6. Representations of $\mathfrak{sl}_3(\mathbb{C})$

We already showed how to find and construct all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$, and how to apply the results to representations of Lie groups, whose Lie algebras have $\mathfrak{sl}_2(\mathbb{C})$ as their complexification, e.g., $\text{SU}_2$ and $\text{SO}_3$.

We will proceed to treat more complicated (semisimple) Lie algebras. We start in this section by considering $\mathfrak{sl}_3(\mathbb{C})$. The representations of $\mathfrak{sl}_3(\mathbb{C})$ are needed for example in quantum chromodynamics (QCD), the theory of strong interactions that govern the atomic nuclei. Besides their direct relevance, the analysis of the structure and representations of $\mathfrak{sl}_3(\mathbb{C})$ will serve as a wonderful example of what happens with semisimple Lie algebras in full generality.

We will follow a similar strategy as in the case of $\mathfrak{sl}_2(\mathbb{C})$ to analyze the structure of $\mathfrak{sl}_3(\mathbb{C})$ and its representations. We only require some new ideas, or rather reinterpretations of a few concepts and arguments. These ideas turn out to be powerful — with them, we will be able to handle any semisimple Lie algebra.

6.1. The Lie algebra $\mathfrak{sl}_3(\mathbb{C})$

Recall that $\mathfrak{sl}_3(\mathbb{C})$ is the set $\mathfrak{sl}_3(\mathbb{C}) = \{ M \in \mathbb{C}^{3 \times 3} \mid \text{tr}(M) = 0 \}$ of traceless (complex) three-by-three matrices, equipped with the Lie bracket $[M_1, M_2] = M_1 M_2 - M_2 M_1$. As a (complex) vector space, it is eight dimensional $\dim(\mathfrak{sl}_3(\mathbb{C})) = 8$.

Indeed, the nine entries $X_{i,j}$, $1 \leq i, j \leq 3$, of a matrix $X \in \mathfrak{sl}_3(\mathbb{C})$ can be chosen arbitrarily subject to just one linear condition, $\text{tr}(X) = X_{1,1} + X_{2,2} + X_{3,3} = 0$.

**Remark II.27.** For calculations below, we recall the definition and properties of the elementary matrices $E^{kl}$. For a general dimension $n \in \mathbb{N}$ and for $1 \leq k, l \leq n$, the elementary matrix $E^{kl} \in \mathbb{K}^{n \times n}$ is the matrix whose $(k,l)$-entry is one, and all other entries are zeroes, $E^{kl}_{ij} = \delta_{k,i} \delta_{l,j}$. The products of such matrices are

$$E^{kl} E^{k'l'} = \delta_{l,k'} E^{k'l'},$$

as is verified by the following direct calculation

$$(E^{kl} E^{k'l'})_{ij} = \sum_m E^{kl}_{im} E^{k'l'}_{mj} = \sum_m \delta_{k,i} \delta_{l,m} \delta_{k',m} \delta_{l',j} = \delta_{l,k'} \delta_{k,i} \delta_{l',j} = \delta_{l,k'} E^{k'l'}. $$

The $n^2$ elementary matrices $E^{kl}$ form a basis of $\mathfrak{gl}_n(\mathbb{K})$, and the brackets in $\mathfrak{gl}_n(\mathbb{K})$ (and thus also in any Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{K})$) read

$$[E^{kl}, E^{k'l'}] = E^{kl} E^{k'l'} - E^{k'l'} E^{kl} = \delta_{l,k'} E^{k'l'} - \delta_{k',l} E^{k'l}. $$

In our analysis of $\mathfrak{sl}_3(\mathbb{C})$, we will follow steps modelled on those that we took in the analysis of $\mathfrak{sl}_2(\mathbb{C})$ in the previous lecture. For $\mathfrak{sl}_2(\mathbb{C})$, our analysis relied first of all on a good choice of basis $H, E, F$ — we split any representation (including the adjoint representation on $\mathfrak{sl}_2(\mathbb{C})$ itself) to eigenspaces of $H$, and figured out how $E$ and $F$ acted on the eigenspaces. The task now is to find the appropriate generalizations.
The good idea turns out to be not to pick just one element to diagonalize, but rather to take an entire subspace \( \mathfrak{h} \subset \mathfrak{sl}_3(\mathbb{C}) \) to be diagonalized simultaneously. Such a simultaneous diagonalization in any representation succeeds if all the needed operators commute with each other, which is guaranteed if \( \mathfrak{h} \) is an abelian subalgebra of \( \mathfrak{sl}_3(\mathbb{C}) \). We choose \( \mathfrak{h} \) to consist of all diagonal matrices in \( \mathfrak{sl}_3(\mathbb{C}) \), i.e.,

