ELEC-E8116 Model-based control systems
Intermediate exam 1. 24. 10. 2019

- *Write the name of the course, your name, your study program, and student number to each answer sheet.*
- *There are three (3) problems and each one must be answered.*
- *No literature is allowed. A function calculator can be used.*

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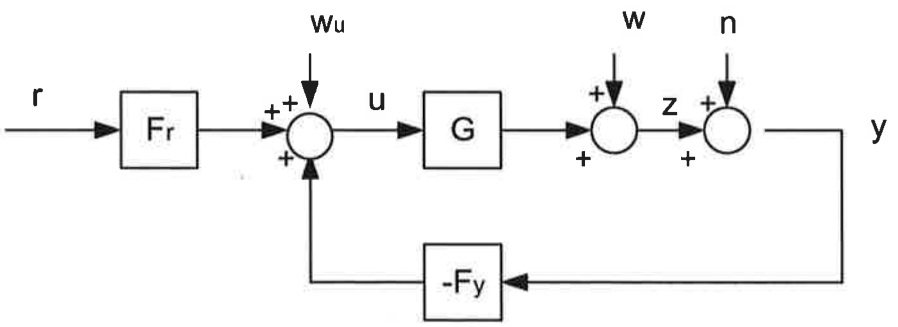
1. Explain briefly the following concepts

- Singular value decomposition and singular values
- Conservative control law
- Input directions of a multivariable system
- Internal stability
- "Push through"-rule

2. Consider a multivariable control configuration.



Write the equations describing the system and identify

a. the loop transfer function, the closed loop transfer function, the sensitivity function and the complementary sensitivity function

b. Show that at each frequency $S + T = I$. Also show that $LS = SL$.

c. Give definition(s) of *bandwidth*. In terms of control performance, what does the concept mean? Explain the relevance in *loop shaping* design.

TURN PAPER

3. Two systems are given by the following transfer function matrices. In both cases do the following:

- determine the number of inputs and outputs
- determine the pole and zero polynomials
- determine the poles and zeros and their multiplicities
- what is the number of states in the minimal realization? (You do not have to form the realizations).
- what can be said about stability?

$$G_1(s) = \begin{bmatrix} \dfrac{2(s+1)(s+2)}{s(s+3)(s+4)} & \dfrac{s+2}{(s+1)(s+3)} \end{bmatrix} \qquad G_2(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{s+3}{(s+1)(s-2)} \\ \dfrac{10}{(s-2)} & \dfrac{5}{s+3} \end{bmatrix}$$

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Intermediate exam 1 24.10.2019 / Solutions

1. (One point each item)

- Singular value decomposition and singular values

For any real or complex matrix $\underset{n \times m}{G}$

there exists the singular value decomposition

$$\underset{n\times m}{G} = \underset{n\times n}{U} \underset{n\times m}{\Sigma} \underset{m\times m}{V^*} \qquad U^*U = \underset{n\times n}{I} \qquad V^*V = \underset{m\times m}{I}$$

U, V unitary matrices

(assume $n \geq m$)

Σ is real, where the main diagonal contains the singular values

$$\underline{\sigma} = \sigma_1 \leq \sigma_2 \leq \cdots \sigma_m = \bar{\sigma}$$

Also $\sigma = \sqrt{\lambda(G^*G)}$

- Conservative control law

A control law that works in principle, but is too sluggish (performance is bad). For example, a controller which gives 0 output all the time might be stable indeed, but it is not good for sure.

- Input directions of a multivariable system

$G = U \Sigma V^*$ (SVD)

$\Rightarrow GV = U\Sigma \underbrace{V^* V}_{=I} = U\Sigma$

$\Rightarrow G\left[v_1 \mid v_2 \mid \cdots \mid v_m\right] = \left[Gv_1 \mid Gv_2 \mid \cdots \mid Gv_m\right]$

$= \left[u_1 \mid u_2 \mid \cdots \mid u_m\right] \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \ \sigma_m \end{bmatrix}$

$= \left[\sigma_1 u_1 \mid \sigma_2 u_2 \mid \cdots \mid \sigma_m u_m\right]$

(it has been assumed that $m \leq n$ $\Rightarrow$ m non-zero singular values)

$\Rightarrow \boxed{Gv_i = \sigma_i u_i}$

there are enough

$\boxed{v_i \text{: input direction}}$

u_i: output direction

σ_i: strength of that input-output

- Internal stability

No input in a closed-loop system can cause any signal grow without limit.
(All possible input/output pairs are BIBO stable)

- Push-through rule

$$\underset{n\times m}{A}\,(\underset{m\times m}{I}+\underset{m\times n}{B}A)^{-1} = (\underset{n\times n}{I}+AB)^{-1}A$$

(Result is enough; no proof was required)

2. See lecture slides, beginning of Chapter 3.

a) Loop transfer function $L(j\omega) = G(j\omega)\,F_y(j\omega)$

Closed loop transfer function $G_{cl} = (I+GF_y)^{-1}GF_r$
$= SGF_r$

Sensitivity function $S = (I+GF_y)^{-1} = (I+L)^{-1}$

Complementary sensitivity function

$T = (I+GF_y)^{-1}GF_y = (I+L)^{-1}L$
$= SL$

a. 2p
b. 1p
c. 2p

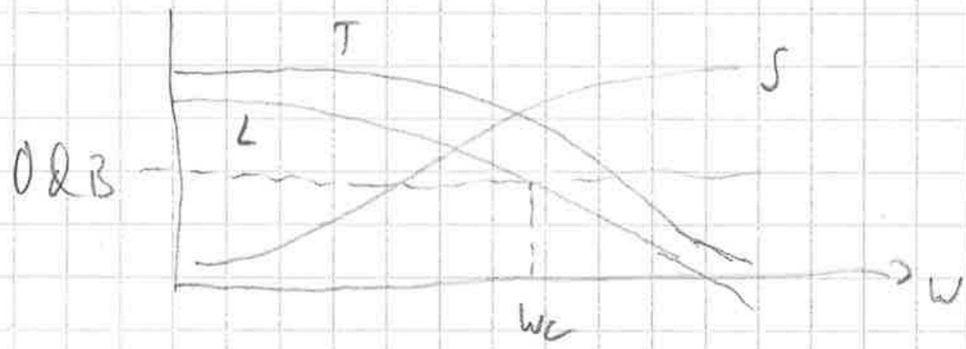
b. $S+T = S+SL = S(I+L) = SS^{-1} = I$

one way to do it.

$$LS = L(I+L)^{-1} = (I+L)^{-1}L = SL$$

push-through

c.



w_c: where L crosses the $0dB$ line (gain crossover frequency)

w_B: where S crosses $-3dB$ line $(=\frac{1}{\sqrt{2}})$ from below

w_{BT} where T crosses $-3dB$ line from above

Under assumptions that hold almost always

$$w_B < w_c < w_{BT}$$

Bandwidth gives a measure how fast the system is and up to which input frequencies control is effective.

In loop shaping we try to design the controller such that (among other things) the desired bandwidth can be obtained

3. See Problem 3 in Exercise 3. The interesting solution with Matlab details can be found in the solutions.

The 1st system is stable (not asymptotically stable)
2 inputs, 1 output

The 2nd system is unstable
2 inputs, 2 outputs

(Each question 1 point)