In short: In this course, we study the
Elastic stability structures – Rakenteiden stabiilius

The content is conceptually very short having only three fundamental concepts to study:
1) Equilibrium – tasapaino
2) Its stability properties – sen laatu
3) Sensitivity of such equilibrium to imperfections – herkkyys häiriöille

Many applications of structural stability of typical structural elements commonly used in civil engineering will be studied.

These are the two lectures of the first week

Content

0. **Basic concepts**
   Equilibrium, Stability
   The energy criterion of stability

First week

1. Flexural buckling (nurjahdus)
2. Lateral-torsional buckling (kiepahdus)
3. Torsional buckling (vääntönurjahdus)
4. Buckling of thin plates
5. Buckling of shells (lommahdus)

following weeks

One topic per week

Feb	24	25	26	27	28	29	1
March	2	3	4	5	6	7	8
	9	10	11	12	13	14	15
	16	17	18	19	20	21	22
	23	24	25	26	27	28	29
March	30	31	1	2	3	4	5

Lecturer
Djebar Baroudi, Dr.
Civil Engineering Department
Aalto University   version 27.2.2020

Content of this 1ˢᵗ week two lectures:

These are the two lectures of the first week

Content

0. **Basic concepts**

    Equilibrium, Stability

    The energy criterion of stability

1. Flexural buckling (nurjahdus)
2. Lateral-torsional buckling (kiepahdus)
3. Torsional buckling (vääntönurjahdus)
4. Buckling of thin plates
5. Buckling of shells (lommahdus)

following weeks

Main topics

- Content of this course
- Literature & additional course material
- Practicalities

- **Introduction**
- What equilibrium and stability mean?
- The key questions
- How stability is investigated?

- Stability loss as a phenomenon
  - Examples of loss of stability
- Basic concepts of stability
  - Static & dynamic stability
- Structural design and stability
- Methods of stability study

- **Energy criteria of stability**
- Lagrange–Dirichlet Stability theorem

- **Equilibrium paths**
- Critical equilibrium points, bifurcation, limit points
- Stability of an equilibrium
- Linear Buckling Analysis
- Non-Linear Buckling Analysis (GNA)

- **Types of bifurcation instabilities**
- **Effect of imperfections on the post-buckling behavior**

- **Illustration examples**

Topics of the lectures and homework

Content

First week

0. **Basic concepts**
   Equilibrium, Stability
   The energy criterion of stability

1. **Flexural buckling (nurjahdus)**
2. **Lateral-torsional buckling (kiepahdus)**
3. **Torsional buckling (vääntönurjahdus)**
4. **Buckling of thin plates**
5. **Buckling of shells (lommahdus)**

Literature

- CHAI H. YOO & SUNG C. LE . *STABILITY OF STRUCTURES - Principles and Applications*, 2011 Elsevier **e-textbook** (our course main textbook)

Additinal reading:

- S.P. Timoshenko & J.M. Gere. *Theory of Elastic Stability. 2nd Ed., 1985.* (Classical textbook)
- Juha Paavola. *Structural Stability.* Lecture note - 2018 (pdf in MyCourses) . This is a must-read for those interrested in general and systematic energetic approach for elastic stability.
- *Structural Stability* (Lecture notes by Prof Markku Tuomala, in Finish). This is a complete textbook with plenty of solved exercises.
- Lecturer weekly provided additional material

Not compulsory

Week 1-2 topics

Fundaments of Stability → [The PDF:] https://mycourses.aalto.fi/pluginfile.php/664042/course/sect

(I provide the pdf I have written. This **stability key topics** are missing in the textbook!)

Practicalities

- Reading assignments

- Lecture augmented slides

- Detailed content

- Reading assignments

MyCourses SCHOOLS

- CIV-E4100 - Stability of Structures L, 24.02.2020-09.04.2020
- Participants
- Grades
- » Course home page
- » Course in short, parcticalities and litterature [2020]
- » **Lectures and readings [2020]**
- » For Aalto users
- » For Aalto users [HW - 2020]
- » Homework and course points [2020]
- » Exams, grades & solutions [2020]
- » Material [old, 2018]
- » Lecture Slides [old, 2018]

BEGIN WEEK 1-2

===

Week 1-2: **[2] Chapter 4, from 4.1 till section 4.11.1:** *The fundaments of Elastic Stability*

(I provide the pdf I have written. This **stability key topics** are missing in the textbook!)

Week 1-2 topics

[PDF Slides] : https://mycourses.aalto.fi/pluginfile.php/952895/course/section/133502/Week_1_Lecture_Slides_DB.pdf

- **Introduction**
  - What is stability? Phenomenon, elastic stability
  - Structural design and stability – some standards

- **Basic concepts**
  - Equilibrium, equilibrium paths
  - Critical points: bifurcation, limit point
  - Stability of equilibrium (branches)

- **Energy criteria of stability**

- About post-buckling analysis

- **Types of bifurcational instabilities**

- **Sensitivity to imperfections**

- **Illustrative examples**

===

From *e*-textbook ----- you can read also,

Week 1-2: **[1] Chapter 1:** *Buckling of columns* ...

& especially, section 1.6: *Introduction to calculus of variation* (variaatiolaskenta)

Practicalities

Assignment week 1 [2020] - Fundaments of Elastic Stability

Fundaments of Elastiic Stability:

- Buckling load,
- energy criteria of stability
- equilibrium paths,
- post-critical analysis,
- effect of imperfections
- asymptotic post-buckling analysis

The next home work, we will concentrate Stability behaviour of various CIV-structures.

The key consepts and method are invariant and independent on the dgeree of complexity of the structure.

An introduction to the calculus of variations can be found at:

1): https://www.open.edu/openlearn/ocw/pluginfile.php/1118521/mod_resource/content/3/Introduction%20to%20the%20calculus%20of%20variations_ms327.pdf

2): http://courses.theophys.kth.se/5A1305/var.pdf

HW_1_2020.pdf   19 February 2020, 8:18 PM

Grading summary

MyCourses   SCHOOL

- CIV-E4100 - Stability of Structures L, 24.02.2020-09.04.2020
- Participants
- Grades
- » Course home page
- » Course in short, parcticalities and litterature [2020]
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- » **Homework assignments [HW - 2020]**
- » Homework and course points [2020]
- » Exams, grades & solutions [2020]
- » Material [old, 2018]
- » Lecture Slides [old, 2018]
- » For the Lecturer only

» **Homework assignments [HW - 2020]**

HW_1_2020.pdf

» Homework assignments [HW - 2020]

...mework and course points

HW_1_2020.pdf   19 February 2020, 8:18 PM

Grading summary

Participants	27
Drafts	0
Submitted	0
Needs grading	0
Due date	Wednesday, 4 March 2020, 11:55 PM

CIV-E4100

A? MyCourses SCHOOLS

Passing the course

- Having obtained from HW-assignements >= 40% of compulsory points togather with passing successfully the written exam.

- when the **written** exam is successfully passed, then the homework points rise the examination grade (arvosana) at most by 1 grade if **homework points >= 2/3 of homework compulsory maximum points**.

- There will be organised only two examinations

Assignments

- readings from textbook and the additional lecturer's pdf-material

- doing weekly homework (probably five topics) *student delivery*: solutions

- one computer analysis: linear buckling and post-buckling analysis: *student delivery*: solutions and report

The purpose of assignments is to train and deepen active learning. All the cession of exercises are guided.

Axial buckling failure in service.

Some videos on stability of structures

https://www.youtube.com/watch?v=OoORi_2Vkcg&app=desktop

1: Lateral torsional buckling of I-beam   (kiepahdus)

Comment: Good experiment with load-displacement curves

The student can clearly see the transition from bending in the vertical plane to bending in the horizontal plane and torsion

(24.02.2010)

https://www.youtube.com/watch?feature=youtu.be&v=cYRicTk-Q08&app=desktop
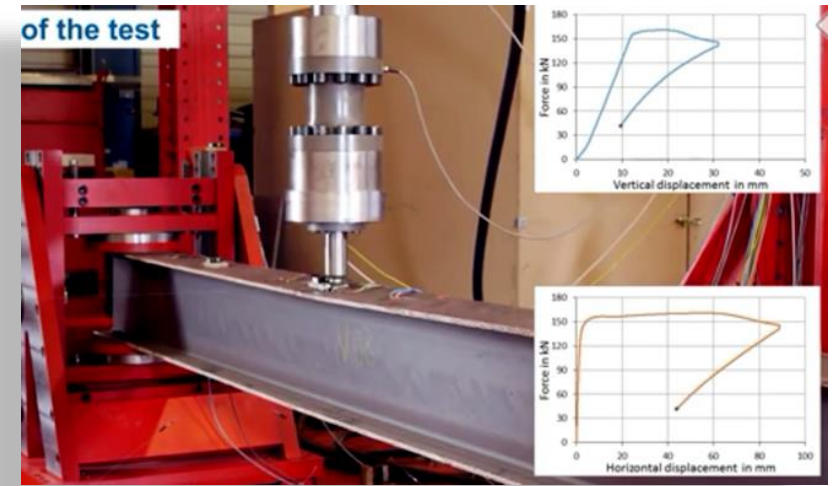
(24.02.2010)

2: Torsional buckling of L-shape cross-section (angle) column   (vääntönurjahdus)

Comment: Good experiment with a funny professor.

Note that, the apparent (torsional) rigidity gets dramatically reduced close to the buckling load

(24.02.2010)

loaded

buckled

https://www.youtube.com/watch?v=0WN8RP7Bz6Q

no load yet

Hohohoh, it buckles

buckled

3: buckling of upper cord of a truss

Comment: Note the FAST and DYNAMICAL transition from the primary equilibrium (unbuckled) to the secondary equilibrium state (buckled)

Some videos on stability of structures
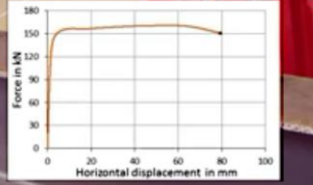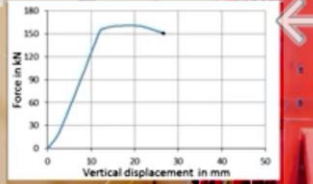
https://www.youtube.com/watch?v=OoORi_2Vkcg&app=desktop
(24.02.2010)

Lateral torsional buckling of
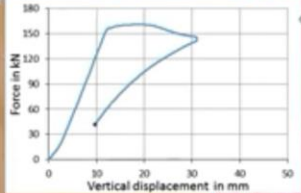I-beam
(kiepahdus)

Progress of the test

Cross-section
motion: combined
bending and torsion

2. Bucklig occurred
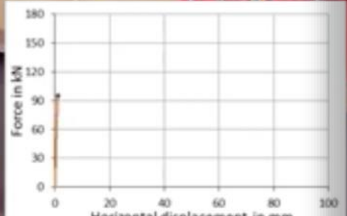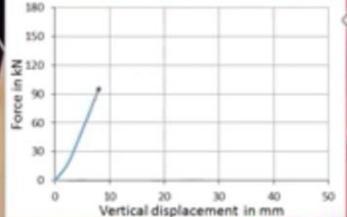Continue loading
Geometrically non-linear

3. unloading
Elastic linear response

Progress of the test
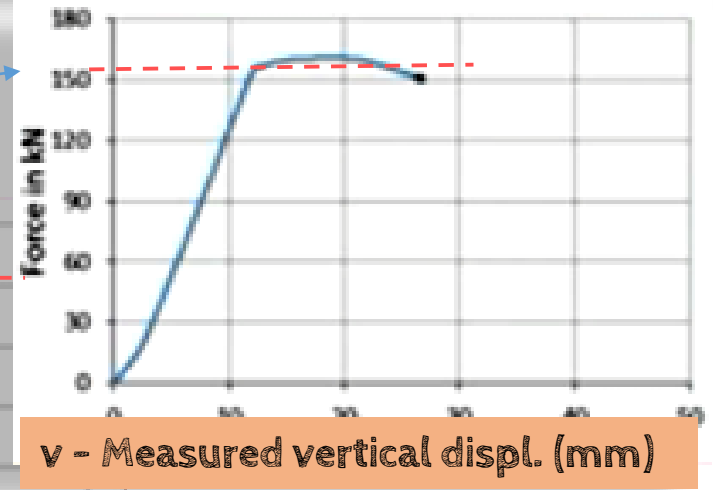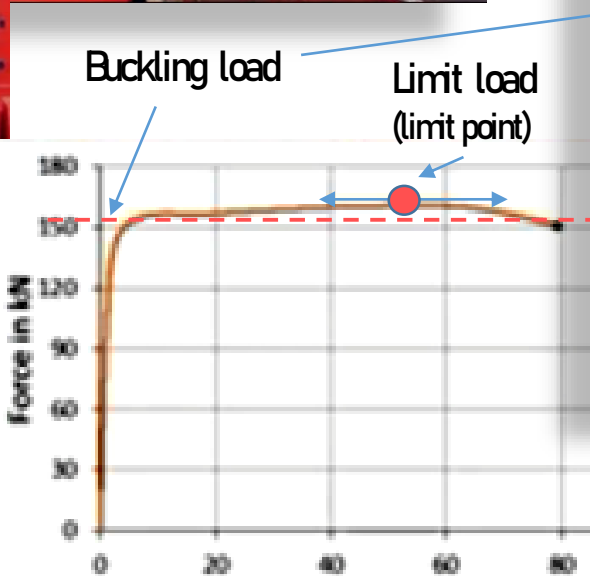
1. Starts loading
Elastic linear response

Buckling load

Limit load
(limit point)

v – Measured vertical displ. (mm)

w – Measured horizontal displ. (mm)

Experimental load-
displacement curves
= equilibrium paths

Systems with multiple equilibrium states

- In such systems, the structure can move from one equilibrium state to another if enough perturbed
- For structural engineers such motion is not desired and is called loss of stability

Open: equilibrium position #1
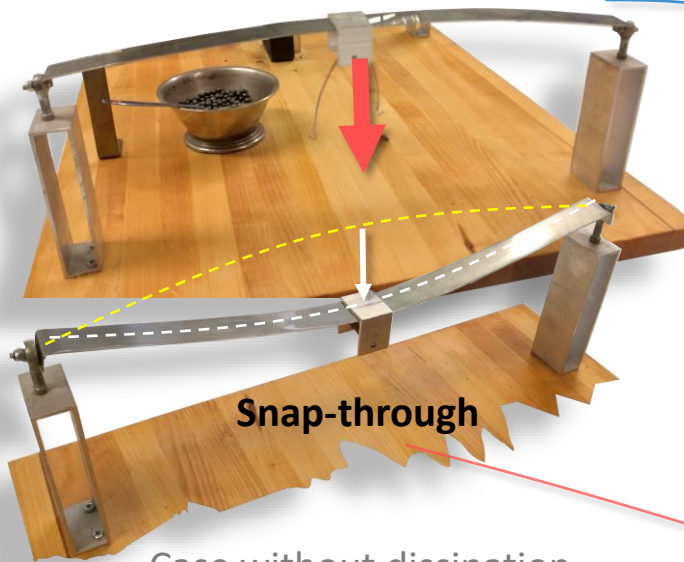
Closed: equilibrium position #2

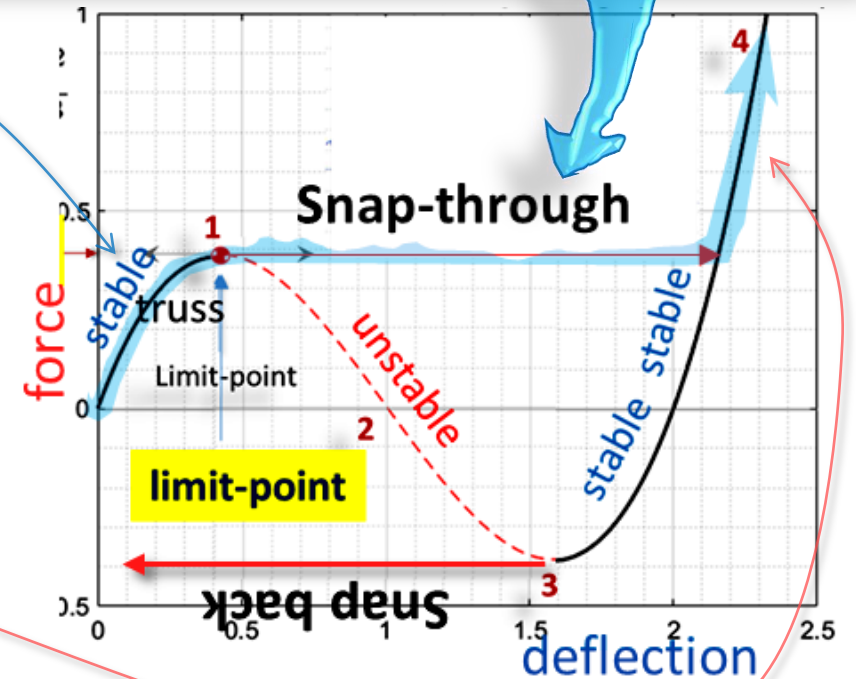Simple bi-stable system

equilibrium state #1

equilibrium state #2

Shallow arch
Model in our coffe-room

Snap-through

Case without dissipation

Snap-through of shallow arch

Snap-through

truss

Limit-point

limit-point

stable

unstable

stable stable

Snap back

force

deflection

Stability is a concept related to systems having *more than one equilibrium states*. Simplest examples are the *bistable* systems, with two equilibrium states where the system can rest in either one two states

The key stability question in structural design
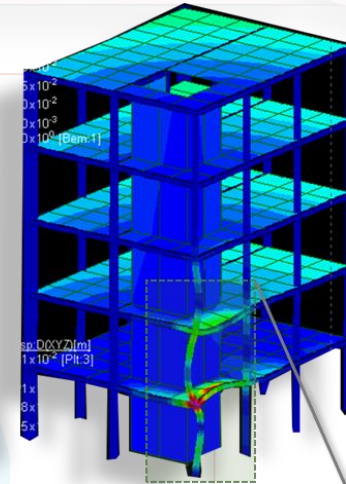
Equilibrium? Yes. ✅ $\delta(\Delta\Pi) = 0$

But, is it **stable**? **No** 🚫 $\delta^2(\Delta\Pi) < 0$

Figure 3.32: *Equilibrium and Stability - the key concept.*

Soil material instability

Lecture slides for internal use only

D. Baroudi, Dr.

Version: 16.2.2020

The key stability question in structural design

BASIC CONCEPTS

Equilibrium? Yes.
But, is it **stable?** No.