\[
\mathfrak{h} = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \middle| a_1, a_2, a_3 \in \mathbb{C}, \ a_1 + a_2 + a_3 = 0 \right\}. \tag{II.13}
\]

All diagonal matrices indeed commute with each other, so \([\mathfrak{h}, \mathfrak{h}] = 0\), and the simultaneous diagonalization of the action of all \( H \in \mathfrak{h} \) is possible.

Since we are not considering the diagonalization of a single linear operator, but an entire space of operators, the concept of eigenvalue needs to be appropriately generalized. If \( V \) is a representation, and \( v \in V \) is a simultaneous eigenvector for the action of all \( H \in \mathfrak{h} \), then we have

\[
Hv = \mu(H)v \quad \forall H \in \mathfrak{h}, \tag{II.14}
\]

where \( \mu(H) \) denotes the eigenvalue of the action of \( H \in \mathfrak{h} \). Obviously \( \mu(H) \) depends linearly on \( H \), and so defines a linear functional \( \mu : \mathfrak{h} \to \mathbb{C} \), i.e., an element \( \mu \in \mathfrak{h}^* \) of the dual of \( \mathfrak{h} \). This is the appropriate generalization of eigenvalues and eigenvectors.

We call \( \mu \in \mathfrak{h}^* \) a weight and \( v \in V \) satisfying (II.14) a weight vector (of weight \( \mu \)). Analogously to the decomposition (II.9), any finite-dimensional representation \( V \) of \( \mathfrak{sl}_3(\mathbb{C}) \) has a decomposition

\[
V = \bigoplus_{\mu} V_{\mu}, \tag{II.15}
\]

where \( \mu \) runs over weights \( V \), a priori some finite collection of linear functionals \( \mu \in \mathfrak{h}^* \), and \( V_{\mu} \) are the corresponding weight spaces for \( \mathfrak{h} \)

\[
V_{\mu} = \left\{ v \in V \mid \forall H \in \mathfrak{h} : Hv = \mu(H)v \right\}. \tag{II.16}
\]

We have \( \text{dim}(\mathfrak{h}) = 2 \), and to be concrete we can take a basis \( H^{1,2} = E^{1,1} - E^{2,2}, H^{2,3} = E^{2,2} - E^{3,3} \) for \( \mathfrak{h} \). It is convenient to write the dual elements as linear combinations of \( \eta^i, i = 1, 2, 3 \), defined on all diagonal \( 3 \times 3 \)-matrices by

\[
\eta^i \left( \sum_{j=1}^{3} a_j E^{j,j} \right) = a_i.
\]

As a basis of the dual, we can then take for example \( \eta^1 - \eta^2 \) and \( \eta^2 - \eta^3 \), but we remark that all \( \eta^i, i = 1, 2, 3 \), make sense as elements of \( \mathfrak{h}^* \).\(^9\)

**Example II.28.** The space \( V = \mathbb{C}^3 \) is naturally a representation of \( \mathfrak{sl}_3(\mathbb{C}) \): any element \( X \in \mathfrak{sl}_3(\mathbb{C}) \) is a \( 3 \times 3 \)-matrix, which we let act on any vector \( v \in V = \mathbb{C}^3 \) by matrix multiplication \( Xv \). This three-dimensional representation is called the standard representation of \( \mathfrak{sl}_3(\mathbb{C}) \).

