Taloustieteen matemaattiset menetelmät
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Problem Set 7: Solutions

1. **Solution**

   We can rewrite all the equations in the form $x_{t+1} = ax_t + b$. When $a \neq 1$, the solution is

$$x_t = a^t \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$$

   (a) $x_t = 5 \cdot 2^t - 4$; unstable because $a > 1$.

   (b) $x_t = \left(\frac{1}{3}\right)^t + 1$; stable because $|a| < 1$.

   (c) $x_t = -\frac{3}{5}\left(-\frac{3}{2}\right)^t - \frac{2}{5}$; unstable because $|a| > 1$.

   (d) Here $a = 1$ so we cannot use the formula above. By repeated substitution, we can see that the solution is $x_0 = 3$, $x_1 = 0$, $x_2 = -3$, $x_3 = -6$, and so on. In general, $x_t = 3 - 3t$, which is unstable ($x_t \to -\infty$ as $t \to \infty$).

2. **Solution**

   (a) We have

$$A\boldsymbol{v}_1 = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix} = 4\boldsymbol{v}_1$$

   and

$$A\boldsymbol{v}_2 = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\boldsymbol{v}_2.$$

   (b) It follows immediately from what we did in (a) that the corresponding eigenvalues for eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are $\lambda_1 = 4$ and $\lambda_2 = -1$, respectively.

   (c) We know that

$$P = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}.$$

   We also know that D is a diagonal matrix and that its diagonal entries are eigenvalues of A, so

$$D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}.$$

Next we need to find P^{-1}. Because $PP^{-1} = I$,

$$PP^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This yields a system of four equations:

$$\begin{aligned} 2a + c &= 1 \\ 2b + d &= 0 \\ 3a - c &= 0 \\ 3b - d &= 1. \end{aligned}$$

By solving this system we get $a = \frac{1}{5}$, $b = \frac{1}{5}$, $c = \frac{3}{5}$, and $d = -\frac{2}{5}$, so we can write P^{-1} as

$$P^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}.$$

Now

$$PDP^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = A.$$

(d) We know that $A^n = PD^nP^{-1}$ (Lecture 18, p. 15). Also, because D is a diagonal matrix, we can write

$$D^n = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}^n = \begin{pmatrix} r_1^n & 0 \\ 0 & r_2^n \end{pmatrix}.$$

Thus

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 4^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 2 \times 4^n + 3 \times (-1)^n & 2(4^n - (-1)^n) \\ 3(4^n + (-1)^{n+1}) & 3 \times 4^n - 2(-1)^{n+1} \end{pmatrix}. \end{aligned}$$

3. **Solution**

Write the system of difference equations in matrix form:

$$\boldsymbol{z_{t+1}} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = A\boldsymbol{z_t}$$

and the initial conditions can be written as $\boldsymbol{z_0} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Use the characteristic polynomial of A to find the eigenvalues:

$$\det(A - rI) = \det \begin{bmatrix} 2-r & -2 \\ -1 & 3-r \end{bmatrix} = (2-r)(3-r) - 2 = 0.$$

By solving this we get the eigenvalues $r_1 = 1$ and $r_2 = 4$.

Next we can use the eigenvalues to find the eigenvectors. Eigenvector $\boldsymbol{v_{r_i}}$ satisfies $(A - r_i I)\boldsymbol{v_{r_i}} = \mathbf{0}$, $i = 1, 2$. We know that $r_1 = 1$, so

$$\begin{bmatrix} 2-1 & -2 \\ -1 & 3-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This yields a system of two equations:

$$\begin{aligned} v_1 - 2v_2 &= 0 \\ -v_1 + 2v_2 &= 0. \end{aligned}$$

The solution for this system of equations is $v_1 = 2v_2$, so the first eigenvector is

$$\boldsymbol{v_{r_1}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The second can be found in a similar way. It is $\boldsymbol{v_{r_2}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Now we know that P is