Figure 3.32: *Equilibrium* and *Stability* - the key concept.

Soil material instability

Lecture slides for internal use only
D. Baroudi, PhD

Version: 25.2.2019

# What are the key question in stability investigation?

Here the content of this course in four points through questions that will be addressed:

or limit load

1. can we predict the buckling (critical) load?

2. what happens at the bifurcation (or limit) point?
   (*i.e.*, after the buckling)   Post-buckling behavior

3. can we determine the post-critical branches?
   What would be their shape?   Nature of stability?   Stable, instable?

How much ?

4. what imperfection-sensitive is the structure under study?

Cable-stayed tower

A model

k_1

N_1

k_2

N_2

q

k_3

N_3

k_4

N_4
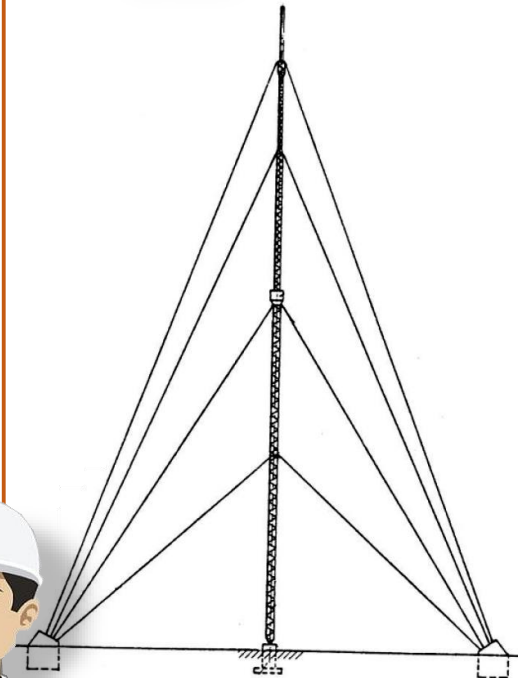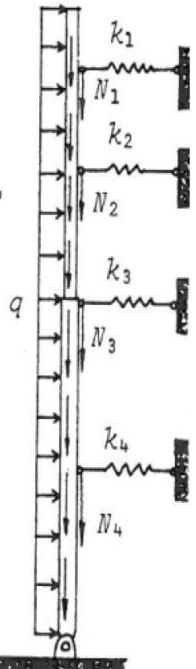
Stability?

Stability of an equilibrium

• Static stability:

➤ Consider a structure which is initially in equilibrium
➤ Introduce an arbitrary tiny perturbation;
➤ Remove the perturbation

▪ Then what happens?

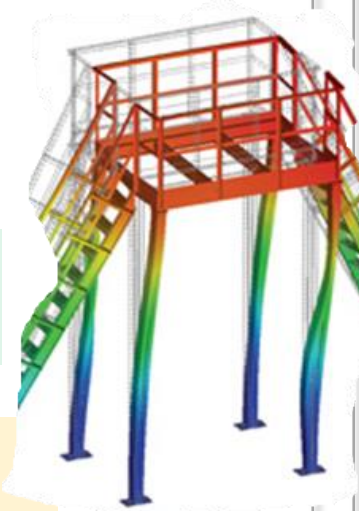YES: → The equilibrium stable

DOES The structure sustains or comes back to its initial equilibrium configuration or very close to it?

NO: → The equilibrium unstable

Equilibrium and Stability

A tiny perturbation

Symmetry breaking

Straight primary equilibrium
Is this position stable?

• Dynamic stability:

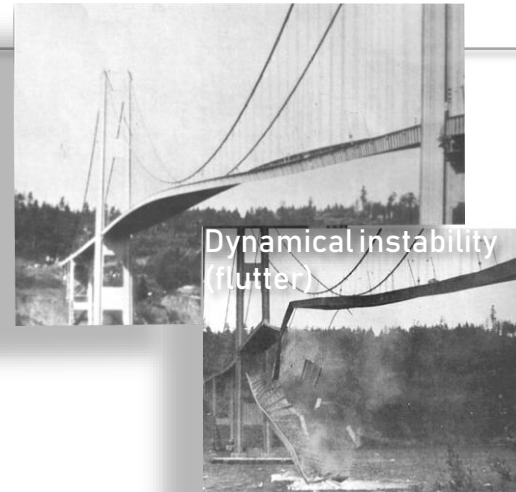➤ **Dynamic stability** considers time history of the motion after an arbitrary tiny perturbation

▪ Then what happens?

DOES the change in amplitude keep not enhanced in time, after such perturbation?

Dynamical instability (flutter)

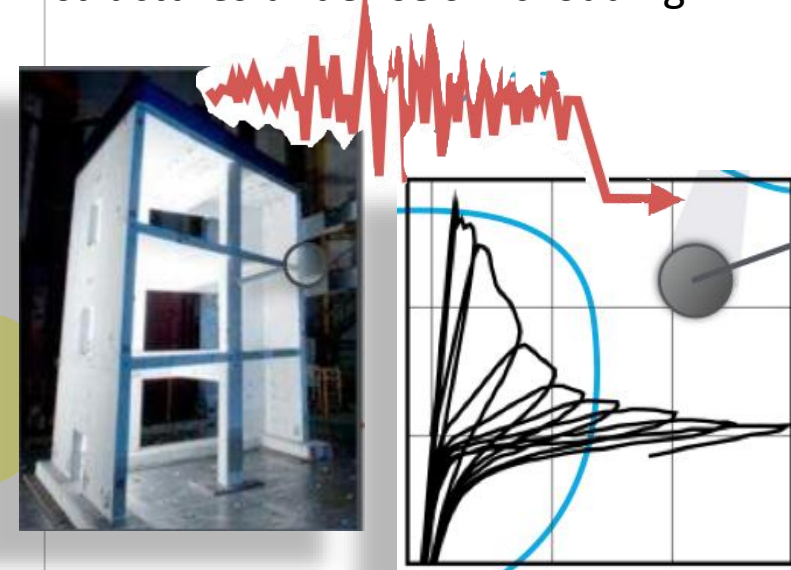⟹ **YES**: dynamically stable

Methods of stability study

In our course, we will systematically the energy approach.

There exits three (analytical) methods for studying the stability of an equilibrium:

1) **Bifurcation approach –** write the equilibrium equations in a deformed configuration and determine the onset of buckling

2) **Energy approach –** the change of total potential energy of the system between two neighboring equilibrium states is used to derive the equations of equilibrium and to study its stability

3) **Dynamic approach –** the equations of motion of the system are established. i) Natural frequencies decreasing to zero, correspond to the onset of instability or ii) investigate how an initial perturbation develops with time

4) **For design purposes**, detailed stability behaviour of structures can be analysed **numerically** by performing a **geometrical and material non-linear** (GNMA) analysis on the real structure with the inherent real and possible imperfections using capability of the Finite Element technology. Such analysis provides full load-displacement curves which are used to identify bifurcations or/and limit points for determining the limit loads

5) **Experimental approach** is needed since models are only approximations and very often, they are a very incomplete approximations. For some structures, experiments are of primary importance

Experimental: stability of concrete structures under seismic loading

Experiment: NASA NESC Shell Buckling

https://www.nasa.gov/offices/nesc/home/Feature_NESC_Shell_Buckling_Investigation_Continues_to_Make_Gains_Feb_2014.html

Buckling of thin-walled cylinders

Basic Concepts

'Two types' of stability

Treated in this course

Not treated in this course

Static Stability (SS):

Stability of static equilibrium configurations of a mechanical system (Euler 1707-1783)

ex. structures: columns, beam-columns, frames, plates, shells …

Assumptions for static stability criteria:

- **Elastic* material** (strain energy exists) displacements and rotations, not necessarily small

- **Loads are conservative**** (= derivable from a potential)

    Examples of conservative forces: *gravity* and *hydrostatic loads, elastic force, …*
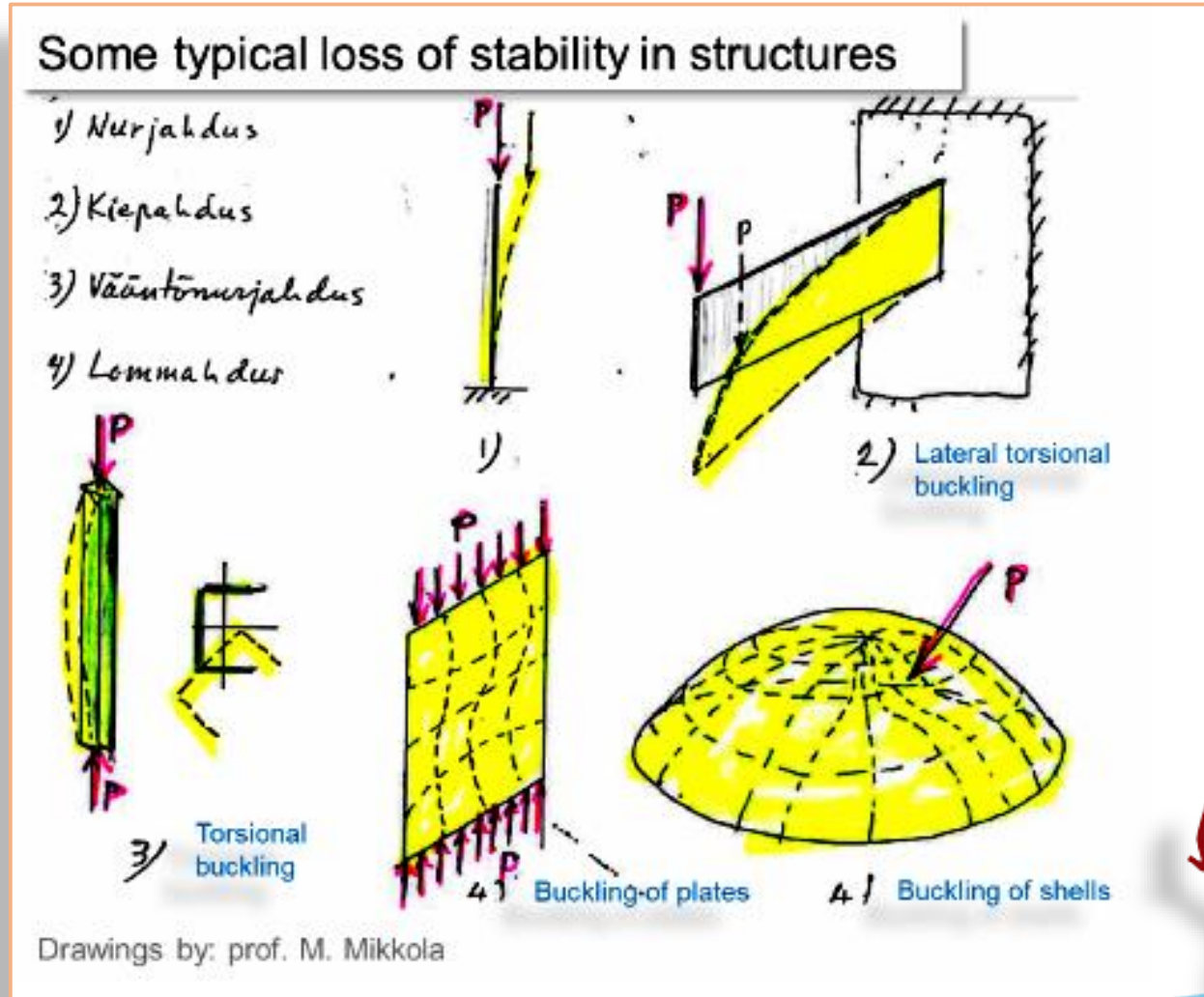
Examples of non-conservative forces: friction, (drag) hydro- & aerodynamic and jet-propulsion loads, gyroscopic forces, following forces…

Dynamic Stability (DS):

Stability of the motion of dynamic systems (Lyapunov 1857-1918)

(*e.g.*, a train trajectory, system with following forces: reaction propulsion rackets, korkeushyppääjän sauva, aerodynamic forces on structures: **flutter** bridges, …)

Ex. non-conservative forces:

Ça y est! La lance est branchée…Et maintenant, ouvrons les vannes.

Jet propulsion force

Stability loss as we see it … consequences

Photo: Hannu Hirsi, 2016 Germany

Loss of stability of a column. Original temporary renforcement method.

Some typical loss of stability in structures

1) Nurjahdus
2) Kiepahdus
3) Vääntönurjahdus
4) Lommahdus

1)

2) Lateral torsional buckling

3) Torsional buckling

4) Buckling of plates

4) Buckling of shells

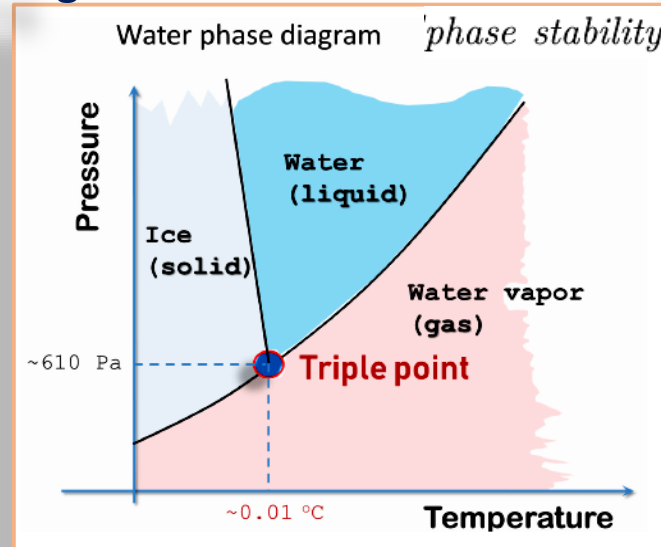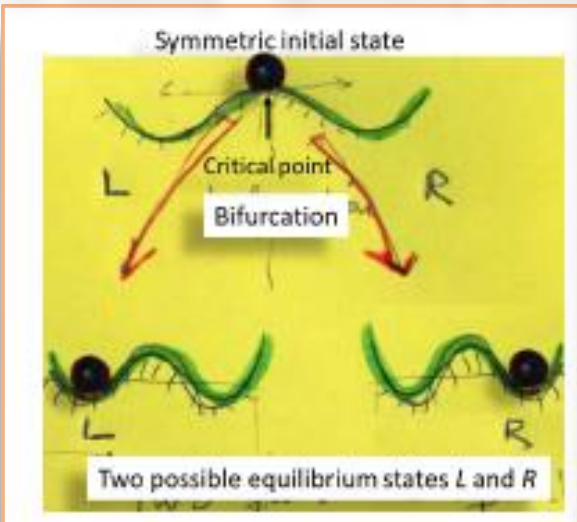Drawings by: prof. M. Mikkola

What is stability as a phenomenon?

Stability is a fundamental property of (dynamical systems) systems having *more than one equilibrium states* where the system can rest in one or in the other states

- These **equilibrium** states correspond to **local minima** in **potential energy** of the system

- Between two local minima a local maximum should exist. The state at this critical point is *unstable*. This local maxima is termed as *potential barrier*.

- A tiny external perturbation can make the system switch to another equilibrium state if enough energy input is given to jump the barrier separating the local minima

Loss of Stability = symmetry breaking

Open: equilibrium position #1

Closed: equilibrium position #2

Simple bi-stable system

Photo: Hannu Hirsi, 2016 Germany

Loss of stability of a column. Original temporary renforcement method.

Symmetric initial state

Critical point

Bifurcation

L   R

L   R

Two possible equilibrium states *L* and *R*

Water phase diagram *phase stability*

Pressure

Ice (solid)

Water (liquid)

Water vapor (gas)

~610 Pa — Triple point

~0.01 °C   Temperature

Simple bi-stable simple mechanical system

Dynamics of snap-through with initial stable pre-buckled shape (Numerical simulation, Baroudi 2019)

a)

Transverse force *F*

Vertical tip-position

b)

Total potential energy

unstable

Vertical tip-position

stable   stable

Bi-stable states

c)

$P > P_E$

stable — 0 ms

stable — 10 ms

unstable — 13 ms

unstable — 14 ms

stable — 19 ms

e)

What is stability as a phenomenon?
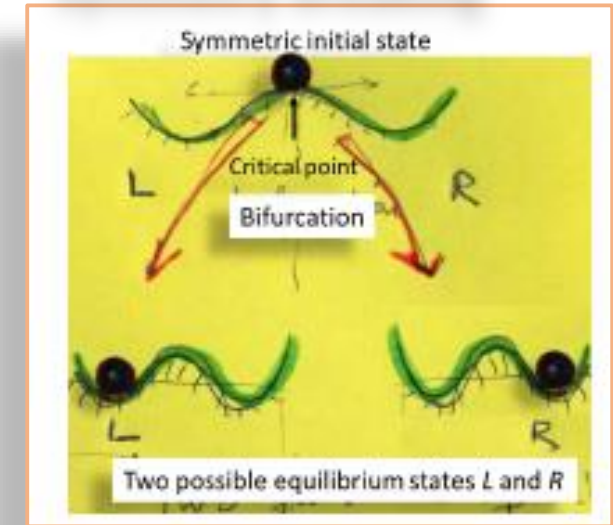
Stability loss as a **symmetry breaking** phenomena

- In **physics, loss of stability** belongs to the class of **symmetry breaking** phenomena …

  ➤ where action of **infinitely small perturbations** (fluctuations) on the **system** being **close to a critical point** …
  ➤ leads to **sudden branching** *via* **bifurcation** to some other **neighboring state**

- **Loss of stability is dynamic by nature**

  Examples
  - ✓ *snap-through* of a shallow arch
  - ✓ *resonance* in parametric excitation of stay-cables of a bridge or cable of guyed tower,
  - ✓ Flutter (a dynamic instability of an elastic structure interacting with a fluid flow … Tacoma Bridge)
  - ✓ … ..