The standard basis vectors \( e_1, e_2, e_3 \in \mathbb{C}^3 \) are weight vectors, with respective weights \( \eta^1, \eta^2, \eta^3 \). The weight space decomposition of the standard representation \( \mathbb{C}^3 \) of \( \mathfrak{sl}_3(\mathbb{C}) \) is thus

\[
\mathbb{C}^3 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 = (\mathbb{C}^3)_{\eta^1} \oplus (\mathbb{C}^3)_{\eta^2} \oplus (\mathbb{C}^3)_{\eta^3}.
\]

\(^9\)Acting on \( \mathfrak{h} \), the elements \( \eta^1, \eta^2, \eta^3 \) are not linearly independent, of course, since \( \eta^1(H) + \eta^2(H) + \eta^3(H) = 0 \) holds for any traceless diagonal matrix \( H \).
Example II.29. Recall that if $V$ is a representation of a Lie algebra $g$, then the dual $V^*$ becomes a representation by defining, for any $X \in g$ and $\varphi \in V^*$, the dual element $X.\varphi$ as $v \mapsto -\varphi(X.v)$ for all $v \in V$.

The dual $V^*$ of the standard representation $V = \mathbb{C}^3$ of $\mathfrak{sl}_3(\mathbb{C})$ is thus a three-dimensional representation. Let $\varphi_1, \varphi_2, \varphi_3 \in V^*$ be the dual basis to the standard basis $e_1, e_2, e_3 \in V$, i.e. $\varphi_j(e_i) = \delta_{ij}$ for all $i, j \in \{1, 2, 3\}$. If $H \in \mathfrak{h}$, then

$$(H.\varphi_j)(e_i) = -\varphi_j(H.e_i) = -\varphi_j(\eta^i(H)e_i) = -\eta^j(H)\delta_{ij} = -\eta^j(H)\varphi_j(e_i),$$

which implies that $H.\varphi_j = -\eta^j(H)\varphi_j$. The basis vectors $\varphi_1, \varphi_2, \varphi_3$ are thus weight vectors, with respective weights $-\eta^1, -\eta^2, -\eta^3$, and the weight space decomposition of the dual of the standard representation of $\mathfrak{sl}_3(\mathbb{C})$ is

$$V^* = \mathbb{C}\varphi_1 \oplus \mathbb{C}\varphi_2 \oplus \mathbb{C}\varphi_3 = (V^*)_{-\eta^1} \oplus (V^*)_{-\eta^2} \oplus (V^*)_{-\eta^3}.$$

In particular (unlike for $\mathfrak{sl}_2(\mathbb{C})$), a representation of $\mathfrak{sl}_3(\mathbb{C})$ and its dual are generally not isomorphic to each other (even the weights in $V$ and $V^*$ are different).

Example II.30. The adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$ is the vector space $V = \mathfrak{sl}_3(\mathbb{C})$ equipped with the adjoint action: for $X \in \mathfrak{sl}_3(\mathbb{C})$ and $Y \in V = \mathfrak{sl}_3(\mathbb{C})$, we set

$$\text{ad}_X(Y) = [X,Y],$$

which defines $\text{ad}: \mathfrak{sl}_3(\mathbb{C}) \to \text{End}(\mathfrak{sl}_3(\mathbb{C}))$.

This is an eight-dimensional representation of $\mathfrak{sl}_3(\mathbb{C})$.

We will next address the weight space decomposition in this case.

6.2. Representations of $\mathfrak{sl}_3(\mathbb{C})$

We will use the following two facts about finite dimensional representations of $\mathfrak{sl}_3(\mathbb{C})$.

Fact II.18. On any finite dimensional representation $V$ of $\mathfrak{sl}_3(\mathbb{C})$, the actions of all $H \in \mathfrak{h} \subset \mathfrak{sl}_3(\mathbb{C})$ are simultaneously diagonalizable.

The proof of this fact follows from general theory of semisimple Lie algebras, but it is also not difficult to deduce from the corresponding fact for $\mathfrak{sl}_2(\mathbb{C})$. The simultaneous eigenspaces are the weight spaces (II.16) in the decomposition (II.15).

Fact II.19. Any finite-dimensional representation of $\mathfrak{sl}_3(\mathbb{C})$ is a direct sum of its irreducible subrepresentations.

This fact follows from general theory of semisimple Lie algebras, which we will treat later.