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

One can solve P^{-1} as in Exercise 2c. It is

$$P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Next, make a change of variables $\boldsymbol{z} = P\boldsymbol{Z}$. Now the system can be written as

$$\boldsymbol{z_{t+1}} = A\boldsymbol{z_t} \Leftrightarrow P\boldsymbol{Z_{t+1}} = AP\boldsymbol{Z_t} \Leftrightarrow \boldsymbol{Z_{t+1}} = P^{-1}AP\boldsymbol{Z_t}$$

so

$$\boldsymbol{Z_{t+1}} = D\boldsymbol{Z_t}.$$

Now we can see that $\boldsymbol{Z_1} = D\boldsymbol{Z_0}$, $\boldsymbol{Z_2} = D\boldsymbol{Z_1} = D(D\boldsymbol{Z_0}) = D^2\boldsymbol{Z_0}$ etc., so

$$\boldsymbol{Z_t} = D^t\boldsymbol{Z_0}, \quad \text{where} \quad D^t = \begin{bmatrix} r_1^t & 0 \\ 0 & r_2^t \end{bmatrix}.$$

Make a change of variables again: $\boldsymbol{z} = P\boldsymbol{Z} \Leftrightarrow \boldsymbol{Z} = P^{-1}\boldsymbol{z}$ and write the system as

$$\begin{aligned} P^{-1}\boldsymbol{z_t} &= D^t P^{-1}\boldsymbol{z_0} \\ \boldsymbol{z_t} &= PD^t P^{-1}\boldsymbol{z_0}. \end{aligned}$$

We know that $P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$, $D^t = \begin{bmatrix} 1^t & 0 \\ 0 & 4^t \end{bmatrix}$, and $\boldsymbol{z_0} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus,

$$\begin{aligned} \boldsymbol{z_t} &= PD^t P^{-1}\boldsymbol{z_0} \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^t & 0 \\ 0 & 4^t \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \boldsymbol{z_0} \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^t & 0 \\ 0 & 4^t \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 1^t & -4^t \\ 1^t & 4^t \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 - 4^t \\ 1 + 4^t \end{bmatrix}. \end{aligned}$$

4. **Solution**

Write the system of difference equations as

$$\boldsymbol{w_{t+1}} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 4 & -2 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = A\boldsymbol{w_t}$$

Use the characteristic polynomial of A to find the eigenvalues:

$$\begin{aligned} \det(A - rI) &= \begin{vmatrix} 4-r & -2 & -2 \\ 0 & 1-r & 0 \\ 1 & 0 & 1-r \end{vmatrix} = (4-r)(1-r)^2 + 2(1-r) \\ &= (r^2 - 5r + 6)(1-r) = (3-r)(2-r)(1-r) = 0 \end{aligned}$$

The eigenvalues are $r_1 = 1$, $r_2 = 3$ and $r_3 = 3$. Next we can use the eigenvalues to find the eigenvectors. If $\boldsymbol{v_{r_i}}$ is an eigenvector, then $(A - r_i I)\boldsymbol{v_{r_i}} = \boldsymbol{0}$, $i = 1, 2, 3$. To find the first eigenvector, write

$$\begin{bmatrix} 4-1 & -2 & -2 \\ 0 & 1-1 & 0 \\ 1 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This yields the equations $3v_1 - 2v_2 - 2v_3 = 0$ and $v_1 = 0$, so $v_2 = -v_3$. Thus, the first eigenvector is

$$\boldsymbol{v_{r_1}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

The other eigenvectors can be found in a similar way: $\boldsymbol{v_{r_2}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{v_{r_3}} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

The general solution can be written as (see Lecture 18, p. 10):

$$\boldsymbol{w_t} = c_1 r_1^t \boldsymbol{v_{r_1}} + c_2 r_2^t \boldsymbol{v_{r_2}} + c_3 r_3^t \boldsymbol{v_{r_3}},$$

where c_1, c_2 and c_3 are constants. Therefore, the general solution of this exercise is

$$\boldsymbol{w_t} = c_1 1^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_2 2^t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 3^t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

5. **Solution**

Exercise 5 can be solved in a similar way as the previous exercise. Write the system of difference equations in matrix form:

$$\boldsymbol{w_{t+1}} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = A\boldsymbol{w_t}.$$

The eigenvalues are $r_1 = 3$, $r_2 = 4$ and $r_3 = 1$, and the eigenvectors are $v_{r_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_{r_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $v_{r_3} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

The general solution is

$$\boldsymbol{w_t} = c_1 3^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 4^t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 1^t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$