A tiny perturbation

Symmetry breaking

Straight primary equilibrium Is this position stable?

Symmetry breaking

Symmetric initial state

Critical point
Bifurcation

L   R

Two possible equilibrium states *L* and *R*

Theory of supersymmetry:
*Physicists believe that just after the **big bang**, all of the forces of nature were identical and all elementary particles were the same.   But within an 'instant', symmetry was broken . . . and then . . . we and the universe are here …*
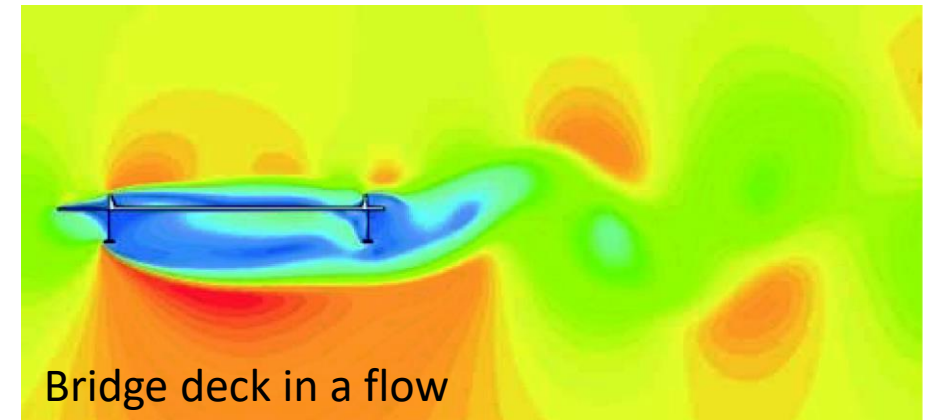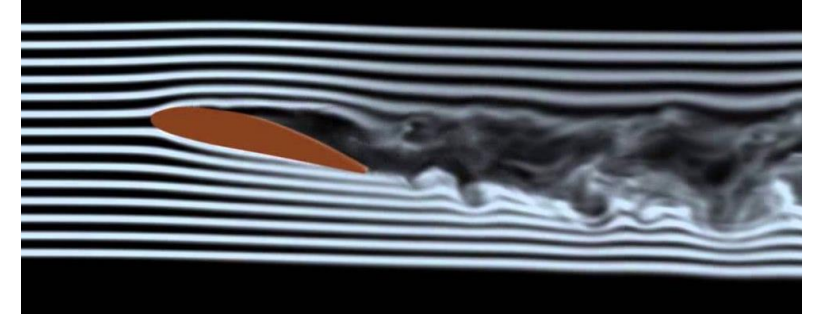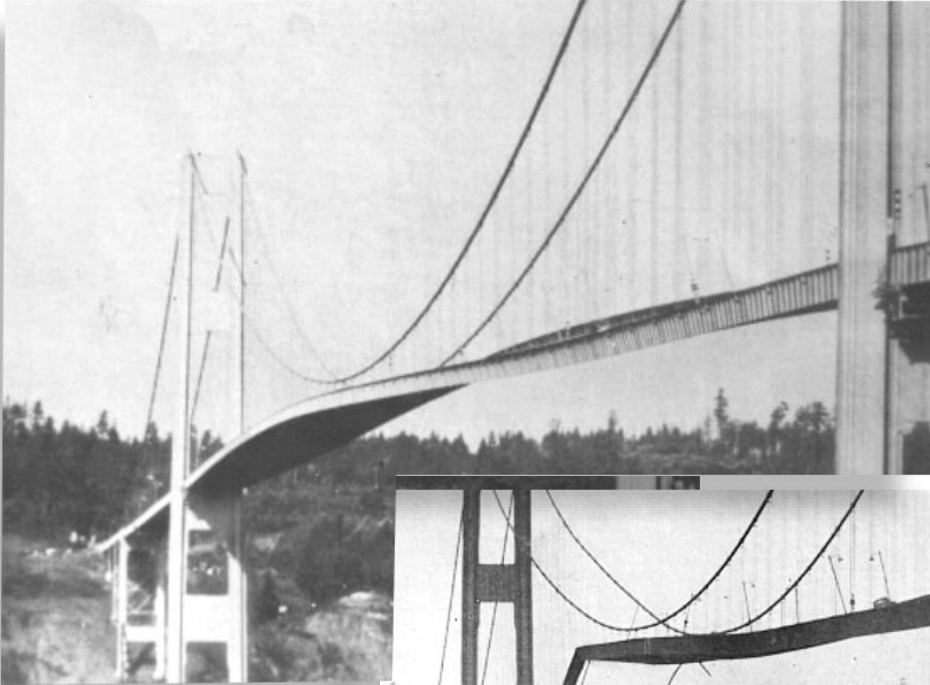
It seems that we are the consequence of a stability loss of the primary universe!

I ask physicists what does '*instant*' means before our time even existed!

What is stability as a phenomenon?

Oscillation and subsequent collapse of the Tacoma Narrows Bridge.

Bridge deck in a flow

Flutter
Coupling structure-fluid motion
Bending and torsional modes … have same frequency

Lyapunov dynamic stability criteria is naturally in use in structural dynamics

What is stability as a phenomenon?

Stability loss as a dynamical process

- **Dynamical systems** are generally described by non-linear differential equation set

$$\begin{cases} \dot{x}(t) = f(x(t)), \ t > 0 \\ x(0) = x_0. \end{cases}$$

Equilibrium point

- The system posses equilibrium points (states) x_e defined by $f(x_e) = 0$

x_e is an equilibrium point, if $f(x_e) = 0$

$$\Downarrow$$

$$\dot{x}_e = 0 \text{ so, with zero velocity}$$

$$\implies \text{ the point is at rest}$$

Lyaponov stability: *If at an equilibrium point x_e, two solutions (time series) having initial conditions close to each other remains close to each another for ever then the equilibrium point x_e is Lyaponov stable.*

Lyapunov dynamic stability:

equilibrium point, x_e is *Lyapunov stable*,

$$\forall \epsilon > 0, \ \exists \delta \text{ such that } \|x(0) - x_e\| \le \delta \implies \|x(t) - x_e\| \le \epsilon, \ \forall t \ge 0.$$
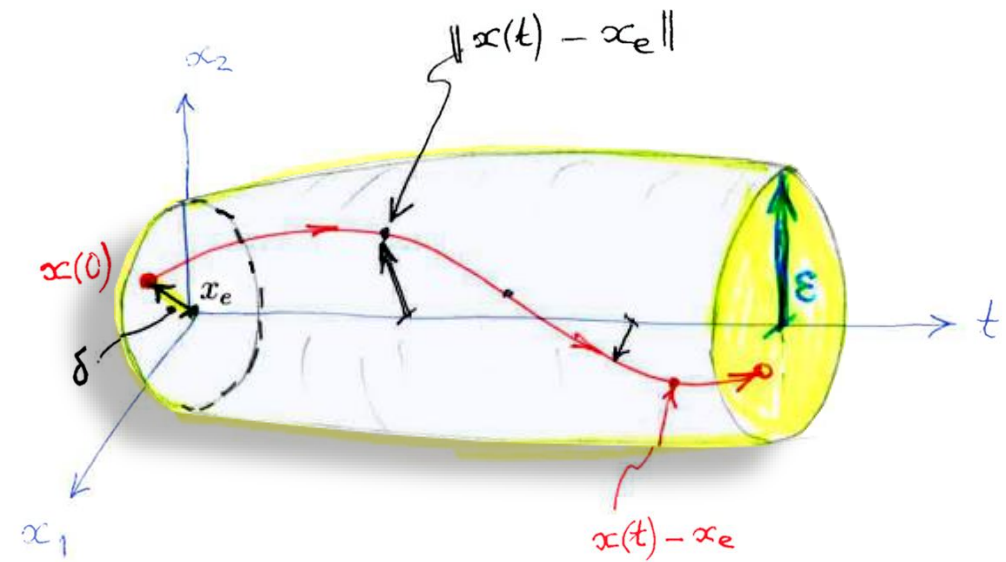
What is stability as a phenomenon?

- **Dynamical systems** are generally described by **non-linear differential equation set**

$$\begin{cases} \dot{x}(t) = f(x(t)), \ t > 0 \\ x(0) = x_0. \end{cases}$$

- The system posses equilibrium points (states)

$$x_e \text{ defined by } f(x_e) = 0$$

- The discrete **equation of motion** of a mechanical system, can be *recast* in terms of a canonical **non-linear dynamical problem**

$$\begin{cases} \mathbf{M\ddot{u} + C\dot{u} + Ku = f}, \text{ linear case} \\ \mathbf{M\ddot{u} = f(u}(t), t), \quad \text{non-linear case.} \end{cases}$$

velocity $\mathbf{v} = \dot{\mathbf{u}}$ as a change of variable

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{v} \end{bmatrix} \implies \underbrace{\begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{v} \end{bmatrix}}_{f(x(t))}$$

Lyapunov dynamic stability:

Lyaponov stability: *If at an equilibrium point x_e, two solutions (time series) having initial conditions close to each other remains close to each another for ever then the equilibrium point x_e is Lyaponov stable.*

Lyapunov dynamic stability criteria is naturally in use in structural dynamics

For the linear case:

$$\mathbf{f} := \mathbf{f} - \mathbf{C\dot{u}} - \mathbf{Ku}$$

What is stability as a phenomenon?

- The discrete **equation of motion** of a mechanical system, can be *recast* in terms of a canonical **non-linear dynamical problem**
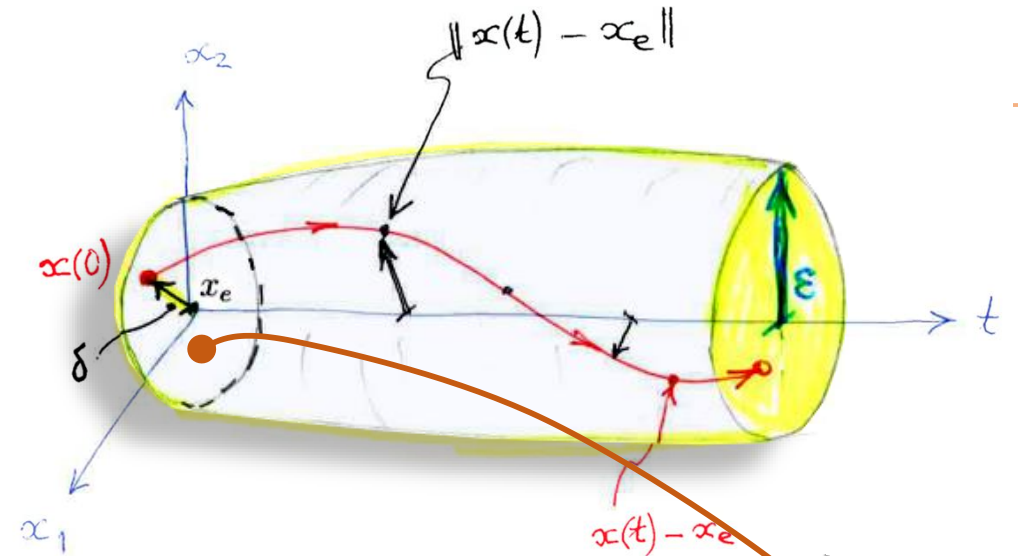
$$\begin{cases} \mathbf{M\ddot{u}} + \mathbf{C\dot{u}} + \mathbf{Ku} = \mathbf{f}, & \text{linear case} \\ \mathbf{M\ddot{u}} = \mathbf{f}(\mathbf{u}(t), t), & \text{non-linear case.} \end{cases}$$

velocity $\mathbf{v} = \dot{\mathbf{u}}$ as a change of variable

For the linear case:

$$\mathbf{f} := \mathbf{f} - \mathbf{C\dot{u}} - \mathbf{Ku}$$

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{v} \end{bmatrix} \implies \underbrace{\begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{v} \end{bmatrix}}_{f(x(t))}$$

Lyapunov dynamic stability criteria is naturally in use in structural dynamics

Includes also aero-elastic forces from the interaction of the structure with flow

Flutter leads ... to collapse

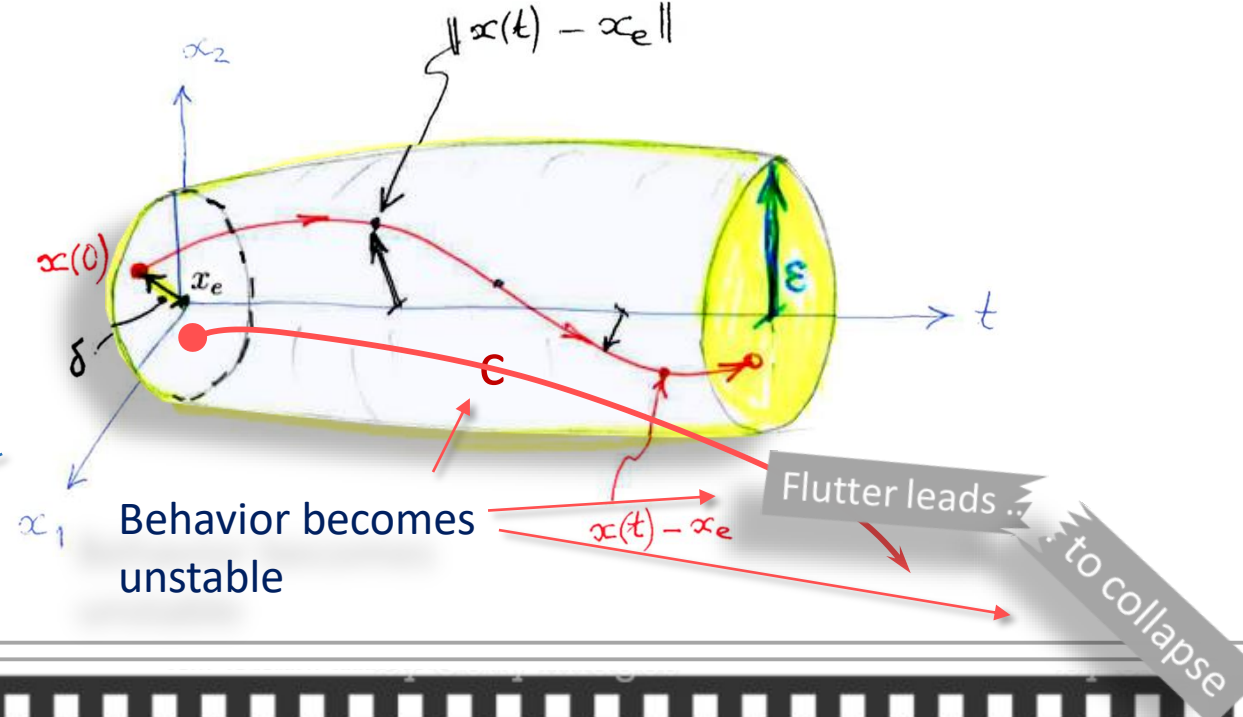✓ **Flutter** - a dynamic instability of an elastic structure interacting with a fluid flow ... Tacoma Bridge

Discrete **equation of motion** of mechanical systems

$$\mathbf{M\ddot{u}} = \mathbf{f}(\mathbf{u}(t), t), \quad \text{non-linear}$$

$$\underbrace{\begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{v} \end{bmatrix}}_{f(x(t))}$$

Includes in forces **f**

aero-elastic forces from the interaction of the structure with flow

Lyapunov *dynamic stability criteria*

$\|x(t) - x_e\|$

x_2

$x(0)$

x_e

δ

ε

t

x_1

Behavior becomes unstable

$x(t) - x_e$

Flutter leads ... to collapse

Tacoma Bridge

Example of dynamic instability: Flutter : *instability* of an elastic structure resulting from structure-fluid interaction

Structural design and stability

Standards: design of steel structures

- Local buckling .... **EN 1993-1-5**
- Flexural buckling .... **EN 1993-1-1** hot rolled columns
- Lateral torsional buckling .... **EN 1993-1-1** beams
- Lateral

- Flexural torsional buckling ....
- Local-global .... **EN 1993-1-3**
- Distortional .... **EN 1993-1-5**
- Shear buckling

- Shell buckling .... **EN 1993-1-6**
  - Linear elastic Bifurcation Analysis (LBA) (= linear buckling analysis)
  - Geometrically Non-linear Analysis (GNA)
  - Geometrically Non-linear Analysis with Imperfections
  - ... LA , LBA , GNA , GNIA, ...   (= post-buckling analysis for perfect structure and structure with imperfections)

  * * *

Standards: design of wood structures

- Stability issues & imperfections .... **EN 1995-1-1**

Standards: design of concrete structures

- Sect. 5.8 Second order effects with axial load..... **EN 1992-1-1**

Some standards related to stability issues in structural design.
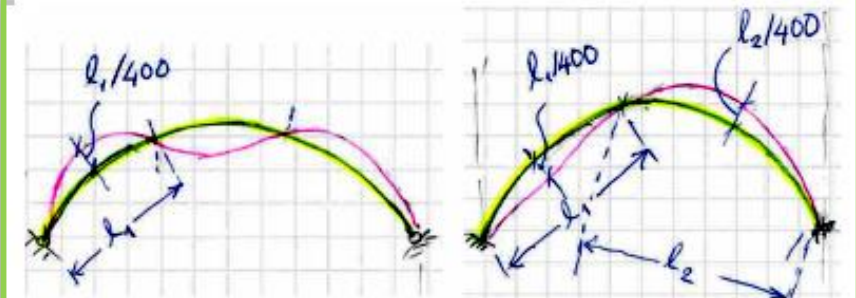
a) loading is symmetric   b) loading is antisymmetric

Example of initial shape imperfections in an arch (Standards: design of wood structures - EN 1995-1-1)

Some typical loss of stability in structures

1) Nurjahdus
2) Kiepahdus
3) Vääntönurjahdus
4) Lommahdus

1)   2) Lateral torsional buckling

3) Torsional buckling   4) Buckling of plates   4) Buckling of shells
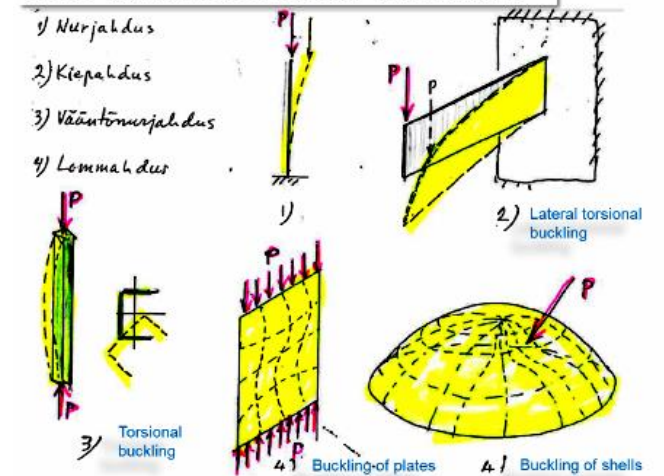
Drawings by: prof. M. Mikkola

Example of initial shape imperfections in wooden arches to be accounted in the structural analysis.

