ELEC-E8116 Model-based control systems

Intermediate exam 2. 12. 12. 2019 / Solutions

- *Write the name of the course, your name, your study program, and student number to each answer sheet.*
- *There are three (3) problems and each one must be answered.*
- *No literature is allowed. A function calculator can be used.*

Each problem gives the maximum of 5 points.

In the optimal route the controls must be optimal with respect to any state, irrespective of how that state has been reached.

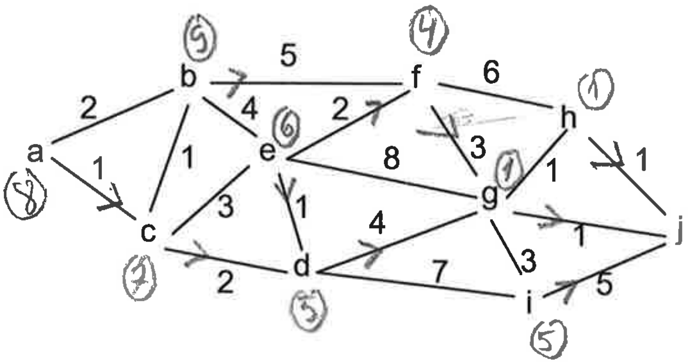
1. **a**. Explain the concept "Principle of Optimality"
   b. In the below figure the cost of moving from one node to another is given by the numbers; movement is allowed only from left to right.

For example

$J_g^* = \min\{J_{gh}+J_h^*, J_{gj}, J_{gi}+J_i^*\}$
$= \min\{1+1, 1, 3+5\}$
$= 1 = J_{gj}$

$J_{aj}^* = 8$

a-c-d-g-j



Use *dynamic programming* to solve the following problem: Find the minimum cost path from node *a* to the desired final state *j*. (Copy the diagram on your paper and show clearly how you have solved the problem using dynamic programming).

2. For the system

$$\dot{x}(t) = 2u(t)$$

($x(0) = x_0$) calculate the control law, which minimizes the criterion

$$J = \frac{1}{2}\int_0^\infty (x^2(\tau) + u^2(\tau))d\tau$$

What is the optimal cost? What is the closed-loop trajectory of the system? Verify the value of optimal cost by substituting $x^*(t)$ and $u^*(t)$ into the integral formula of J and then calculating its value.

TURN PAPER

3. Consider a SISO-process with the transfer function

$$G(s) = \frac{s + \frac{1}{T_1}}{\left(s + \frac{1}{T_2}\right)\left(s + \frac{1}{T_3}\right)}, \quad T_1 < 0, \quad T_2 < 0, \quad T_3 > 0$$

Explain, what kind of fundamental limitations on control performance can be stated for this system? (Present also some calculations; the formulas in the end of the problems can be of help.)

Some formulas, which might be useful:

$$\int_0^\infty \log |S(i\omega)| \, d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}\,(\mathrm{p_i})$$

$$|W_T(p_1)| \le 1 \quad \Rightarrow \quad \omega_0 \ge \frac{p_1}{1 - 1/T_0}$$

$$|W_S(z)| \le 1 \quad \Rightarrow \quad \omega_0 \le \left(1 - 1/S_0\right) z$$

$$\dot{x} = Ax + Bu$$

$$J(t_0) = \frac{1}{2} x^T(t_f) S(t_f) x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left(x^T Q x + u^T R u\right) \, dt$$

$$S(t_f) \ge 0, \quad Q \ge 0, \quad R > 0$$

$-\dot{S}(t) = A^T S + SA - SBR^{-1}B^T S + Q, \; t \le t_f,$ boundary condition $S(t_f)$

$$K = R^{-1} B^T S$$

$$u = -Kx$$

$$J^*(t_0) = \frac{1}{2} x^T(t_0) S(t_0) x(t_0)$$

2. $A=0,\ B=2,\ Q=R=1$

$-\dot{s} = -4s^2+1 = 0$ because the optimization horizon is infine

$\Rightarrow s = \underset{(-)}{+}\frac{1}{2} = \frac{1}{2}$ positive definite solution

$u^* = -1\cdot 2\cdot \frac{1}{2}x = -x$ optimal control

$\dot{x} = -2x\ ,\ x(0)=x_0$

$sX(s)-x_0 = -2X(s) \Rightarrow (s+2)X(s) = x_0 \Rightarrow X(s) = \frac{x_0}{s+2}$

$\Rightarrow x^*(t) = e^{-2t}x_0$ optimal trajectory

$J^* = \frac{1}{2}x_0^2\cdot\frac{1}{2} = \frac{1}{4}x_0^2$ optimal cost

Verification of the optimal cost

$$J^* = \frac{1}{2}\int_0^\infty \left(e^{-4t}x_0^2 + e^{-4t}x_0^2\right)dt = x_0^2\int_0^\infty e^{-4t}\,dt = -\frac{1}{4}x_0^2\int_0^\infty -4e^{-4t}\,dt$$

$$= -\frac{1}{4}x_0^2 \Big/_0^\infty e^{-4t} = -\frac{1}{4}x_0^2(0-1) = \frac{1}{4}x_0^2 \quad \text{ok.!}$$

3. $T_2 < 0 \Rightarrow$ unstable system (RHP pole)

$T_1 < 0 \Rightarrow$ non-minimum phase system (RHP zero)

Interpolation formulas:

$$p_1 = -\frac{1}{T_2} > 0$$

$$z = -\frac{1}{T_1} > 0$$

"Bandwidth" $$w_0 \geq \frac{-\frac{1}{T_2}}{1-\frac{1}{T_0}} = \frac{1}{\frac{T_2}{T_0} - T_2}$$

$$w_0 \leq \left(1-\frac{1}{S_0}\right)\left(-\frac{1}{T_1}\right) = \frac{1}{T_1 S_0} - \frac{1}{T_1}$$

T_0 and S_0 are the maximum values of the desired complementary sensitivity and sensitivity functions in the parametrization

$$\frac{1}{W_T} = \frac{1}{\frac{1}{T_0} + \frac{s}{w_0}}, \quad \frac{1}{W_S} = \frac{S_0 s}{s + w_0 S_0}$$

For any controller $F_y(s) \to L(s) \to S(s)$ the sensitivity function fulfills

$$\int_0^{\infty} \ln|S(i\omega)|\, d\omega = -\frac{\pi}{T_2}$$