PHYS-C0252 - Quantum Mechanics Part 10
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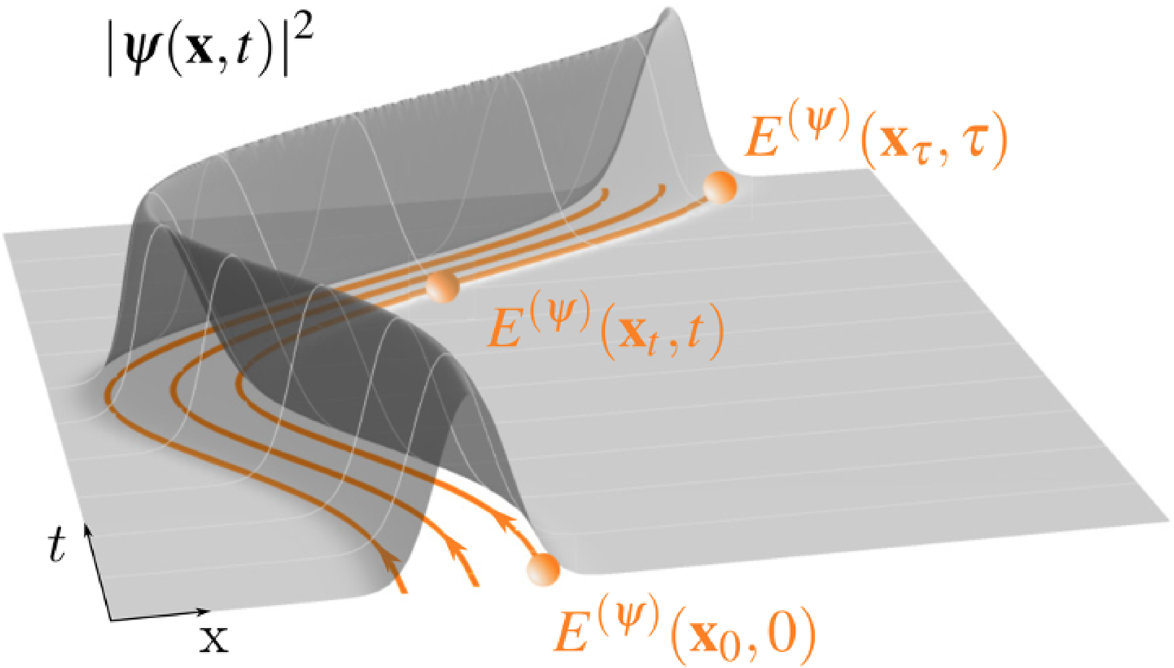
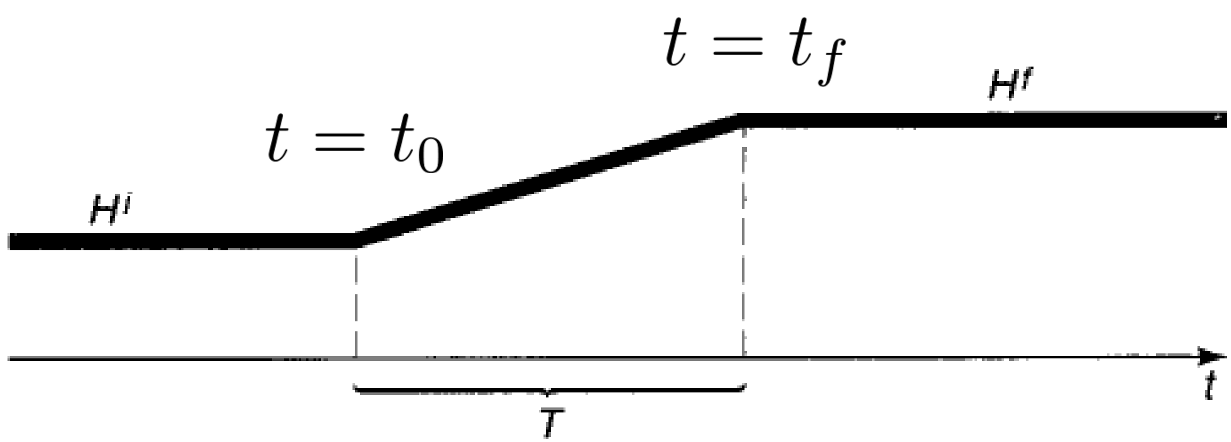