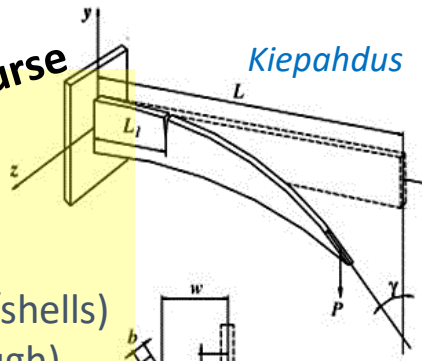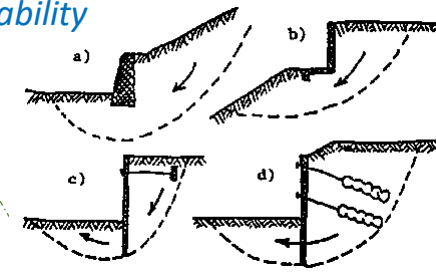# Examples of loss of stability

Slope stability
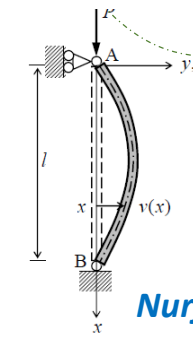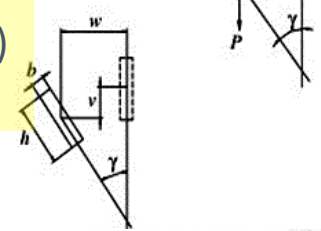
Rakenteiden epästabiiliusilmiöitä: ← *This course*
- sauvojen **nurjahdus** (flexural buckling)
- **vääntönurjahdus** (torsional buckling)
- **kiepahdus** (lateral torsional buckling)
- levyjen ja kuorien **lommahdus** (buckling of plates/shells)
- laakeiden kaarien ja kuorien **läpilyönti** (snap through)

Kiepahdus

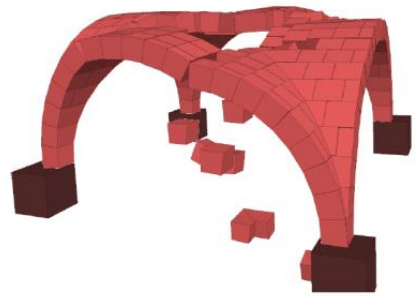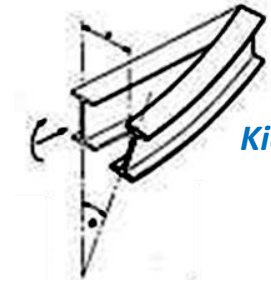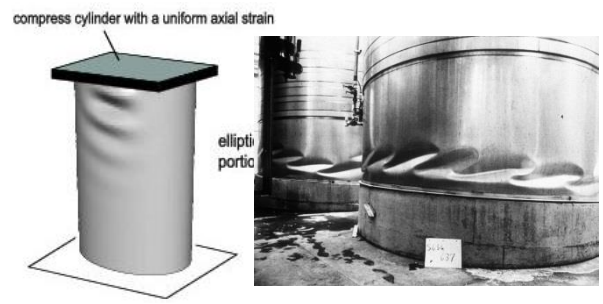*Nurjahdus*

Lommahdus

compress cylinder with a uniform axial strain

Kiepahdus

(Leikkaus-)lommahdus

Läpilyönti

Load

Vertical Deflection
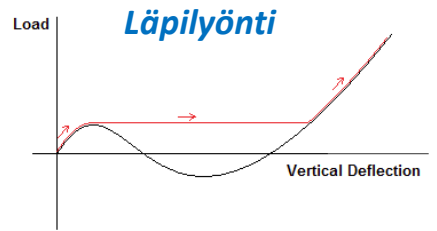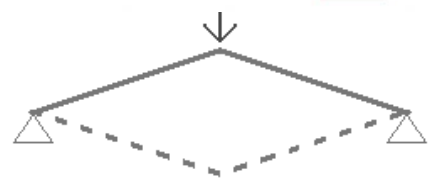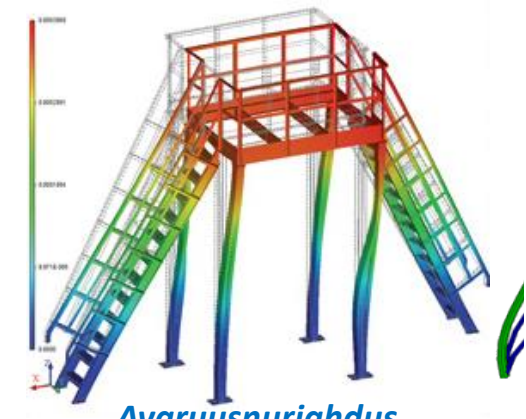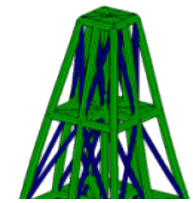
Lommahdus

Avaruusnurjahdus
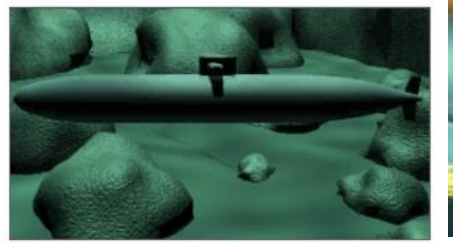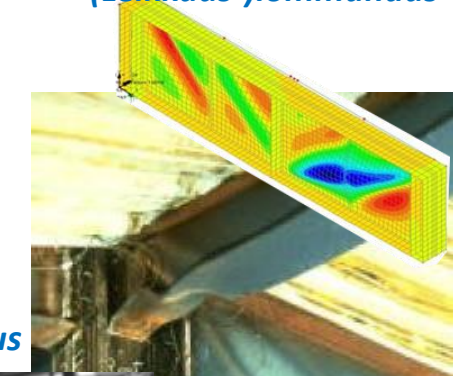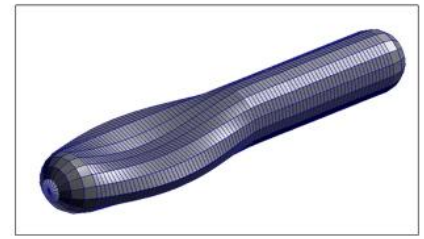
Examples of loss of stability

Rakenteiden stabiiliusilmiöitä - sekalaista
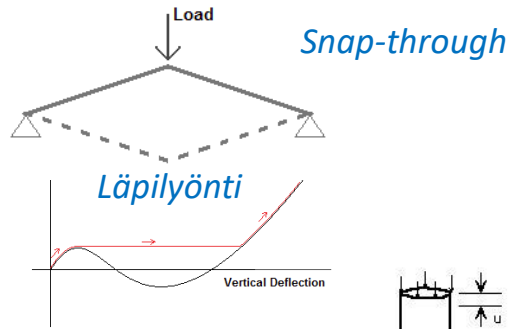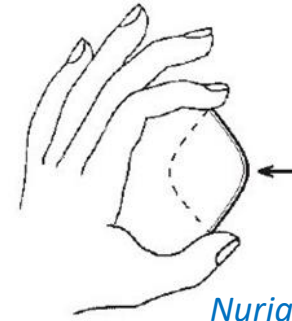
Snap-through

Läpilyönti

Kimmoton nurjahdus
Inelastic buckling

Nurjahdus

Läpilyönti
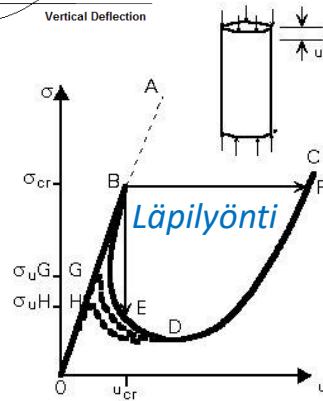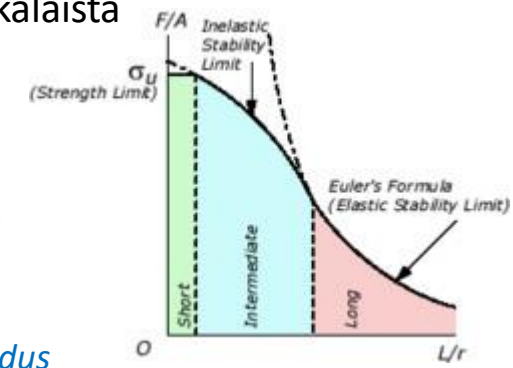
Lommahdus

Lommahdus

Cracks in concrete soffit

Local buckling in top flange
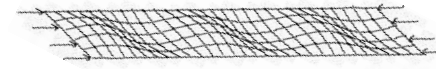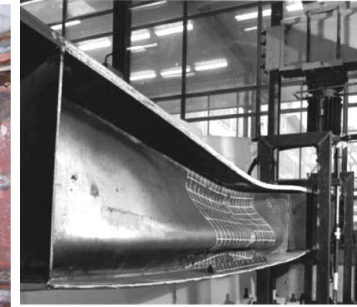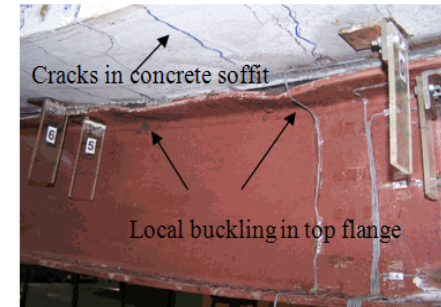
Figure 2. Vertical web buckling.

Figure 6 Post-buckling behaviour of an imperfect axially compressed cylindrical shell

Lommahdus

Construction Materials

Unattached Metal Deck or Plywood

C-Joist

Kiepahdus
Lateral buckling

Unbraced compression flange

Lommahdus

vrt. kiepahdukseen – vaikkakin alapaarre on vedossa
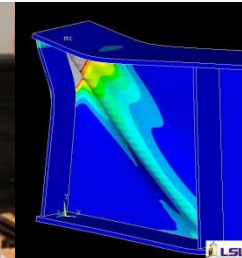
Maapohja petti ... maan leikkausmurto
Soil Foundation Bearing capacity Failure – China

Material instabilities

Material instabilities

An anthology of errors accumulation

美丽的
莲花河畔景苑大楼
是怎样倒塌的

on the south an underground garage was being dug 4.6 m depth

莲花河畔景苑施工方案
先造楼，后造地下车库

Why in the Caesar Bridge (55 years BC), the piles are not parallel!

the excavated dirt was being piled up on the north, 10m height

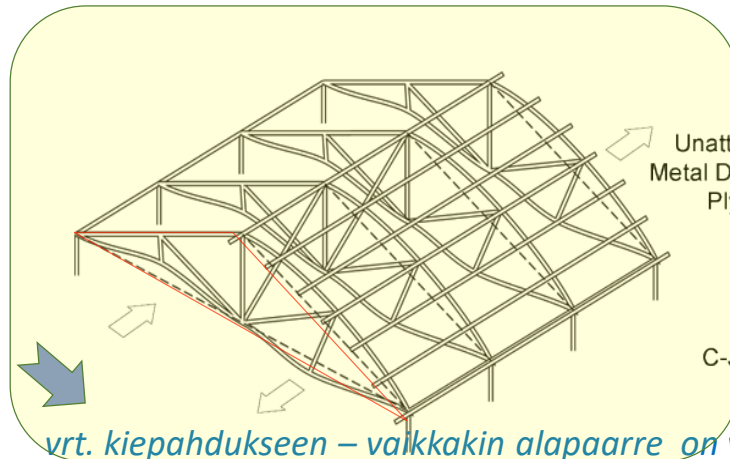on ehdottomasti väärä paikka varastoida kaivauksen maa-massa...

Then the rains came

大楼发生倾斜

The building experienced uneven lateral pressure north

resulted in a lateral pressure of 3,000 tonnes, which was greater than what the un-reinforced pilings could tolerate. Thus, the building toppled completely over in a southerly direction

创造世界房屋倒塌奇迹

Slope stability

Homogeneous soil

Soil after slope failure

Soil Foundation Bearing Capacity Failure

© 1998, Alan J. Scott

SUMMARY: 13 stories built on the surface of the ground . . . with no basement and attached to hollow concrete pilings with NO steel reinforcing. (*Where have all the true engineers gone?*)

Static: Transcosna Grain Elevator Canada (Oct. 18, 1913)

West side of foundation sank 24-ft

Soil Foundation Bearing Capacity Failure
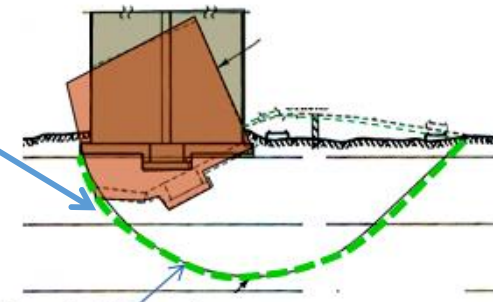
Todennäköinen leikkauspinnat
Shear-slip surface

China (Jiujiang) foot (or ramp) bridge collapse - buckling

Ref: https://www.youtube.com/watch?v=0WN8RP7Bz6Q

$$P_{cr} = \mu \pi^2 \frac{EI}{L^2}$$

$$\frac{4\pi^2 EI}{L^2}$$

P_{cr}

Examples of stability failure

Why instability is an unwanted event in design of structures?

Buckling failure happens – watch it, you'll see how fast & unpredictable

- **At once and global (in the whole structure or its member)**
- **Unpredictable (you do not have time to see the signs that is too late already) ... happens between 7th s and 9th s!**

Examples of stability failure

Why instability is an unwanted event in design of structures?