6.2.1. The adjoint representation and roots for $\mathfrak{sl}_3(\mathbb{C})$

In particular, the adjoint representation $V = \mathfrak{sl}_3(\mathbb{C})$ admits a decomposition to weight spaces

$$\mathfrak{sl}_3(\mathbb{C}) = \bigoplus_{\mu} (\mathfrak{sl}_3(\mathbb{C}))_{\mu},$$

as we will verify now. The (abelian) subalgebra of diagonal matrices clearly consists of vectors that have eigenvalue 0 for the adjoint action of any other diagonal matrix,
so we have \( \mathfrak{h} \subset (\mathfrak{sl}(\mathbb{C}))_0 \). For an elementary matrix \( E^{ij} \), and diagonal matrix \( H = \sum_k a_k E^{kk} \), we calculate

\[
[H, E^{ij}] = \sum_k a_k [E^{kk}, E^{ij}] = \sum_k a_k (\delta_{ki} E^{kj} - \delta_{jk} E^{ik})
\]

which shows that the one-dimensional subspace \( \mathbb{C}E^{ij} \), for \( i \neq j \), is a simultaneous eigenspace for all \( H \in \mathfrak{h} \), with eigenvalues given by the weight \( \eta^i - \eta^j \in \mathfrak{h}^* \). This in fact concludes the weight space decomposition: the eight-dimensional space \( \mathfrak{sl}(\mathbb{C}) \) has six one-dimensional weight spaces of different non-zero weights, and the two-dimensional subspace \( \mathfrak{h} \) of zero weight:

\[
\mathfrak{sl}(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E^{ij}.
\] (II.18)

The non-zero weights appearing in the adjoint representation are called roots, and denoted traditionally by \( \alpha \). The set of roots is denoted by \( \Phi \): for \( \mathfrak{sl}(\mathbb{C}) \) we have

\[
\Phi = \{ \eta^1 - \eta^2, \eta^1 - \eta^3, \eta^2 - \eta^3, \eta^2 - \eta^1, \eta^3 - \eta^1, \eta^3 - \eta^2 \}.
\] (II.19)

For the adjoint representation, the weight spaces other than \( \mathfrak{h} \) are called root spaces. The decomposition (II.18) is also called the root space decomposition.

6.2.2. Irreducible representations of \( \mathfrak{sl}(\mathbb{C}) \)

Let again \( V \) be a finite-dimensional representation of \( \mathfrak{sl}(\mathbb{C}) \), and assume moreover that it is irreducible. The decomposition \( V = \bigoplus \mu V_\mu \) to weight spaces

\[
V_\mu = \{ v \in V \mid \forall H \in \mathfrak{h} : Hv = \mu(H)v \}
\]
tells exactly how any \( H \in \mathfrak{h} \) acts on \( V \). In view of the root space decomposition (II.18) of \( \mathfrak{sl}(\mathbb{C}) \), the remaining task is to describe how the root vectors \( E^{ij} \), \( i \neq j \), act on \( V \).

Let now \( v \in V_\mu \) be a weight vector of weight \( \mu \in \mathfrak{h}^* \), and consider the action of \( E^{ij} \) on \( v \). Denote by \( \alpha^{ij} = \eta^i - \eta^j \) the corresponding root, and let \( H \in \mathfrak{h} \).

Fundamental calculation (second time):

\[
H(E^{ij}v) = E^{ij}(Hv) + [H, E^{ij}]v = E^{ij}(\mu(H)v) + \alpha^{ij}(H)E^{ij}v = (\mu + \alpha^{ij})(H)E^{ij}v.
\]

This calculation shows that if \( v \) is a weight vector with weight \( \mu \), then \( E^{ij}v \) is a weight vector with weight \( \mu + \alpha^{ij} \) (although not necessarily a non-zero vector). In other words, for any \( \mu \) and for any \( i \neq j \) we have

\[
E^{ij} : V_\mu \to V_{\mu + \alpha^{ij}}.
\]

As with \( \mathfrak{sl}(\mathbb{C}) \) we can immediately conclude something about the differences of any two weights appearing in an irreducible representation.

