ELEC-E5440 Statistical Signal Processing.
Homework set #2
due December 14, 2020, at 18:00

1. MS and MAP Estimators

Consider the model $Y = X+N$, where X and N are random variables with density functions

$$f_X(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-1)$$

and

$$f_N(n) = \frac{1}{2}e^{-|n|}$$

respectively, and $\delta(\cdot)$ is the unit impulse function. Find $\hat{x}_{\text{MS}}$, the minimum mean-square error estimator and $\hat{x}_{\text{MAP}}$, the maximum a posteriori estimator of X from the observation Y.

2. MS and MAP Estimators

Suppose that Θ is a random parameter and that, given $\Theta = \theta$, the real random variable X has the conditional density:

$$p(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}, \quad x = 0, 1, \ldots, n, \quad 0 \leq \theta \leq 1 \tag{1}$$

Suppose further that Θ has the prior density:

$$p(\theta) = 3(1-\theta)^2, \quad 0 \leq \theta \leq 1 \tag{2}$$

a) Find the maximum a posteriori estimate of θ

b) Find the mean square estimate for θ

Hint: The computation is straight forward, if you consider that the Beta distribution, with parameters α and β has the form:

$$Beta_{\alpha,\beta}(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad \text{with} \quad 0 \leq \theta \leq 1 \quad \text{and} \quad \alpha,\ \beta > 0. \tag{3}$$

where the Gamma function is:

$$\Gamma(k) = \int_0^\infty x^{k-1}e^{-x}dx. \tag{4}$$

Just remind that the Gamma function has the following recursive relationship:

$$\Gamma(k+1) = k\Gamma(k). \tag{5}$$

3. Direction of Arrival estimation using real-world data

Apply the MUSIC and ESPRIT methods to the real-world data in the file *submarine.mat*, which can be found at the course web pages. These data are underwater measurements collected by the Swedish Defence Agency in the Baltic Sea. The 6-element array of hydrophones used in the experiment can be assumed to be a ULA with inter-spacing element equal to 0.9m. The wavelength of the signals is approximately 5.32m. Can you find the submarine(s)? Compare the performance of MUSIC and TLS-ESPRIT.

4. Target tracking using Kalman filter

Kalman filtering example: Target tracking

A radar tracks a target in two-dimensional space. The target state $\vec{x}$ is modeled using a second-order model such that

$$\vec{x} = \begin{pmatrix} x & \dot{x} & \ddot{x} & y & \dot{y} & \ddot{y} \end{pmatrix}$$

where (x, y) is position, $(\dot{x}, \dot{y})$ is velocity, and $(\ddot{x}, \ddot{y})$ is acceleration. The states are predicted and filtered using a Kalman filter. The state transition matrix is derived from kinematics assuming that x and y axes are decoupled such that

$$\vec{F} = \vec{I} \otimes \begin{pmatrix} 1 & \Delta t & \frac{1}{2}\Delta t^2 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{pmatrix}$$

where $\Delta t = 1$ is time interval between measurements and $\otimes$ is Kronecker product. Assume that the target state transitions can be modeled as a Wiener process such that $\vec{x}_{k+1} = \vec{F}\vec{x}_k + \vec{g}w$, where $w \in \mathcal{N}(0, \sigma_w^2)$. Therefore, the process covariance matrix is

$$\vec{Q} = \vec{I} \otimes \sigma_w^2 \begin{pmatrix} \frac{1}{4}\Delta t^4 & \frac{1}{2}\Delta t^3 & \frac{1}{2}\Delta t^2 \\ \frac{1}{2}\Delta t^3 & \Delta t^2 & \Delta t \\ \frac{1}{2}\Delta t^2 & \Delta t & 1 \end{pmatrix}$$

The sensor measures the target position such that the measurement matrix is

$$\vec{H} = \vec{I} \otimes \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

The measurements are corrupted by zero mean Gaussian noise with a covariance matrix

$$\vec{R} = \sigma_v^2 \vec{I}$$

where $\sigma_v = 500$ is the standard deviation of the measurement noise.

Variable	**Description**
meas_pos	Radar measurements of target positions with interval Δt.
real_pos	Real target positions at the measurement time instances.
x_init	Initial state estimate.
P_init	Initial estimation covariance matrix.

Table 1: Description of variables in *exercise_data.mat*.

Exercise data is given as a file *exercise_data.mat* which contains the variables described in Table 1.

a) Implement Kalman filtering steps (prediction and filtering) using MATLAB. Set the parameter $\sigma_w = 0.22$. Visualize measured, predicted, filtered and real target positions in the same figure.

b) The selection of the process noise σ_w is difficult in real-world scenarios because the motion model may not be optimal or σ_w is unknown. Experiment with different σ_w values using 0.1, 2.2 and 100. How does the parameter σ_w affect the prediction and filtering performance when comparing estimation error, posterior error, and measurement error as a function of time? The error used here is Euclidean distance to the real target position. Explain intuitively the results.