Foot bridge (ramp) collapse in Jiujiang City (China's Jiangxi)

0:10 / 1:00

P_{cr} P_{cr}
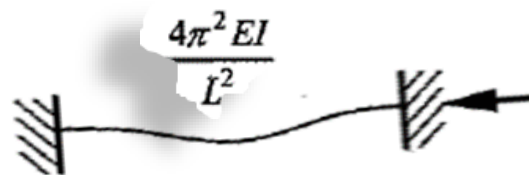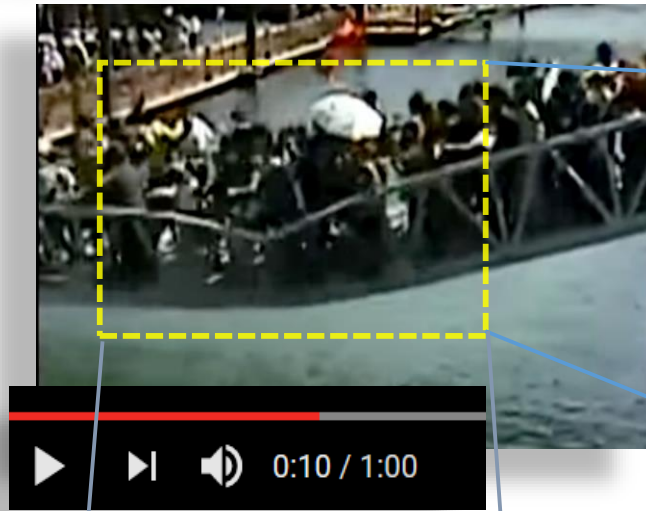
$$\frac{4\pi^2 EI}{L^2}$$

$$P_{cr} = \mu\pi^2 \frac{EI}{L^2}$$

Railway bridge collapse, Russia ~1890
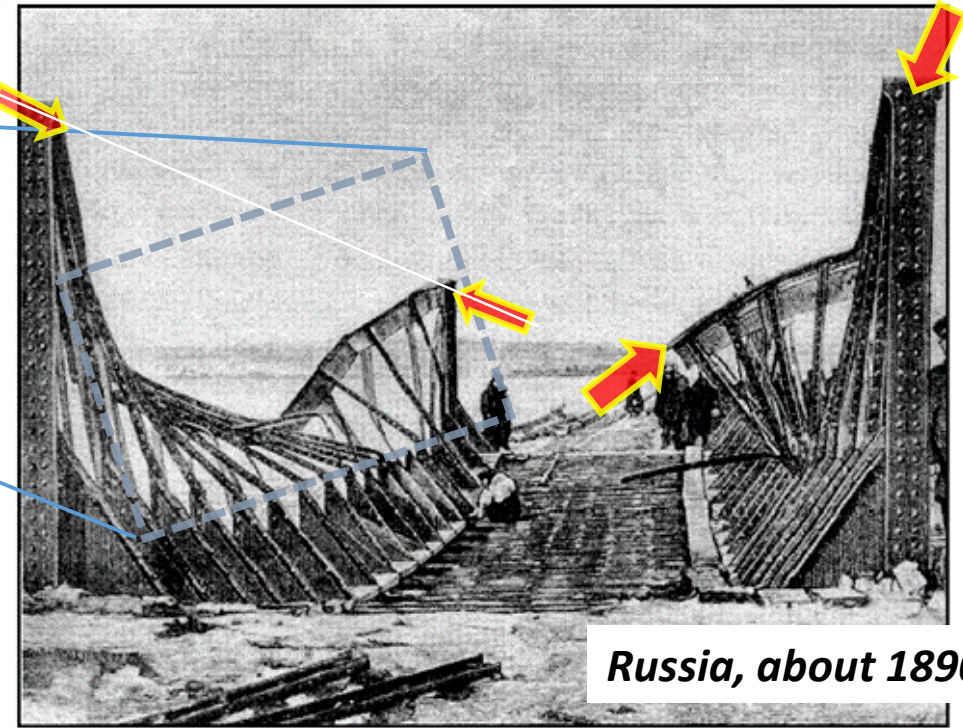
Russia, about 1890

fig. 8.3. - *Flambement d'ensemble de la membrure supérieure des poutres en treillis d'un pont de chemin de fer (Russie, vers 1890).*

buckling

Flambement d'ensemble de la membrure supérieure des poutres en treillis d'un pont de chemin de fer (Russie, vers 1890).

The mechanical cause of the collapse is the same: flexural buckling of compressed upper chord of the truss (yläpaarteen nurjahdus)

Consequences of stability loss

Good engineering final result should always provide the product itself together with *a quantified safety margin* of it operation.

Non-linear analysis — Joints — **Stability loss of primary structure**

Joints

Joints

Joints

consequence

Post-buckling can result in excessive displacements and/or rotations from stability loss can lead to failure of joints or components and of subsequent structures resulting in overall collapse
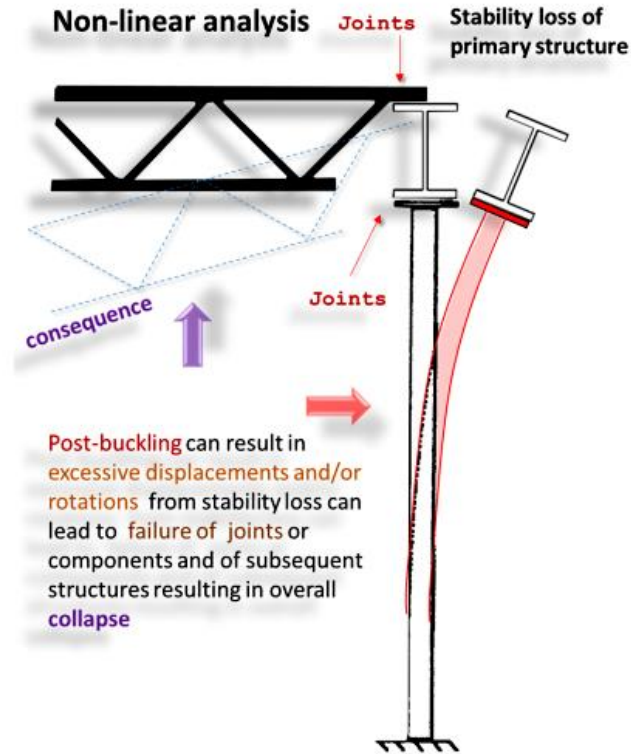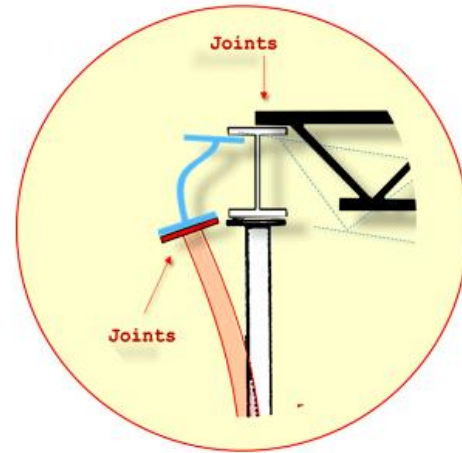
Figure 3.26: Illustration of possible consequence of too large rotations or displacements after stability loss.

Structural design and stability
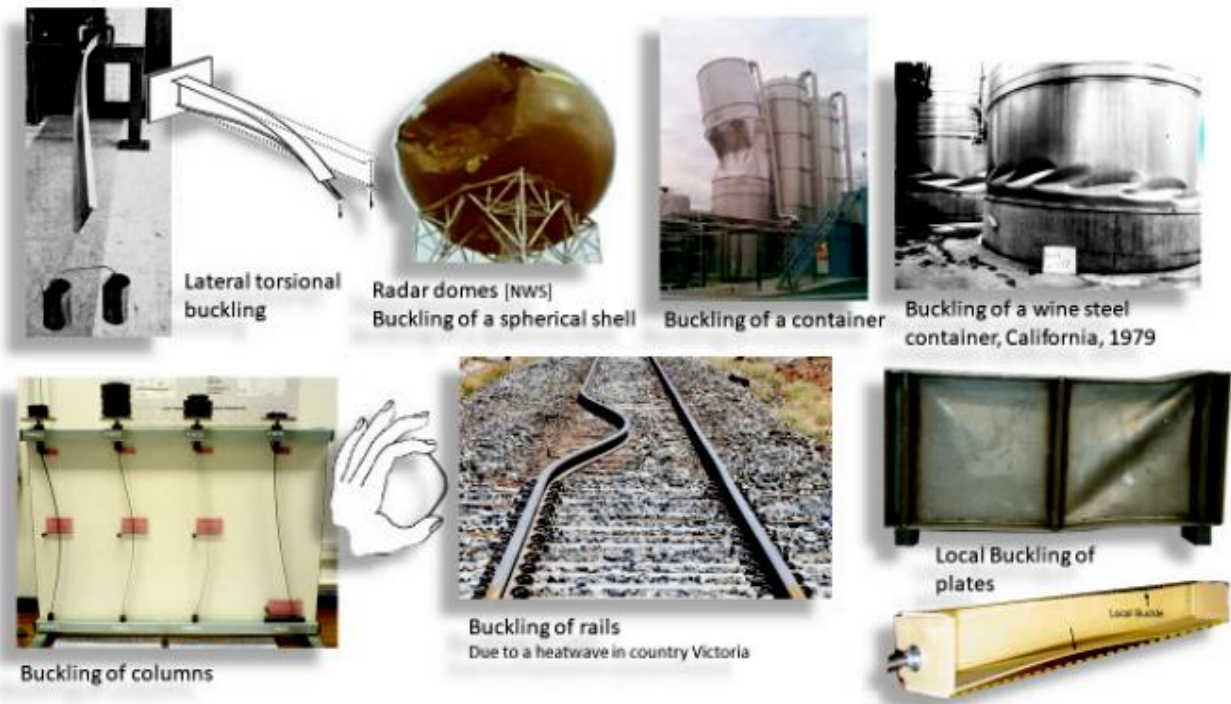
Figure 3.23: Examples of various types of loss of stability in simple structures. From left to right: lateral-torsional buckling, buckling of spherical and cylindrical shells, buckling of slender columns, buckling of a rail-road rail bonded to a support and plate buckling represented by shear buckling of the flanges and compressive buckling of web.
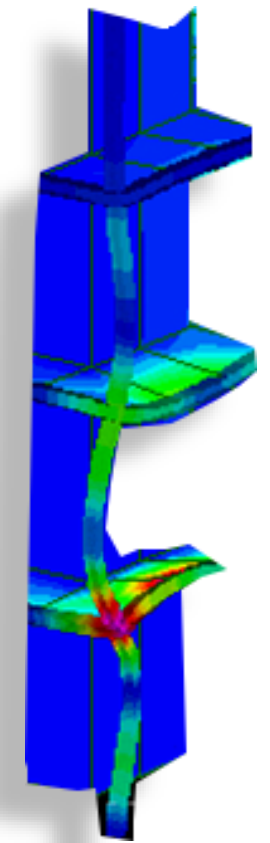
Flexural buckling

Pieter van Musschenbroek

(1692 – 1761)

A Dutch scientist

Physics, mathematics,
philosophy, medicine, astronomy

He did pioneering studies on the buckling of compressed struts

$$P_{failure} \quad \alpha \quad 1/\ell^2$$

➢ Performed experiments on column buckling (1729)

➢ Observed that the maximum compressive load a column can sustain prior to failure is proportional to $1/\ell^2$

experiment

theory

Compare with Euler's buckling load:
(obtained theoretically, 1744)

$$P_{cr} = \frac{\pi^2 EI}{l^2} \quad \sim \quad 1/\ell^2$$

STATIC STABILITY OF STRUCTURES

Exp: Djebar Baroudi, PhD

Djebar BAROUDI, PhD.
Short-version, state: **7.10.2016**

Euler, the Great

7.10.2016

Leonhard Euler

To: Structural Mechanics:

Beams and frame structures

$$\sigma = \frac{F}{A} = \frac{\pi^2 E}{(\ell/r)^2}$$

Writing by Euler himself; can you identify **the formula?**

Modern writing
$$\frac{\pi^2 EI}{L^2}$$

$$\text{eft} = \pi\bar\omega \cdot \frac{E\acute{k}k}{aa},$$

SUR
LA FORCE DES COLONNES.
PAR M. EULER.

cette colonne eft capable de foutenir fans fe plier, le poids que $\text{eft} = \pi\bar\omega \cdot \frac{E\acute{k}k}{aa},$

$a = 2f$
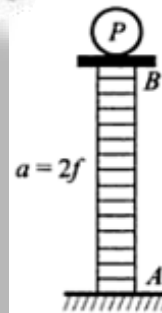
He derived the theoretical critical load for buckling of a column already in **1774!** At that time no one understood the importance of such result.

Leonard Euler

2..718281828459045235360287471352662497757247093699995...

33

Effects of boundary conditions – experimental evidence for Euler's buckling formulas
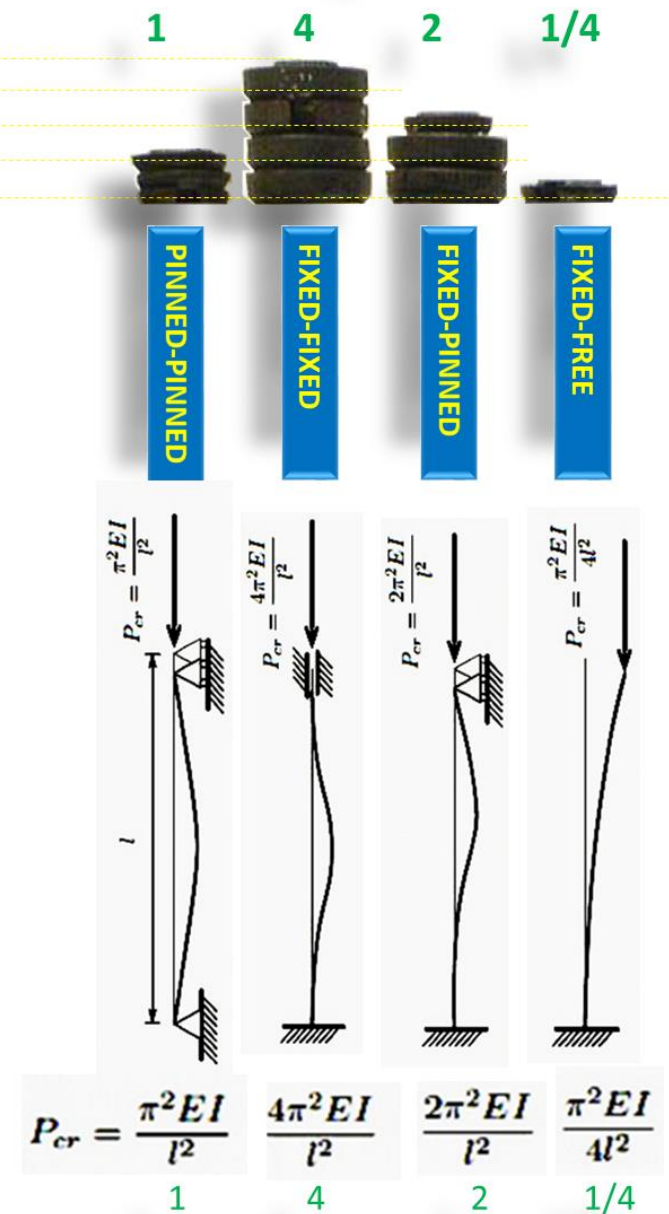
Stability analysis investigates

Photo: D. Baroudi, 2016

- Equilibrium configurations existence of multiples equilibriums or limit points
- Stability of these equilibriums with respect to small perturbations
- Sensitivity with respect to imperfections
  - shape
  - geometry
  - Loads (eccentricity)
  - Material imperfections

In addition to the above points, following questions will be answered:

- Can we predict the critical load?
- What happens at the bifurcation (or limit) point?
- Can we describe or determine the post-critical branches? What would be their shape? Their nature?

Equilibrium? **Yes**. $\quad \Leftarrow \delta\Pi[\mathbf{u}] = 0$

But, is it stable? **No.** $\Leftarrow \delta^2\Pi[\mathbf{u}] < 0$

Fundamental Concepts

- **Stable equilibrium state**

  - Let's perturb a bit a structure Initially resting at an *initial equilibrium state* away from this equilibrium state. If after removal of all disturbing factors the structure *returns to its initial equilibrium state* then the initial equilibrium state is **stable** (for elastic structures)

  - Even when the structure *tends only to return to its initial equilibrium state* one can assume that this initial equilibrium state is stable.
    Such behaviour occurs when material behaviour is plastic or elasto-plastic. In such cases the structure returns only incompletely to its initial configuration because of residual deformations.

  - In case of rigid bodies the **stability of position** is considered

- When the above listed behaviour is not fulfilled we say that the equilibrium is unstable

Photo: D. Baroudi, 2016
stability of position

Small perturbation away from this equilibrium state
Is it stable?

Undeformed shape

Applied pressure

"Snapped through" shape

motion

Lateral torsional buckling

Snap-through

Snap-Through

Snap-Back

Flexural buckling
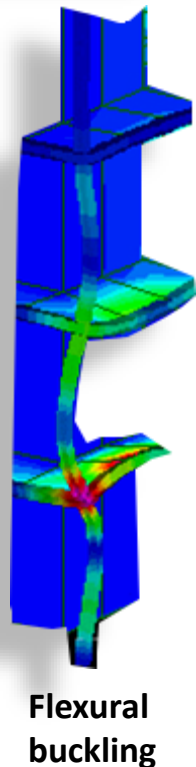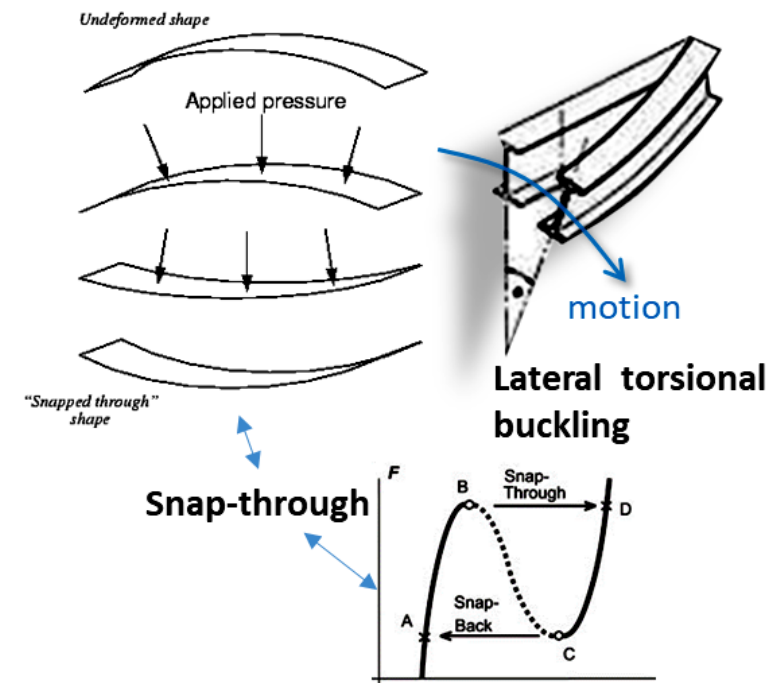
Method for Study of Stability

The stability theorem

$$\Pi' = 0$$
$$\delta\Pi = 0$$

Lagrange-Dirichlet Theorem: *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total potential energy of the system has a strict minimum (i.e., is positive-definite).*

$$\Pi'' > 0$$
$$\delta^2\Pi(u) > 0$$

S- total potential energy (or action)

$$\Delta S = S^* - S^0$$

$\delta S^0 = 0$ $\delta S^* = 0$

Initial equilibrium state   **Slightly perturbed state·**

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$

Figure 3.36: Generic illustration of the adjacent-equilibrium criterion in its variational version (up). (Δu is an infinitesimal perturbation corresponds to w on the lowest plot). Finite element post-buckling analysis (Lower figure). Note the initiation of combined lateral and torsional motion of the

Post-buckling analysis Lateral Torsional buckling, force control [Baroudi, 2019]

Lateral-torsional buckling just starts

Post-buckled equilibrium

Π^*

No buckling yet

Π^0

Clamped cantilever

Primary equilibrium

$\lambda = P/P_{cr}$

$w(\ell)/h$

$h = 10$ cm.