**Observation II.20.** In an irreducible representation of \( \mathfrak{sl}(\mathbb{C}) \), any two weights \( \mu, \mu' \) differ by an integer linear combination of roots, \( \mu' = \mu + \sum_{i \neq j} n_{ij} \alpha^{ij} \) with some \( n_{ij} \in \mathbb{Z} \).
This can be reformulated as saying that the weights in an irreducible lie in some translate of the root lattice

$$\Lambda_R = \sum_{\alpha \in \Phi} \mathbb{Z}\alpha = \mathbb{Z}\alpha^{12} \oplus \mathbb{Z}\alpha^{23}. \quad \text{(II.20)}$$

For the latter expression we used the fact that $\alpha^{13} = \alpha^{12} + \alpha^{23}$, by virtue of which all roots can in fact be expressed as integer linear combinations of $\alpha^{12}$ and $\alpha^{23}$. We call these $\alpha^{12}$ and $\alpha^{23}$ simple roots (a choice has been made here). The set $\Delta = \{\alpha^{12}, \alpha^{23}\}$ of simple roots forms a $\mathbb{Z}$-basis of the root lattice $\Lambda_R$. Roots which are non-negative (resp. non-positive) integer linear combinations of simple roots are called positive roots (resp. negative roots), and their set is denoted by

$$\Phi^+ = \Phi \cap \bigoplus_{\alpha \in \Delta} \mathbb{Z}_{\geq 0} \alpha \quad \text{(resp. } \Phi^- = -\Phi^+).$$

Concretely, here we have $\Phi^+ = \{\alpha^{12}, \alpha^{23}, \alpha^{13}\} = \{\alpha^{ij} \mid i < j\}$.

To continue with comparisons to the case of $\mathfrak{sl}_2(\mathbb{C})$, recall that at this stage we showed that in an irreducible representation, any non-zero vector from the $H$-eigenspace with maximal eigenvalue $\lambda$ generated the entire representation, which was in fact determined by $\lambda$. Such a vector $v$ satisfied $Ev = 0$ and then successive action by $F$ on $v$ was enough to span the representation. What is the correct generalization to the present situation?

The eigenvalues have been replaced by weights $\mu \in \mathfrak{h}^*$, and it is not a priori clear which should be though of as maximal. Let us make an arbitrary looking choice: choose numbers $r_1 > r_2 > r_3$ such that $r_1 + r_2 + r_3 = 0$, and define a linear functional $\ell$ on $\mathfrak{h}^*$ by

$$\ell(a_1 \eta^1 + a_2 \eta^2 + a_3 \eta^3) = a_1 r_1 + a_2 r_2 + a_3 r_3.$$  

The choice made above is such that the positive roots evaluate to positive numbers, in particular for the two simple roots we have $\ell(\alpha^{12}) = r_1 - r_2 > 0$ and $\ell(\alpha^{23}) = r_2 - r_3 > 0$. Let us agree to say that a maximal weight is the one with the largest value of (the real part of) $\ell$. To ensure that there is a unique maximal choice, we assume furthermore $r_1, r_2, r_3$ chosen so that $\ell: \Lambda_R \to \mathbb{R}$ has a trivial kernel ($\ell$ is irrational with respect to the lattice $\Lambda_R$).

Then in a finite-dimensional representation $V$ there exists a unique maximal weight, denote it by $\lambda$. Note that since $\ell(\alpha^{ij}) > 0$ for all $i < j$, we must have $E^{ij}V_\lambda = 0$. The root spaces of the positive roots thus annihilate the weight space with maximal weight. We introduce some terminology:

**Definition II.21.** If $V$ is any representation of $\mathfrak{sl}_3(\mathbb{C})$, then a (non-zero) vector $v \in V$ which satisfies $E^{ij}v = 0$ for all $i < j$, and $Hv = \mu(H)v$ for all $H \in \mathfrak{h}$ and some $\lambda \in \mathfrak{h}^*$ is called a highest weight vector, and the weight $\lambda \in \mathfrak{h}^*$ is called its highest weight.

**Observation II.22.** In any irreducible finite-dimensional representation $V \neq 0$ of $\mathfrak{sl}_3(\mathbb{C})$, there exists a non-zero highest weight vector.

*Proof.* Take $\lambda$ the maximal weight in $V = \bigoplus_{\mu} V_\mu$, and choose a non-zero $v \in V_\lambda$. $\square$

**Example II.31.** In the standard representation $V = \mathbb{C}^3$ of $\mathfrak{sl}_3(\mathbb{C})$, the vector $e_1$ a highest weight vector of highest weight $\eta^1$.  

Example II.32. In the dual $V^*$ of the standard representation the vector $\varphi_3$ a highest weight
vector of highest weight $-\eta^3$.

