; a < |z - z_0| < b\},$$
where $0 \le a < b \le \infty$ and $z_0 \in \mathbb{C}$. Show that
$$\int_{|z-z_0|=r} f(z) \, dz = \int_{|z-z_0|=s} f(z) \, dz$$
whenever $a < r < s < b$ and the circles $|z - z_0| = r$ and $|z - z_0| = s$ are positively oriented. (6p)

(2) Let $\alpha(t) = e^{it}$, $\beta(t) = \frac{5}{3}+e^{it}$, and $\gamma(t) = -1+2e^{it}$ for $0 \le t \le 2\pi$.
  (a) Is the cycle $\sigma = (\alpha, \beta, \gamma)$ homologous to zero in $\mathbb{C}\backslash\{2i, -2i\}$? (2p)

  (b) Are the cycles $\sigma = (\alpha, -\gamma)$ and $\tau = (\beta, \beta)$ homologous in $\mathbb{C} \setminus \overline{\Delta(0, \frac{1}{2})}$? (2p)

  (c) Are the cycles $\sigma = (\alpha, \beta, \gamma)$ and $\tau = (\gamma, \gamma, \beta)$ homologous in $\mathbb{C} \setminus \{0\}$? (2p)

(3) Let $\alpha(t) = e^{-it}$ and $\beta(t) = 3\cos t + i \sin t$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Compute
$$\int_{\beta+\alpha} \frac{\operatorname{Log}(z)}{z(z-2)^2(z-4)} \, dz.$$

(4) Prove that
$$\lim_{r \to \infty} \int_{c-ir}^{c+ir} \frac{1}{z \operatorname{Log}(z)} \, dz = 0$$
when $c > 1$. (*Hint*: Consider $\int_\gamma (z \operatorname{Log}(z))^{-1} \, dz$ where $\gamma(t) = c + re^{it}$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.)