For conservative systems we consider stability of static equilibrium

Stability theorem of **Lagrange-Dirichlet**

$$\Pi' = 0$$
$$\delta\Pi = 0$$

Lagrange-Dirichlet Theorem: *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total potential energy of the system has a strict minimum (i.e., is positive-definite).*
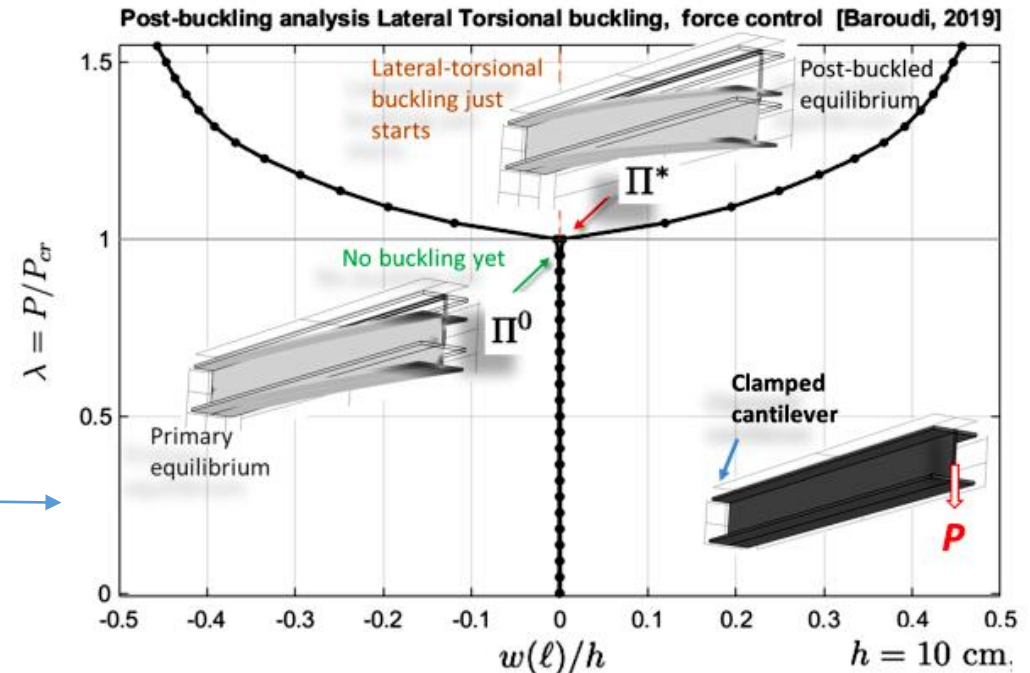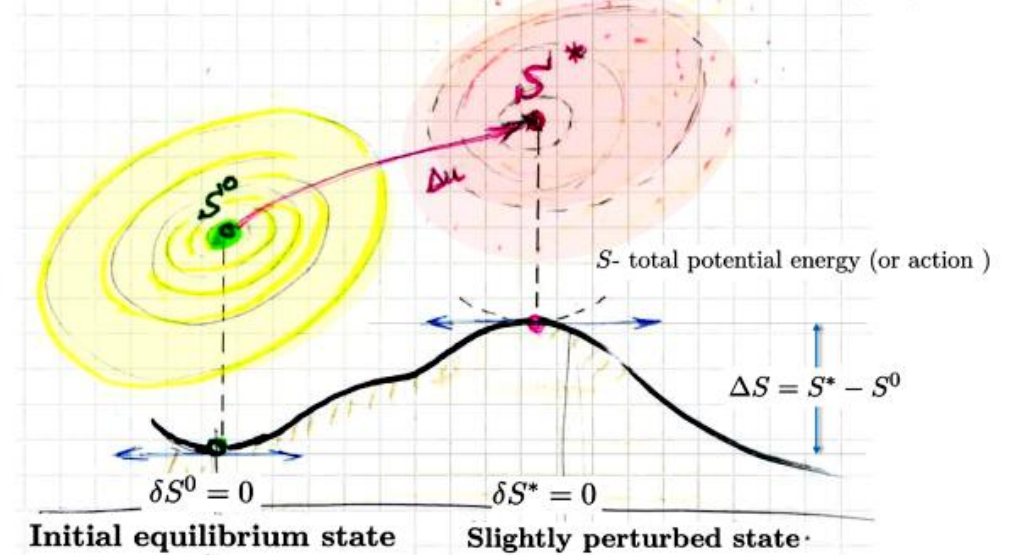
Trefftz condition

for stability of an equilibrium:

$$\begin{cases} \delta^2\Pi(u) > 0, & \text{stable,} \\ \delta^2\Pi(u) = 0, & \text{neutral,} \\ \delta^2\Pi(u) < 0, & \text{unstable.} \end{cases}$$

$$\Pi'' > 0 \quad \delta^2\Pi(u) > 0$$

- Is a *global energy criterion for stability*
- will be used systematically to derive the all the equations of stability (loss) we need for all elastic structure s

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$

Lagrange

Lagrange-Dirichlet theorem and investigate the sign of the

increment $\quad \Delta\Pi = \delta\Pi + \delta^2\Pi + \delta^3\Pi + \delta^4\Pi + \dots$

Dirichlet

(More general than Trefftz)

$\Pi''(u; P)$

Total potential energy

$\Pi'' > 0$ stable

$\Pi'' = 0$ neutral

$\Pi'' < 0$ unstable

Stable   unstable

$+$

P_{cr}

$\Pi''(u; P) = 0$

$-$

P

O

Trefftz is a particular case where the total potential energy increment is expanded only up-to its quadratic terms between the initial and perturbed states

The 3nergy approach permits a geometrical visualization of equilibrium and its stability as topographical maps where it is easy to recognize equilibrium paths and their stability properties

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$

Lagrange

Dirichlet

The geometry of equilibrium and stability

$$\Pi = \frac{1}{2}c\theta^2 - P\ell(1 - \cos\theta)$$

Total potential energy [D. Baroudi, 2018]

equilibrium paths

bifurcation

stable

unstable

stable

Π

Π/o-line zero

$\lambda = P/P_{cr}$

θ

Symmetric bifurcation

D. Baroudi, 201

$\frac{P}{P_{cr}}$

$\frac{P}{P_{cr}}$

D′   Unstable C   D

Stable   Stable

1   B

Stable

A

θ [degs]

180°

90°

45°

P

B

B′

θ

ℓ

$\ell(1-\cos\theta)$

C

A

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$

Energy approach – the MAP
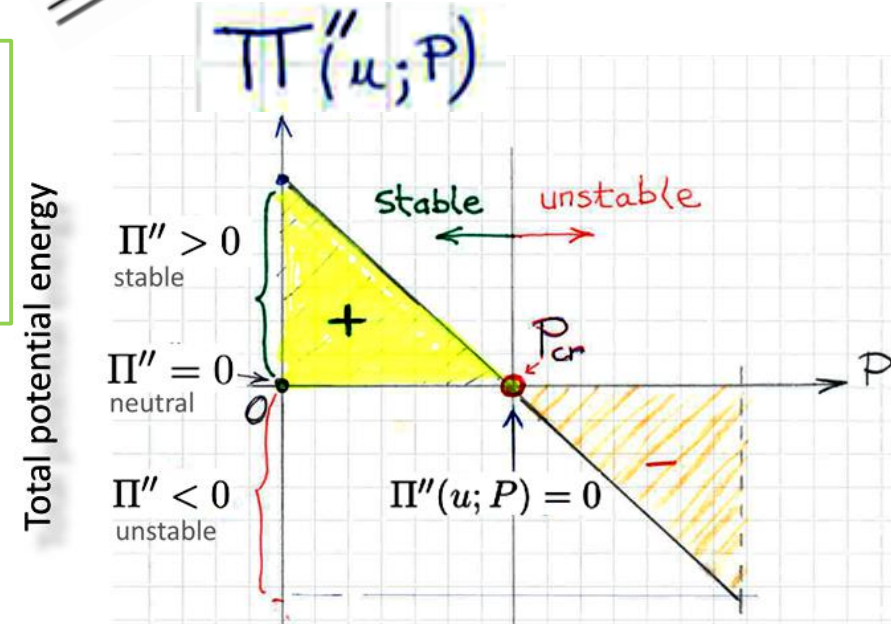
Total potential energy [D. Baroudi, 2018]

Equilibrium paths

Bifurcation

Buckling load

$\lambda = P/P_{cr}$

$\Delta\Pi > 0$ Stable

$\Delta\Pi < 0$ Unstable

$\Delta\Pi = 0$ Indifferent
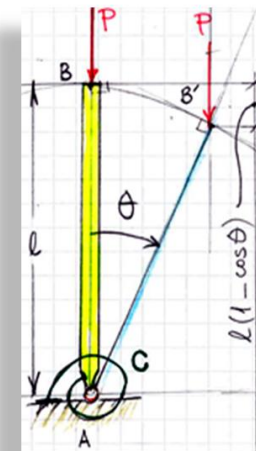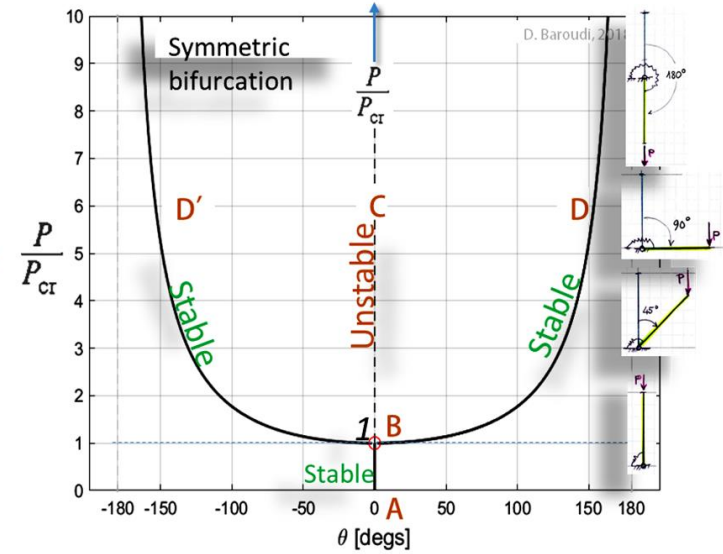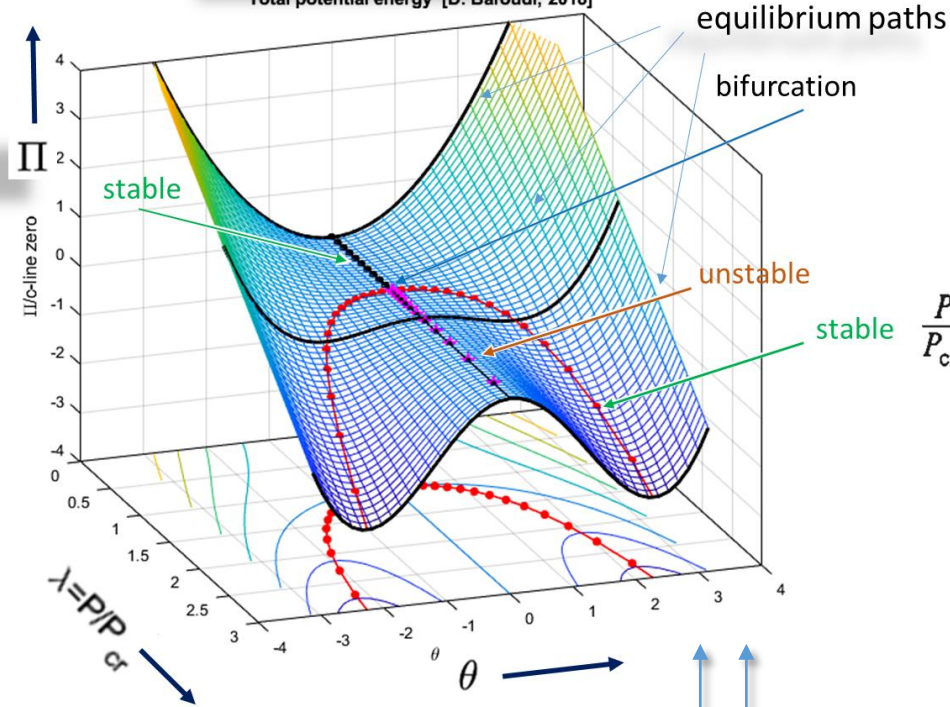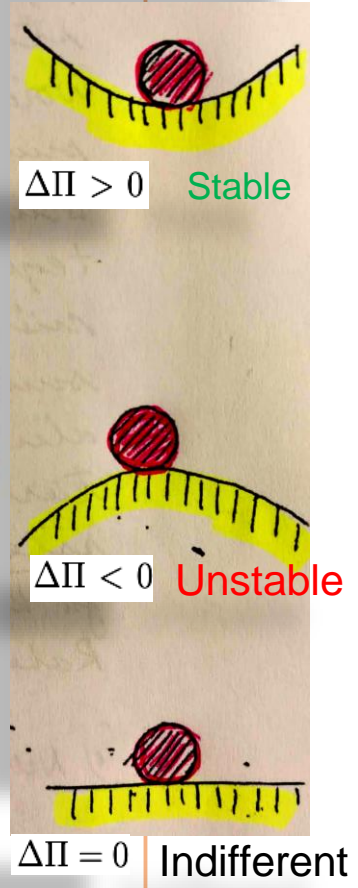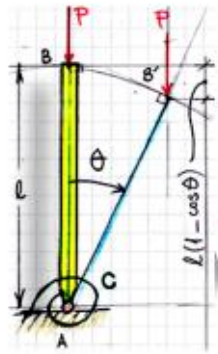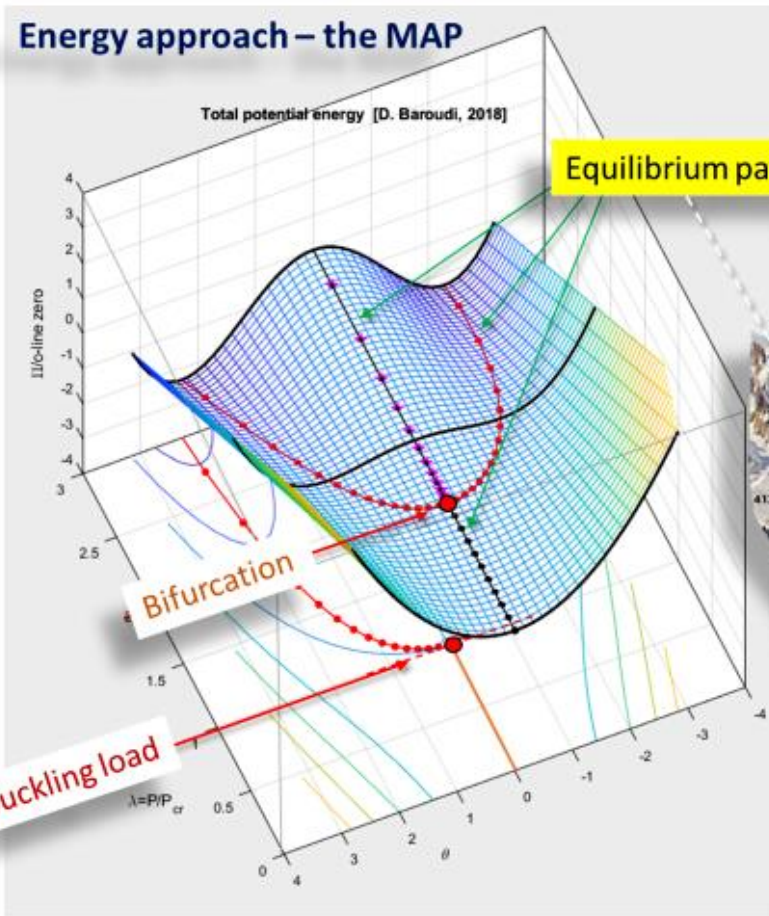
Figure 3.24: *Map* of total potential energy $\Delta\Pi$ of an elastic structure. Equilibrium paths corresponds to locations where $\delta(\Delta\Pi) = 0$ while keeping P constant. Stable equilibrium is achieved there where $\delta^2(\Delta\Pi) > 0$. Note the analogy with the topographical map of piece of Chamonix (Alpes).