Example II.33. In the adjoint representation $\mathfrak{sl}_3(\mathbb{C})$, by Equations (II.12) and (II.17), the vector
$E^{13}$ a highest weight vector of highest weight $\alpha^{13} = \eta^1 - \eta^3$.

A highest weight vector $v \in V_\lambda$ is annihilated by half of the root vectors, and like for $\mathfrak{sl}_2(\mathbb{C})$, applying repeatedly on it the other half of the root vectors, we generate
the entire irreducible representation.

Claim II.23. Let $0 \neq v \in V_\lambda$. Then $V$ is spanned by the vectors obtained by
successively applying $E^{21}$, $E^{32}$, and $E^{31}$ on $v$.

Proof. Let $W$ be the linear span of vectors obtained by successively applying $E^{21}$, $E^{32}$, and $E^{31}$ on $v$. Note that since $E^{31} = -[E^{21}, E^{32}]$, alternatively $W$ could have been defined as the linear
span of vectors obtained by successively applying only $E^{23}$ and $E^{32}$ on $v$. For an inductive
argument, let $W_n$ denote the linear span of vectors obtained by successively applying on $v$ a
word of at most $n$ letters, each equal to $E^{21}$ or $E^{32}$. Then $W$ is the sum of $W_n$, as $n$ ranges
over natural numbers. By definition we have $E^{21}W_n \subset W_{n+1}$ and $E^{32}W_n \subset W_{n+1}$, and
then using the fact that $E^{31} = -[E^{21}, E^{32}]$ we get that $E^{31}W_n \subset W_{n+2}$. Also for any $H \in \mathfrak{h}$
we have $HW_n \subset W_n$, since the vector obtained by applying a word on the highest weight
vector, is a weight vector (of weight $\lambda$ plus the sum of the negative roots corresponding
to the letters of the word), and such vectors span $W_n$. It follows that $W = \sum_n W_n$ is an invariant subspace for the action of all $H \in \mathfrak{h}$ and $E^{21}$, $E^{32}$, and $E^{31}$. It remains to see what
the positive root vectors $E^{12}$, $E^{23}$, and $E^{13}$ do to $W_n$. Moreover, since $E^{13} = [E^{12}, E^{23}]$, it in fact suffices to consider $E^{12}$ and $E^{23}$.

We claim that $E^{12}W_n \subset W_{n-1}$ and $E^{23}W_n \subset W_{n-1}$. The proofs are entirely similar, so consider the first case. The case $n = 0$ is clear, since $W_0 = \mathbb{C}v$ is the one-dimensional space
spanned by the highest weight vector, which is annihilated by $E^{12}$ and $E^{23}$. Proceed by induction on $n$. Suppose that $w$ is a vector obtained by applying on $v$ a word of $n$ letters,
each equal to $E^{21}$ or $E^{32}$. Depending on the last letter, we have either $w = E^{21}w'$ or $w = E^{32}w'$, with $w' \in W_{n-1}$. Consider first the first case. Then

$$
E^{12}w = E^{12}E^{21}w' = (E^{21}E^{12} + [E^{12}, E^{21}])w' = (E^{21}E^{12} + H^{12})w'
$$

$$
= E^{21}E^{12}w' + H^{12}w' \in E^{21}W_{n-2} + W_{n-1} \subset W_{n-1}
$$

where we used the induction assumption $E^{12}W_{n-1} \subset W_{n-2}$ and the fact that $\mathfrak{h}$ preserves $W_{n-1}$. In the second case,

$$
E^{12}w = E^{12}E^{32}w' = (E^{32}E^{12} + [E^{12}, E^{32}])w' = (E^{32}E^{12} + 0)w'
$$

$$
= E^{32}E^{12}w' \in E^{32}W_{n-2} \subset W_{n-1},
$$

where we again used the induction assumption $E^{12}W_{n-1} \subset W_{n-2}$. By induction, we thus establish that $E^{12}W_n \subset W_{n-1}$ and $E^{23}W_n \subset W_{n-1}$, and as a consequence also $E^{12}W_n \subset W_{n-2}$. Therefore $W = \sum_n W_n$ is invariant also for $E^{12}$, $E^{23}$, and $E^{13}$, and is therefore a subrepresentation. \qed