ELEC-E8116 Model-based control systems

Full exam 17. 12. 2019

- *Write the name of the course, your name, your study program, and student number to each answer sheet.*
- *There are five (5) problems and each one must be answered.*
- *No literature is allowed. A function calculator can be used.*
- *Your solutions must contain enough information to see how you have solved the problems.*

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1. **a**. Explain briefly the following concepts in control theory (shortly, only what they are and what they mean from control viewpoint)

- SVD
- LQ
- IMC

b. Write a short description of what the concept "Fundamental restrictions of control" means. What are the most significant ones of them?

2.a. Consider a MIMO system. Draw a schema of the "one-degrees-of-freedom" control configuration. Define the concepts *loop transfer function*, *sensitivity function* and *complementary sensitivity function* for it.

2.b. Consider a SISO-case. Determine the region in the complex plane where $|S| = 1/\sqrt{2}$. Then determine the region where $|T| = 1/\sqrt{2}$. Do these regions have common points? If they do, what are these points? Interpretation from control perspective?

3. Consider a MIMO system with the transfer function matrix

$$G(s) = \frac{1}{(0.2s+1)(s+1)}\begin{bmatrix} 1 & 1 \\ 1+2s & 2 \end{bmatrix}$$

a. Determine the poles and zeros of the above system.
What conclusions can be made with respect to control?

b. Calculate the *Relative Gain Array* RGA at zero frequency in the above example case.
What conclusions can be made with respect to control?

4. The discrete time LQ problem can be formulated as

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$$x_{k+1} = A_k x_k + B_k u_k, \quad k > i$$

$$J_i = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} \left(x_k^T Q_k x_k + u_k^T R_k u_k \right), \quad \text{(final state free)}$$

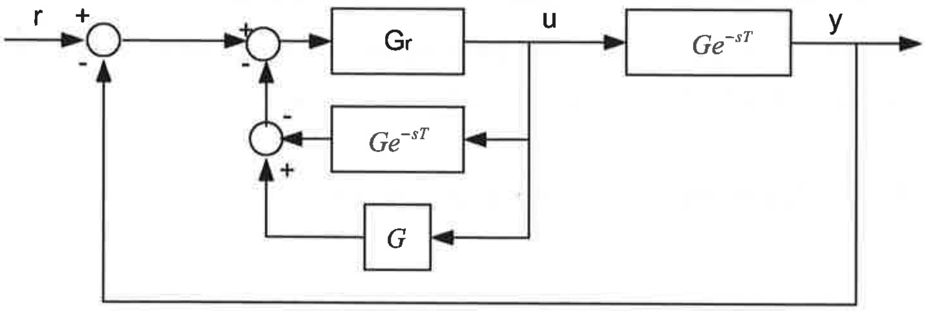
$$S_N \geq 0, \quad Q_k \geq 0, \quad R_k > 0$$

where S_N, $Q_k = Q$, $R_k = R$ are symmetric given matrices (Q and R constant matrices for all k). The process is assumed time-invariant so that also $A_k = A$, $B_k = B$, constant matrices for all k. In solving the discrete-time LQ problem an essential step is to apply the Principle of Optimality to find the "first control step" by minimizing the cost

$$J_{N-1} = \frac{1}{2} x_{N-1}^T Q x_{N-1} + \frac{1}{2} u_{N-1}^T R u_{N-1} + \frac{1}{2} (A x_{N-1} + B u_{N-1})^T S_N (A x_{N-1} + B u_{N-1})$$

Do it! In other words, solve for the "first control" u_{N-1}.

5. Consider the control configuration shown in the figure (known as the *Smith-predictor*). Calculate the closed loop transfer function and verify the idea behind this controller. Change the configuration for the form of the *IMC*-controller. What is the Q parameter? What can be said about stability (no proof required)?



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Full exam 17.12.2019 / Solutions

1a SVD = singular value decomposition

LQ = linear quadratic optimal control problem
linear dynamics, quadratic criterion

IMC = internal model control

See lecture slides

1b. Restrictions in performance, which cannot be overcome by any controller.