Application of **Lagrange-Dirichlet** Stability theorem

Testing stability of an equilibrium:

- a static **equilibrium configuration**

+ *plus*

- **a small perturbation**

Perturb & release

δu

δu -infinitesimal perturbation
keeping loading parameter
unchanged

$\delta^2\pi < 0$
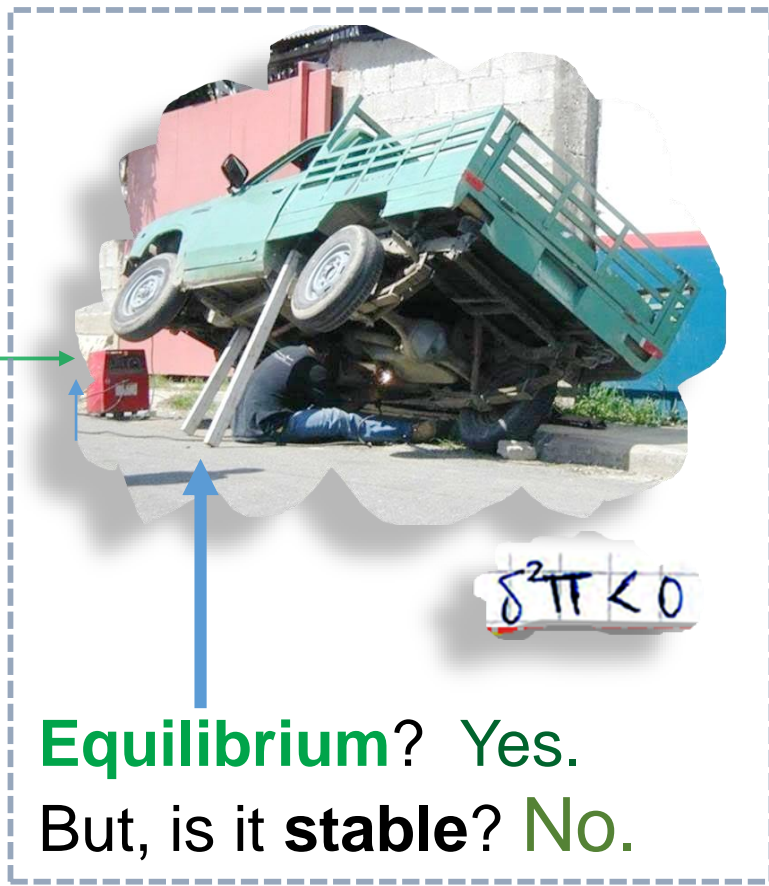
Equilibrium? Yes.
But, is it **stable**? No.

$\delta^2\pi > 0 \implies$ **Stable**

$\delta^2\pi = 0 \implies$ **Neutrally stable**

$\delta^2\pi < 0 \implies$ **Unstable**

Equilibrium Path for Deformed Systems

Illustration example: the simplest systems with one DOF

Equilibrium equation: $Pl \sin \varphi = k\varphi,$

Two independent solutions:

$\varphi = 0$ for any P; ←— primary path (primary equilibrium)

$P = \dfrac{k}{l}\dfrac{\varphi}{\sin \varphi}.$ ←— secondary path
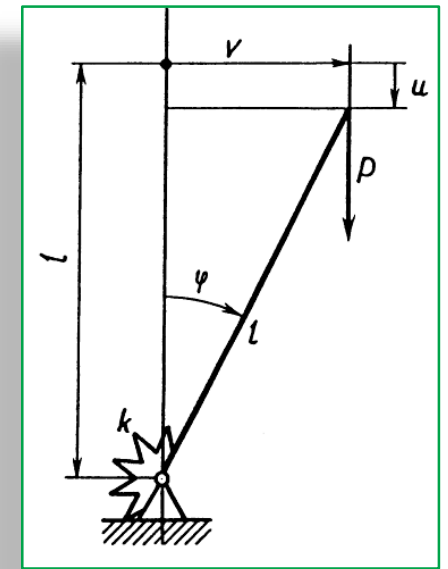
Equilibrium paths: are graphs of equilibrium solutions (= load-displacement curves)

Primary branches = pre-buckled behaviour

A key behavior of a loaded structure can be studied through:

- measuring or deriving or the relation between the *load* and the *displacement* at some characteristic point(s)

- Graphical representations of such relations

are called equilibrium paths (or equilibrium curves)

Secondary branches
Post-buckled behaviour
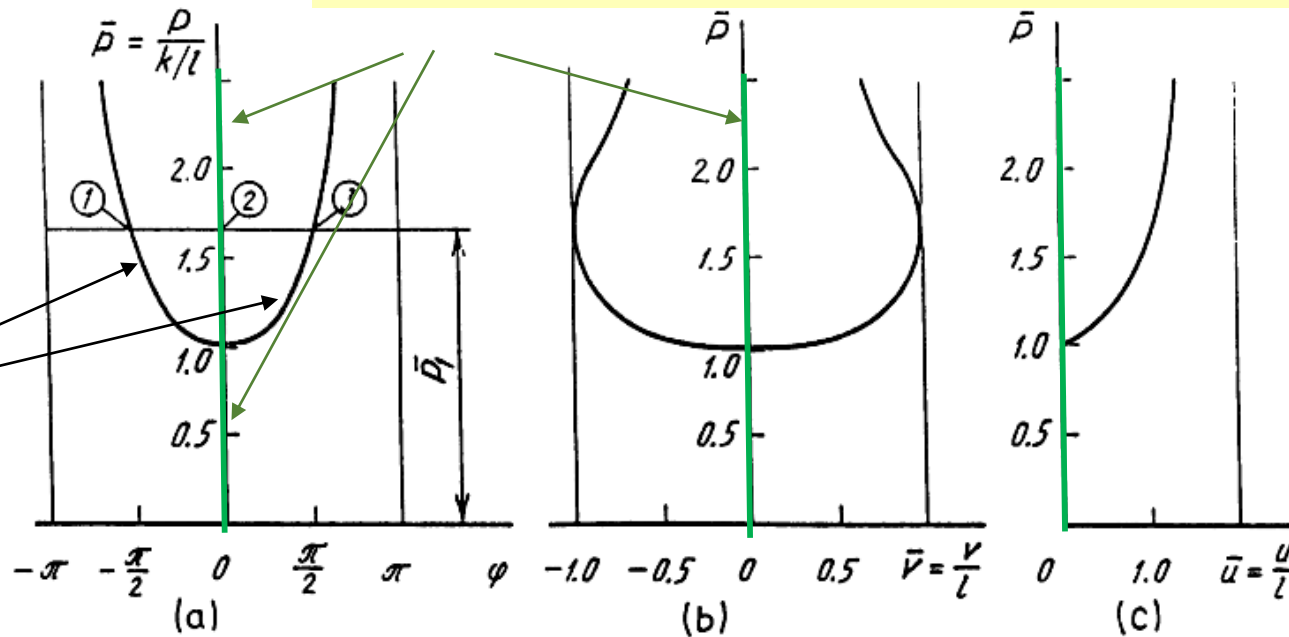
(a) Rotation

(b) Horizontal displacement

(c) Vertical displacement

Ref: this example is from:
N. A. Alfutov
Stability of Elastic Structures

Equilibrium path and stability loss

Critical Equilibrium Points

Generic equilibrium path
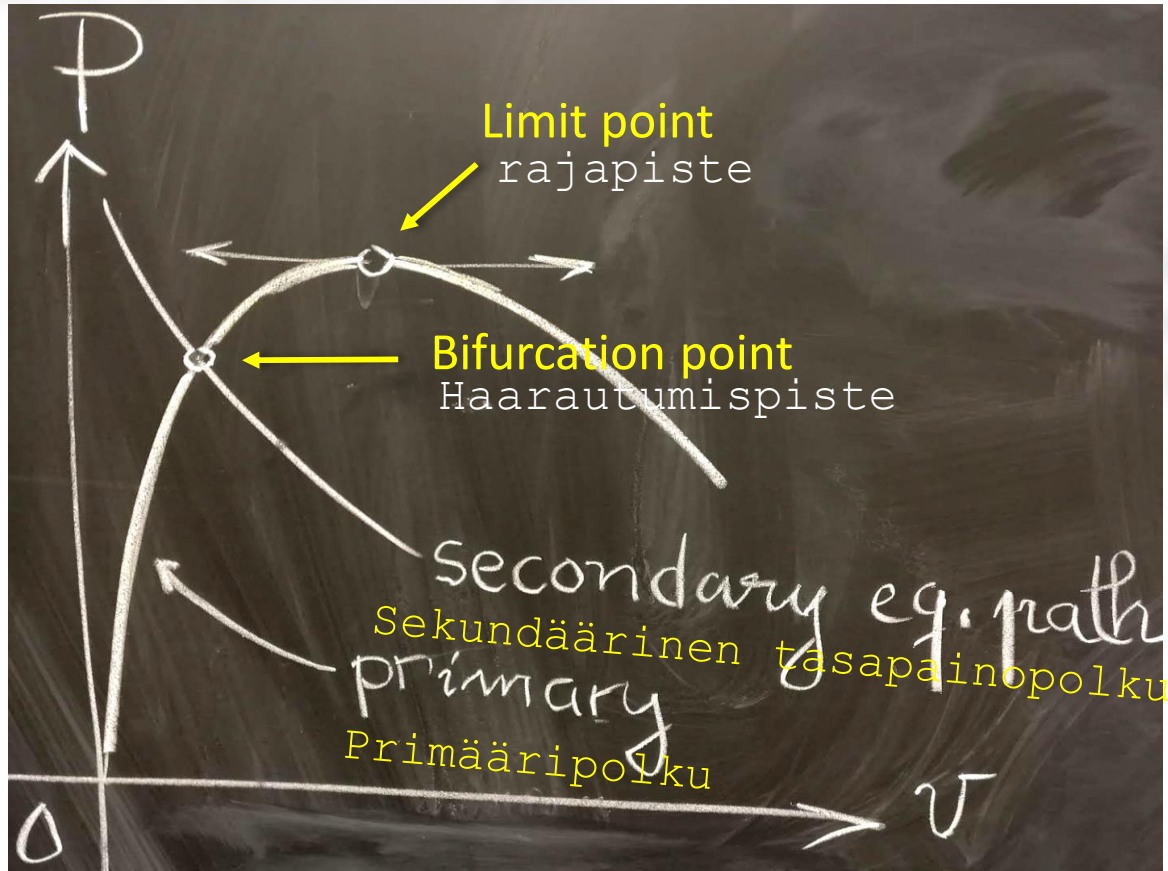
Critical points
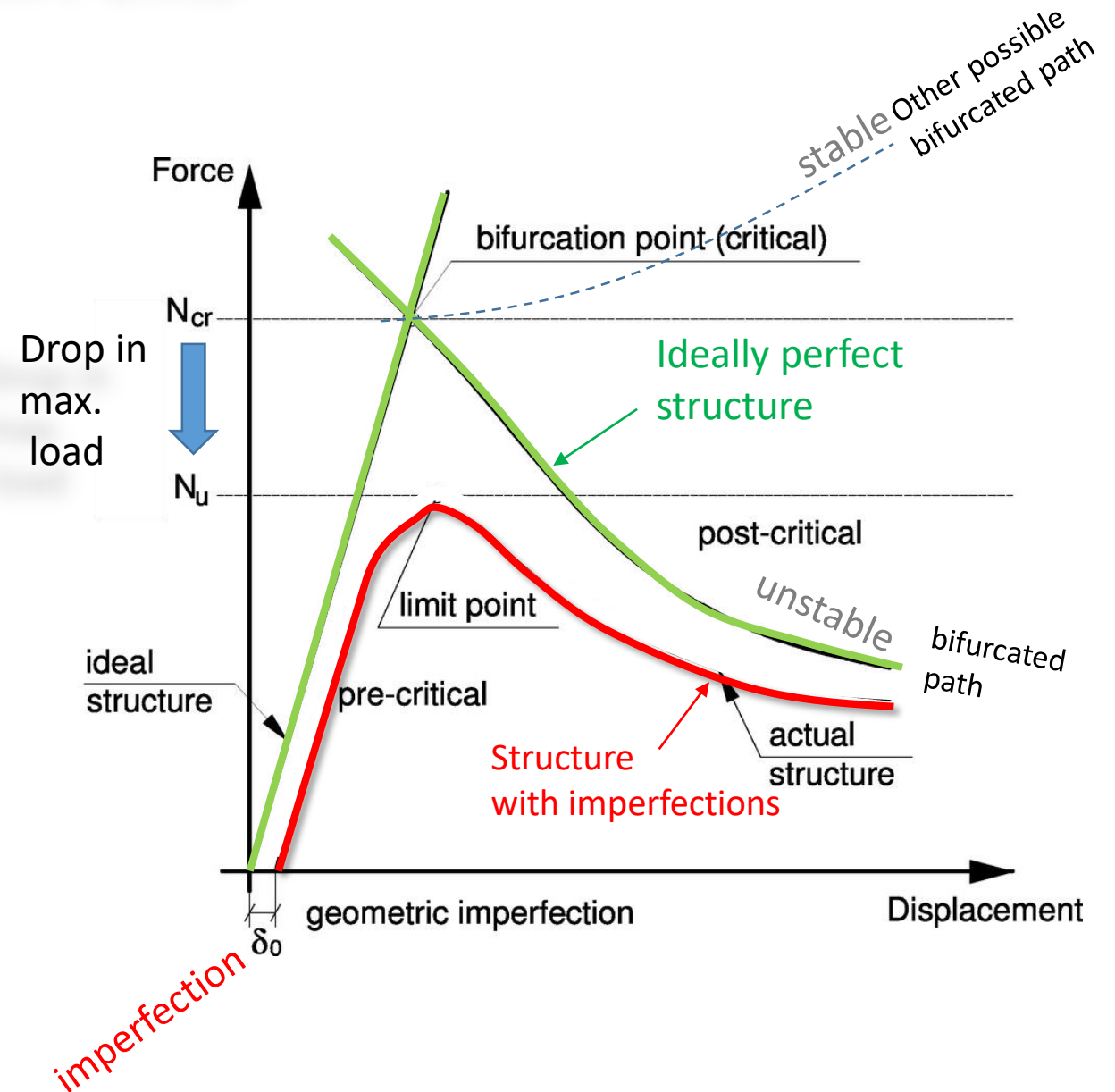
Linear Buckling Analysis
Non-Linear Buckling Analysis (GNA)

- **Linear buckling** analysis = solving an **Eigen-value problem**. Provides the smallest Eigen-value (**buckling load**) for ideally **perfect** structure

- **A non-linear buckling** analysis = a **geometrically nonlinear analysis** which is not anymore an *Eigen-value-problem* provides **limit-load** (or loosely called buckling load) for the imperfect structure and *the displacement-load curve*

- **How to do GNA?** Non-linear static + gradually increasing the loads + accounts for initial geometrical imperfections

- In real problems, all other relevant sources of non-linearity can be introduced: **material non-linearity**, contact, etc... **GMNA**

Linear Buckling analysis
(For ideally perfect structure)

Provides the Buckling Load
smallest Eigen-value

Buckling load of ideally **perfect** structure → N_{cr}

Buckling (or **limit**) **load** of imperfect structure N_u
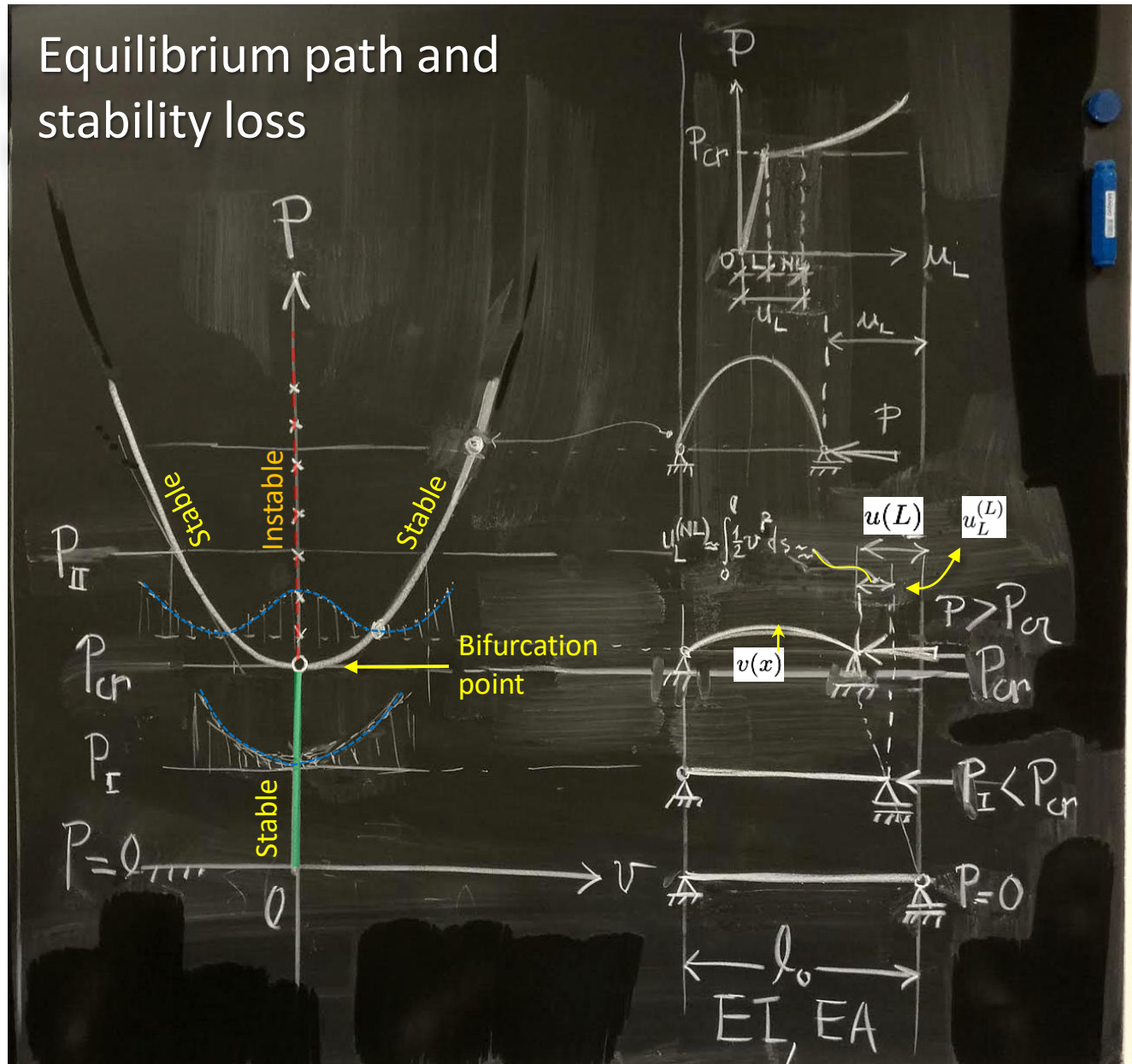
Non-Linear Buckling Analysis (GNA)
(For structure with imperfections)

