

ELEC-E8116 Model-based control systems /exercises 11 Solutions

Problem 1. Consider the general system representation

$$\dot{x} = Ax + Bu + Nw$$

$$z = Mx + Du$$

$$y = Cx + w$$

where it is assumed that

$$D^T [M \ D] = [0 \ I]$$

Show that this assumption can be relaxed by taking

$$\tilde{u} = (D^T D)^{1/2} u + (D^T D)^{-1/2} D^T Mx \quad \text{and}$$

$$z = \tilde{M}x + \tilde{D}\tilde{u}, \quad \tilde{M} = \left(I - D(D^T D)^{-1} D^T \right) M, \quad \tilde{D} = D(D^T D)^{-1/2}$$

Solution. First we show that the value of z remains the same:

$$\begin{aligned} \tilde{M}x + \tilde{D}\tilde{u} &= \left[I - D(D^T D)^{-1} D^T \right] Mx + D(D^T D)^{-1/2} \cdot \left\{ (D^T D)^{1/2} u + (D^T D)^{-1/2} D^T Mx \right\} \\ &= Mx - D(D^T D)^{-1} D^T Mx + D(D^T D)^{-1/2} (D^T D)^{1/2} u + D(D^T D)^{-1/2} (D^T D)^{-1/2} D^T Mx \\ &= Mx + Du = z \end{aligned}$$

Ok.

Then it remains to show that $\tilde{D}^T [\tilde{M} \ \tilde{D}] = [0 \ I]$.

$\tilde{D}^T [\tilde{M} \ \tilde{D}] = [\tilde{D}^T \tilde{M} \ \tilde{D}^T \tilde{D}]$. Let us consider both submatrices separately

$$\begin{aligned} \tilde{D}^T \tilde{M} &= (D^T D)^{-T/2} D^T \left(I - D(D^T D)^{-1} D^T \right) M \\ &= (D^T D)^{-1/2} D^T \left(I - D(D^T D)^{-1} D^T \right) M \\ &= (D^T D)^{-1/2} D^T M - (D^T D)^{-1/2} D^T D (D^T D)^{-1} D^T M \\ &= (D^T D)^{-1/2} D^T M - (D^T D)^{-1/2} D^T M = 0 \end{aligned}$$

$$\begin{aligned}
\tilde{D}^T \tilde{D} &= (D^T D)^{-T/2} D^T D (D^T D)^{-1/2} \\
&= (D^T D)^{-1/2} D^T D (D^T D)^{-1/2} \\
&= (D^T D)^{-1/2} (D^T D)^{1/2} = I
\end{aligned}$$

Ok.

Note that in above $(D^T D)^{-T/2} = ((D^T D)^{-1/2})^T = ((D^T D)^T)^{-1/2} = (D^T D)^{-1/2}$. Remember the rules of matrix calculus.

Problem 2. The *Frobenius norm* of a matrix A is defined as

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{\text{tr}(A^* A)}$$

Show that $\|A\|_F = \sqrt{\sum_i \sigma_i^2(A)}$, where $\sigma_i(A)$ are the singular values of A .

Solution. We do the singular value decomposition of matrix A

$$A = U \Sigma V^* \Rightarrow A^* A = V \Sigma^T \overbrace{U^* U}^I \Sigma V^* = V \Sigma^T \Sigma V^*$$

But generally (see Exercise 2) it holds $\text{tr}(CD) = \text{tr}(DC)$, provided that the dimensions agree of course. Then

$$\begin{aligned}
\text{tr}(A^* A) &= \text{tr}(V \Sigma^T \Sigma V^*) = \text{tr}(\Sigma V^* V \Sigma^T) \\
&= \text{tr}(\Sigma \Sigma^T) = \sum_i \sigma_i^2(A)
\end{aligned}$$