Most significant $S(j\omega) + T(j\omega) = I$

Pole in RHP: lower limit for bandwidth

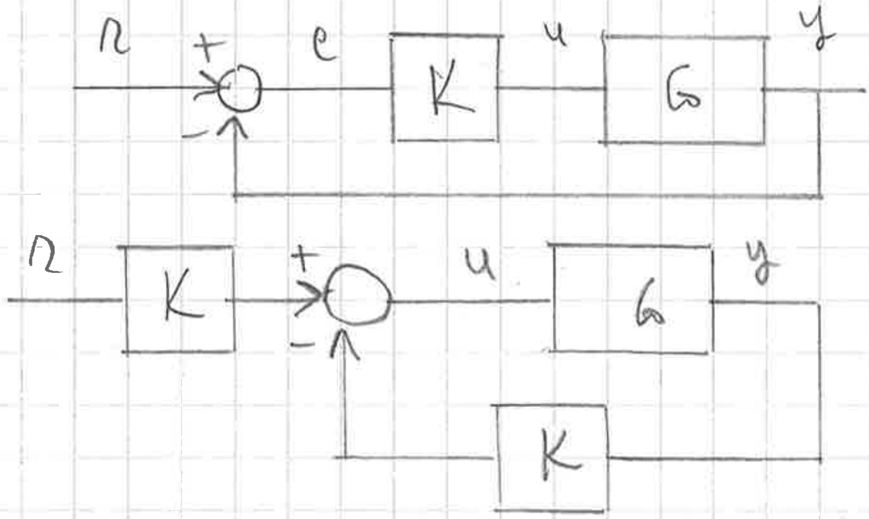
Zero in RHP: upper limit for achievable bandwidth

Time delay in process: ↗

Water bed formula for $S(j\omega)$ etc.

See lecture slides.