7. Time Dependence

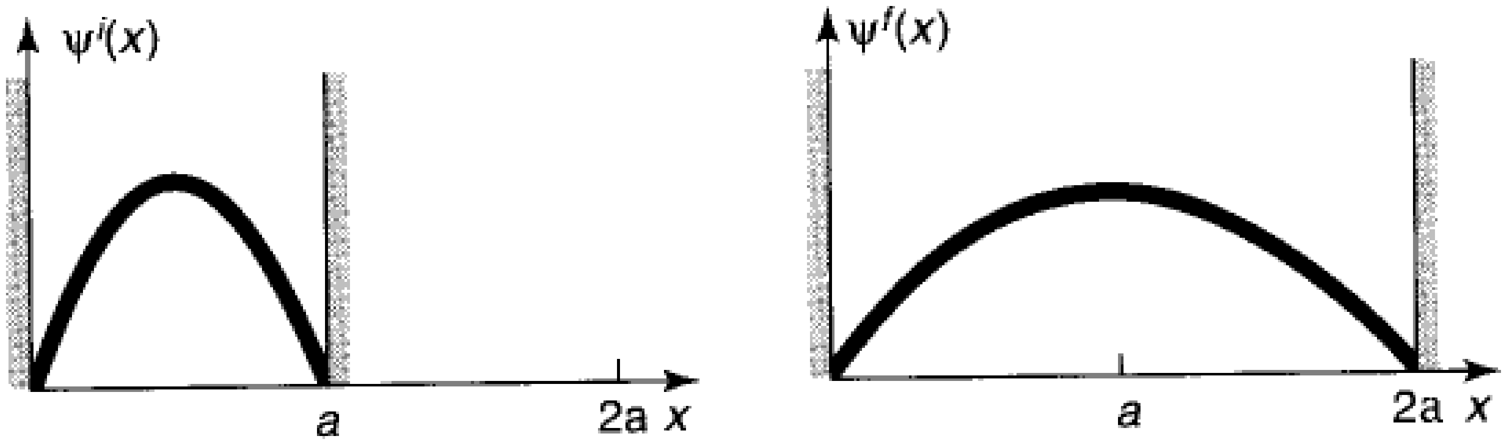
7.1 Adiabatic Theorem

- Assume that we have a system that "very slowly" evolves in time s.t. the Hamiltonian $\hat{H}^i \to \hat{H}^f$



Adiabatic processes carry the system form an initial eigenstate of $\hat{H}^i(t = t_0)$ to that of the final Hamiltonian $\hat{H}^f(t = t_f)$

For example, if for the infinite wall well the wall distance is adiabatically increased from *a* to 2*a*:



$$\psi^i(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a} x\right) \qquad \psi^f(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{\pi}{2a} x\right)$$

For *rapid* (non-adiabatic) processes the final state is some combination of final eigenstates

7.2 Geometric and Berry Phases

- If the Hamiltonian is independent of time, the particle which starts at the *n*th eigenstate

$$\hat{H}\psi_n(x) = E_n\psi_n(x)$$

remains in that eigenstate and picks up a phase factor from the SE

$$\Psi_n(x,t) = \psi_n(x)e^{\imath E_n t/\hbar}$$

If the Hamiltonian is time dependent then we can formally write (but usually not solve)

$$\hat{H}(t)\psi_n(x,t) = E_n(t)\psi_n(x,t)$$

According to the *adiabatic theorem*, the system will remain at the *n*th eigenstate even with time dependence:

$$\Psi_n(x,t) = \hat{U}(t)\psi_n(x,t)$$

To obtain the time-evolution operator for a *time-dependent Hamiltonian*, we have to solve for

$$\imath\hbar\frac{\partial\hat{U}(t,t_0)}{\partial t} = \hat{H}(t)\hat{U}(t,t_0)$$