2a



1 DOF configuration

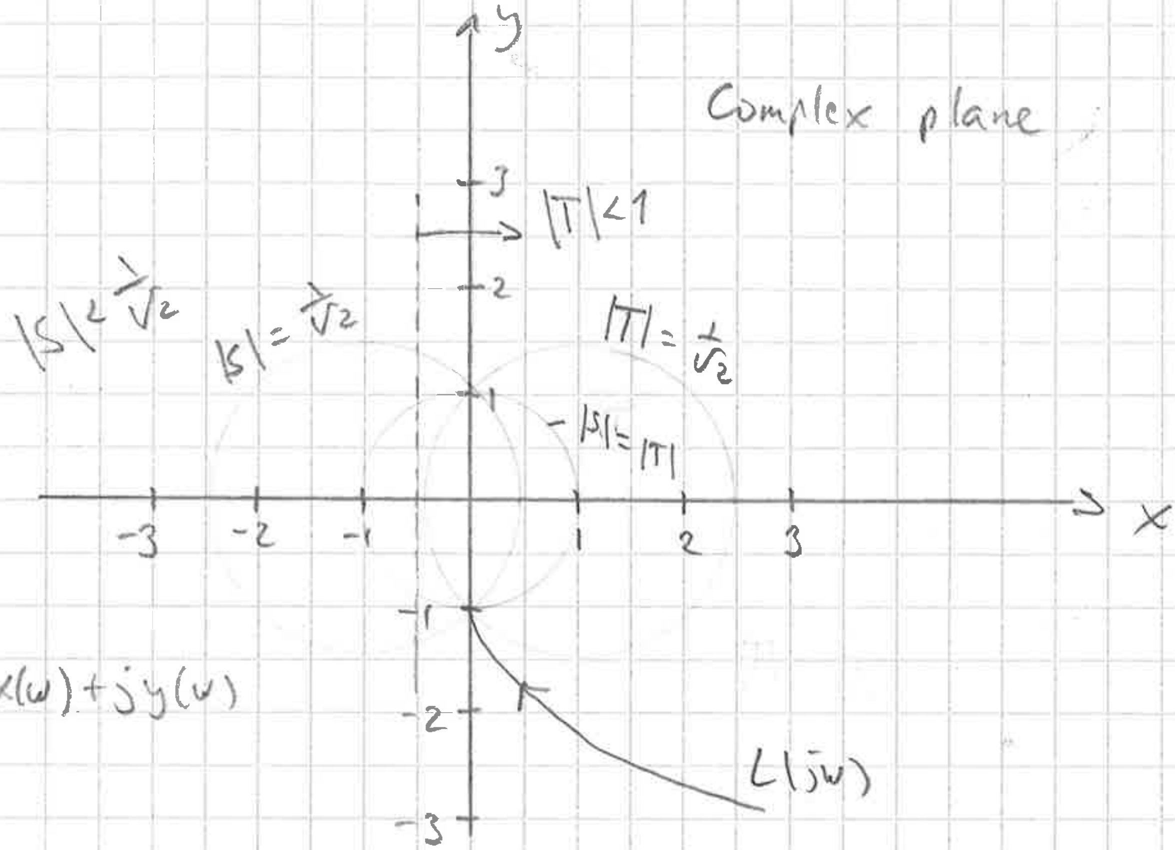
⟺

$L(j\omega) = G(j\omega)\,K(j\omega)$

$S(j\omega) = [I + L(j\omega)]^{-1}$

$T(j\omega) = [I + L(j\omega)]^{-1} L(j\omega)$

26.



$L(j\omega) = x(\omega) + jy(\omega)$

$$S = \frac{1}{1+L} = \frac{1}{1+x+iy}, \quad |S| = \frac{1}{\sqrt{(1+x)^2+y^2}}$$

$$T = \frac{L}{1+L} = \frac{x+iy}{1+x+iy}, \quad |T| = \sqrt{\frac{x^2+y^2}{(1+x)^2+y^2}}$$

$$|S| = |T| \Rightarrow \frac{1}{\sqrt{(1+x)^2+y^2}} = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}} \Rightarrow x^2+y^2=1$$

$$|S| = \frac{1}{\sqrt{2}} \Rightarrow (1+x)^2+y^2 = \frac{1}{\frac{1}{2}} = 2,$$ circle with center $(-1,0)$ radius $\sqrt{2} \approx 1{,}414$

$$|T| = \frac{1}{\sqrt{2}} \Rightarrow \frac{x^2+y^2}{(1+x)^2+y^2} = \frac{1}{2} \Rightarrow (1+x)^2+y^2 = 2x^2+2y^2 \Rightarrow 1+x^2+2x+y^2 = 2x^2+2y^2$$

$$\Rightarrow x^2+y^2-2x=1 \Rightarrow (x-1)^2-1+y^2=1 \Rightarrow (x-1)^2+y^2=2,$$

circle with center $(1,0)$ radius $\sqrt{2} \approx 1.414$

Common points: $\begin{cases}(1+x)^2+y^2=2 \\ x^2+y^2=1\end{cases} \Rightarrow 1+2x=1$
(subtract equations)
$\Rightarrow x=0,\ y=\pm 1 \qquad (0,i),(0,-i)$

Interpretation: Let $L(j\omega)$ go as shown in the figure.

It intersects with $|S| = |T| = \frac{1}{\sqrt{2}}$ at point $(0,-1)$

But this happens at the gain crossover frequency

ω_c, because $|L(j\omega_c)| = 1$. Then also $|S(j\omega_c)| = |T(j\omega_c)| = \frac{1}{\sqrt{2}}$

The phase margin is $PM = 90^\circ$

Note that this does not violate the condition

$\omega_B < \omega_c < \omega_{BT}$ shown in Exercise 7, Problem 2, because there it was assumed that $PM < 90^\circ$.

3. a) $G(s) = \frac{1}{(0.2s+1)(s+1)} \begin{bmatrix} 1 & 1 \\ 2s+1 & 2 \end{bmatrix}$

Minors: $\frac{1}{(0.2s+1)(s+1)}$, $\frac{2s+1}{(0.2s+1)(s+1)}$, $\frac{2}{(0.2s+1)(s+1)}$

$$\frac{2-(2s+1)}{(0.2s+1)^2(s+1)^2} = \frac{-2s+1}{(0.2s+1)^2(s+1)^2}$$

The pole polynomial is the least common denominator of the minors

$$P(s) = (0.2s+1)^2(s+1)^2$$

The minimal realization has 4 states, and the poles are $-1, -1, -5, -5$.

Nothing special in control. Stable process.

The zero polynomial is formed by the maximal minor

$$z(s) = -2s+1$$

and the system has a RHP zero at 0.5.
It causes a fundamental restriction to the closed loop bandwidth. The maximum achievable bandwidth is about $\frac{z}{2} = 0.25\ \frac{\text{rad}}{s}$ which corresponds to time constant 4s.

b) RGA at zero frequency

$$RGA(0) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} .\!* \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-T} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} .\!* \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Coupling $u_1 \to y_1$, $u_2 \to y_2$ seems better, but is not very good either.

4. $\frac{\partial}{\partial x}(Ax) = A$, $\frac{\partial}{\partial x}(x^T A x) = x^T(A+A^T) = 2x^T A$
(A symmetric)

$$J = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u + \frac{1}{2}(x^T A^T + u^T B^T)(SAx + SBu)$$

$$= \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u + \frac{1}{2}x^T A^T S A x + \frac{1}{2}x^T A^T S B u + \underbrace{\frac{1}{2}u^T B^T S A x}_{=\frac{1}{2}x^T A^T S B u} + \frac{1}{2}u^T B^T S B u$$

$$= \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u + \frac{1}{2}x^T A^T S A x + x^T A^T S B u + \frac{1}{2}u^T B^T S B u$$

$$\frac{\partial J}{\partial u} = u^T R + x^T A^T S B + u^T B^T S B = 0$$

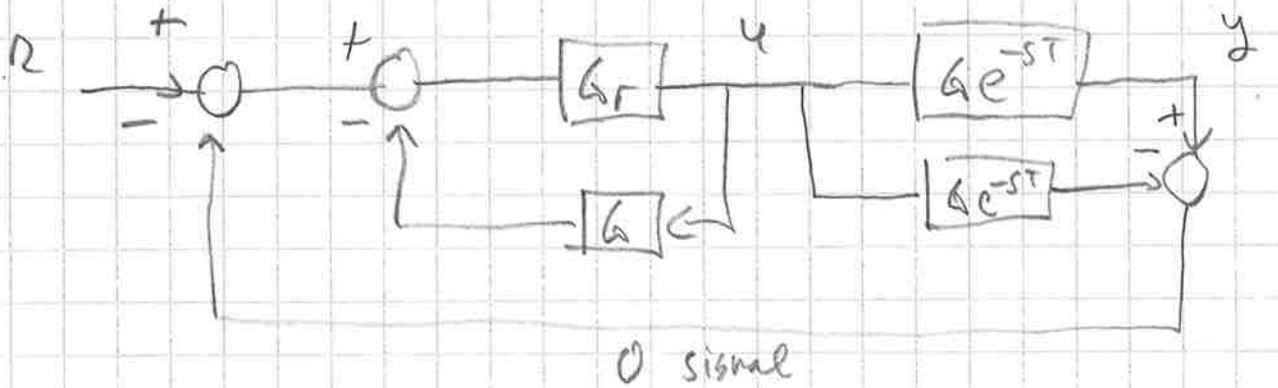
$$\Rightarrow R u + B^T S A x + B^T S B u = 0$$

$$\Rightarrow (R + B^T S B) u = - B^T S A x$$

$$\Rightarrow u = -(R + B^T S B)^{-1} B^T S A x$$

This is the desired "first control" from state x.

5. The most straightforward way is to change the topology into IMC form directly



The closed-loop transfer function is clearly

$$\frac{G_r}{1+G_r G} \cdot G e^{-sT} = \frac{G_r G}{1+G_r G} e^{-sT}$$

Q parameter

The time delay T has been moved outside the loop.

However, deeper analysis reveals that the process G must be stable to make the closed loop internally stable.