The formal solution of this equation for Hamiltonians commuting at all times is (prove by expanding the exp function)

$$\hat{U}(t, t_0) = e^{-\imath \int_{t_0}^{t} dt' \hat{H}(t')/\hbar}$$

Operating on the eigenstates gives simply that

$$\Psi_n(x, t) = \psi_n(x, t) e^{-\imath \int_{t_0}^{t} dt' E_n(t')/\hbar} e^{\imath \gamma(t)}$$

The term

$$\theta_n(x, t) = -\int_{t_0}^{t} dt' E_n(t')/\hbar$$

Is known as the *dynamic phase* and the extra phase factor $\gamma_n(t)$ is the *geometric phase*

We can plug in the solution back to the time-dependent SE to get

$$i\hbar\left[\frac{\partial\psi_n}{\partial t}e^{i\theta_n}e^{i\gamma_n} - \frac{i}{\hbar}E_n\psi_n e^{i\theta_n}e^{i\gamma_n} + i\frac{d\gamma_n}{dt}\psi_n e^{i\theta_n}e^{i\gamma_n}\right]$$

$$= [H\psi_n]e^{i\theta_n}e^{i\gamma_n} = E_n\psi_n e^{i\theta_n}e^{i\gamma_n},$$

and thus $$\frac{\partial\psi_n}{\partial t} + i\psi_n\frac{d\gamma_n}{dt} = 0$$

Taking inner product with ψ_n

$$\frac{d\gamma_n}{dt} = i\langle\psi_n|\frac{\partial\psi_n}{\partial t}\rangle$$

Lue us assume that the time dependence in the Hamiltonian is given by some (classical) function *R*(*t*):

$$\frac{\partial\psi_n}{\partial t} = \frac{\partial\psi_n}{\partial R}\frac{dR}{dt} \quad \Rightarrow \quad \frac{d\gamma_n}{dt} = i\langle\psi_n|\frac{\partial\psi_n}{\partial R}\rangle\frac{dR}{dt}$$

This can be integrated to give

$$\gamma_n(t) = i\int_0^t \langle\psi_n|\frac{\partial\psi_n}{\partial R}\rangle\frac{dR}{dt'}\,dt' = i\int_{R_i}^{R_f}\langle\psi_n|\frac{\partial\psi_n}{\partial R}\rangle\,dR$$

If there are N time-dependent parameters in the Hamiltonian:

$$\frac{\partial \psi_n}{\partial t} = \frac{\partial \psi_n}{\partial R_1}\frac{dR_1}{dt} + \frac{\partial \psi_n}{\partial R_2}\frac{dR_2}{dt} + \cdots + \frac{\partial \psi_n}{\partial R_N}\frac{dR_N}{dt} = (\nabla_R \psi_n) \cdot \frac{d\mathbf{R}}{dt}$$

This can be again integrated to give

$$\gamma_n(t) = i \int_{\mathbf{R}_i}^{\mathbf{R}_f} \langle \psi_n | \nabla_R \psi_n \rangle \cdot d\mathbf{R}.$$

If the Hamiltonian is cyclic with period T

$$\boxed{\gamma_n(T) = i \oint \langle \psi_n | \nabla_R \psi_n \rangle \cdot d\mathbf{R}.}$$

This is a line integral around a closed loop in the parameter space and in general it is not zero. $\gamma_n(t)$ is called the *Berry phase*.

- The Berry phase only depends on the (adiabatic) path taken, not on time!

In contrast, the dynamic phase is time dependent as

$$\theta_n(T) = -\frac{1}{\hbar}\int_0^T E_n(t')\,dt'$$

- The Berry phase is *real valued and it is measurable!*