Structure with imperfections

Ideally perfect structure

bifurcation point (critical)

Important drop

limit point

pre-critical

post-critical

actual structure

geometric imperfection δ_0

Displacement

Force

ideal structure

$\ell_1/400$

a) loading is symmetric

$\ell_1/400$ $\ell_2/400$

ℓ_2

b) loading is antisymmetric

Example of initial shape imperfections in an arch (Standards: design of wood structures - EN 1995-1-1)

Introduce the **imperfections**

GNA = geometrically Non-linear Analysis

Types of bifurcational instabilities

- The nature of post-buckling behavior determines to a large extend safety and the robustness of the structural design

Basic types of bifurcations Haarautuminen

- **Stable symmetric**
  - ✓ Structures having this type of behavior are always *imperfection insensitive* and have consequently *a reserve of resistance*

- **Unstable symmetric**
  - ✓ This gives *imperfection sensitive structures*

- **Asymmetric or unsymmetrical**
  - ✓ This gives much *more imperfection sensitive structures* than above

- **Snap-through**
  - ✓ Such dynamic behavior is pathological not desired behavior and is locally like an asymmetric branching on equilibrium path

Post-buckling behavior

perfect

imperfect

Neutral

Unsymmetric

Symmetric unstable

Symmetric Stable

Bifurcation points

Snap-through

Shallow arch

Thin shell

Axial shortening

$\frac{P}{P_c} = 1 + b(\frac{\delta}{t})^2$

deflection/thickness
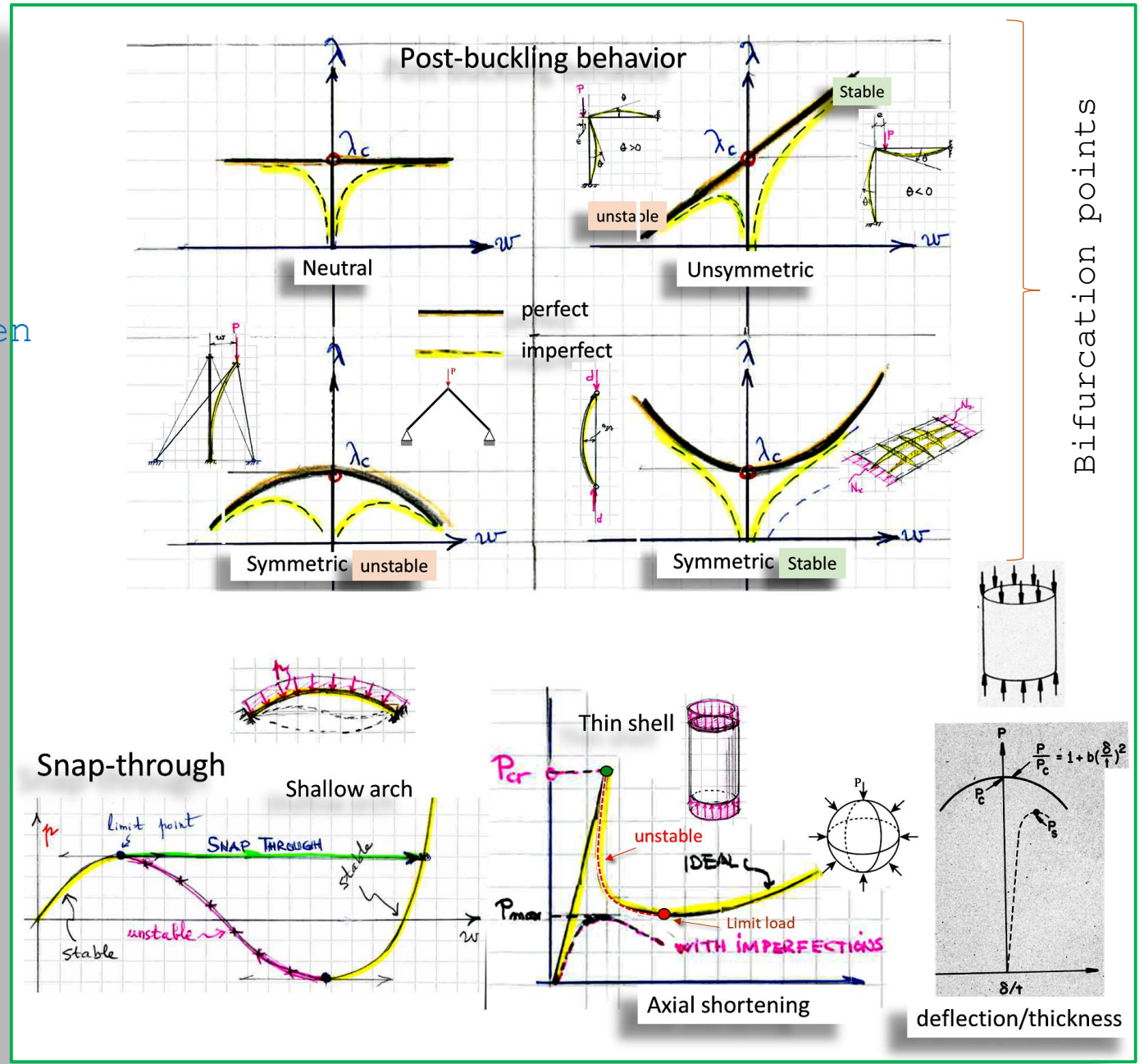
Limit point
Rajapiste

Limit point

Types of bifurcational instabilities

- The nature of post-buckling behavior determines to a large extend safety and the robustness of the structural design

Basic types bifurcations

- **Stable symmetric**
  - ✓ Structures having this type of behavior are always *imperfection insensitive* and have consequently *a reserve of resistance*

- **Unstable symmetric**
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- **Asymmetric or unsymmetrical**
  - ✓ This gives much *more imperfection sensitive structures* than above

Bifurcation diagrams
are not only
theoretical concepts
but they really exists

Roorda's experimental verification

Roorda's (1971) experimental verification of calculated postcritical response in asymmetric bifurcation of a Γ-frame.

Roorda, 1971, *An experience in equilibrium and stability*, Techn. Note No. 3, Solid Mech. Div., University of Waterloo, Canada.

Post-buckling behavior

Axial shortening

Example 1: Buckling of beam-column under self-weight

Dynamic Buckling

Equilibrium
Straight configuration
only stretching

Starts to moves to adjacent dynamic 'equilibrium state'
New flexural configuration
No more stretching, only flexion, in the beginning

Post-buckling behaviour Both curvature and stretch changes are present

Equilibrium
(almost)

Buckling under self-weight

g ↓

time [ms] =   0   4833   14083   19000   20333   21333   23916   54583

TIME (ms) ⇒

Critical load = $\rho g A L = 7.84 \dfrac{EI}{L^2}$ (N)

Figure 1.38 – Buckling dynamics of a column under self-weight under a tiny horizontal perturbation. The first configuration is stressless ($t = 0$) then at $t = 0^+$, the column is let to fall freely under its self-weight. The last shown configuration is close to the equilibrium configuration (Fig. 1.36). The colourful lines are trajectories covered during the same laps of time Δt.

Ref: D. Baroudi, 2018 – *full-nonlinear modelling – systematic use of virtual work principle with D'Alembert principle*
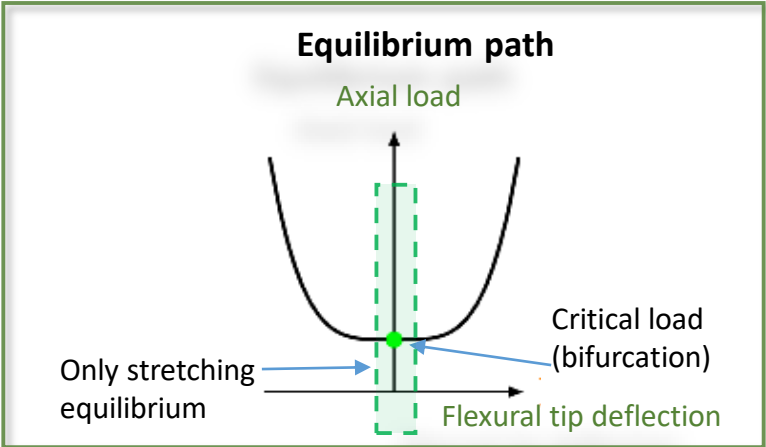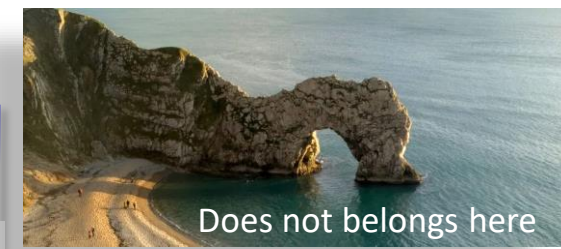
- Here one observes '*small*' quasi-static equilibrium configuration change going from straight to adjacent flexural sate (the green box)

- Such small changes are what we study in this course

- The minimum load neads to move from straight to the adjacent (flexural) equilibrium configuration is called buckling (critical) load

- The flexural new equilibrium configuration is called the buckling mode

Equilibrium path

Axial load

Only stretching equilibrium

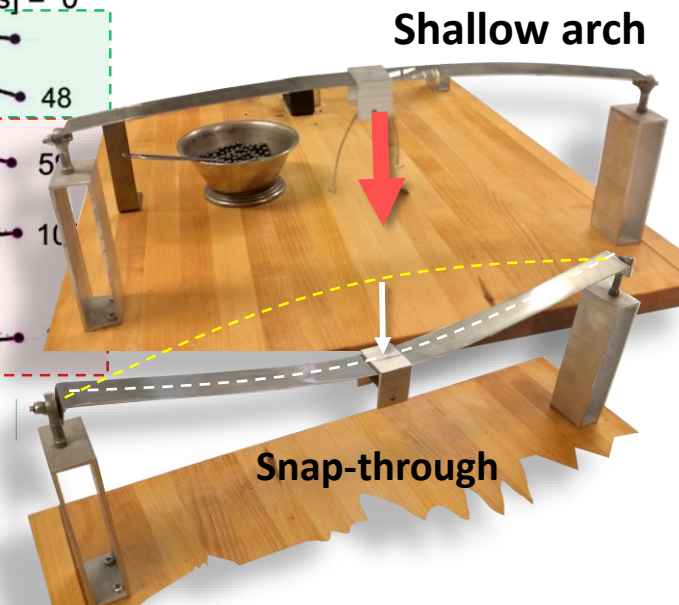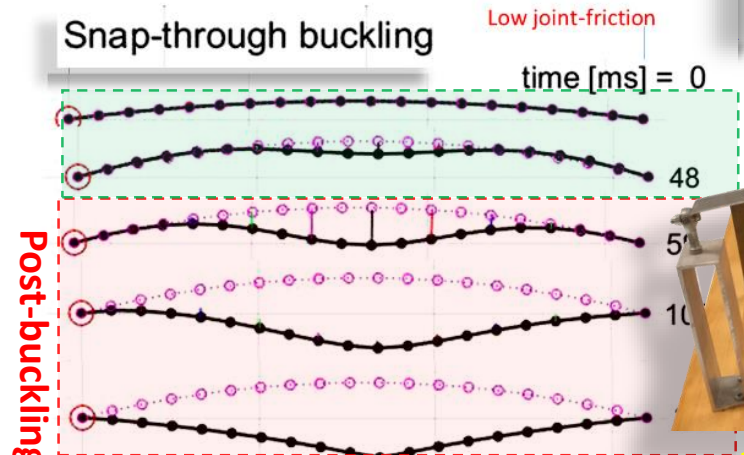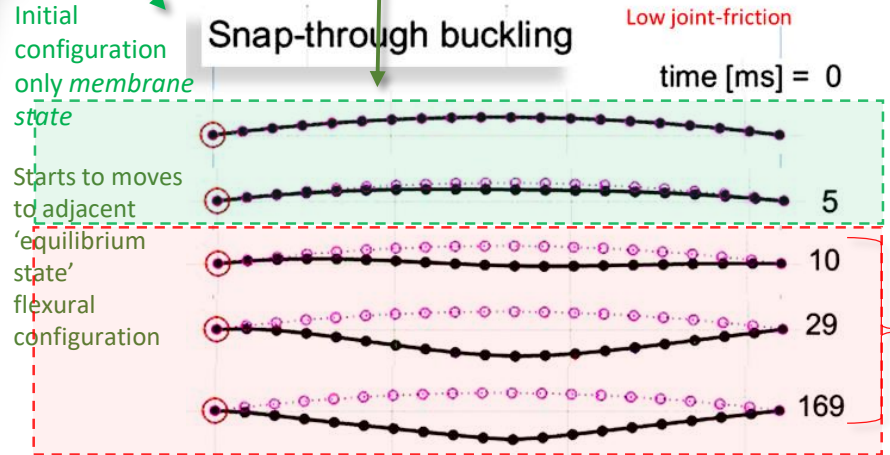Critical load (bifurcation)

Flexural tip deflection

Example 2: **Snap-through of shallow arch** (symmetric mode)

Ref: D. Baroudi, 2018 – *full-nonlinear modelling – dynamics*

Dynamic Buckling

Does not belongs here

Shallow arch

Snap-through

Snap-through buckling
Low joint-friction
time [ms] = 0

Initial configuration only *membrane state*

Starts to moves to adjacent 'equilibrium state' flexural configuration

5

10

29

169

F

TIME (ms)

curvature and membrane state

Post-buckling behaviour Both

Snap-through buckling
Low joint-friction
time [ms] = 0

48

5

10
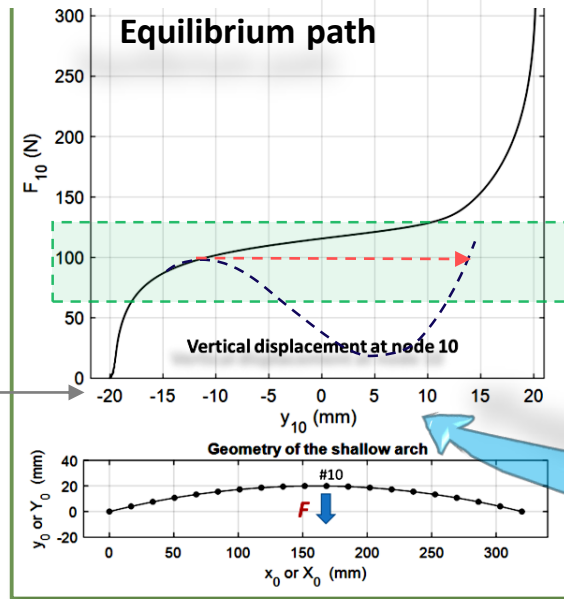
F

Self-weight and linearly increasing external nodal load at node 10 are acting
In the simulation, the system is to dissipative.

Equilibrium path
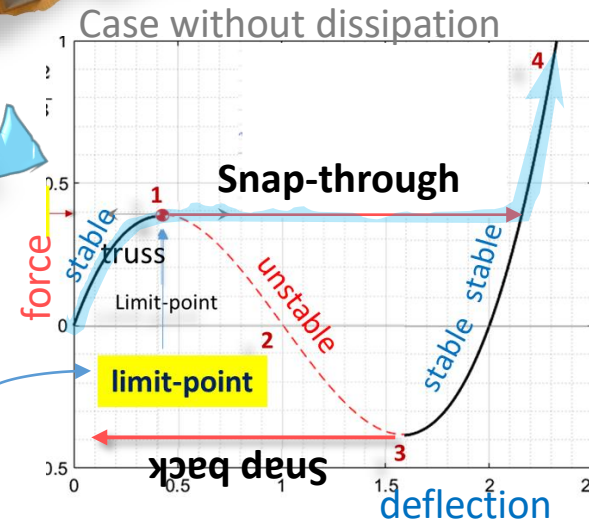
300
250
200
150
100
50
0

F_{10} (N)

Stable

Stable

-20 -15 -10 -5 0 5 10 15 20
y_{10} (mm)

Vertical displacement at node 10

Painovirhe: These are coordinate of the tip point (not displacements)

Geometry of the shallow arch
40
20
0
-20
y_0 or Y_0 (mm)
#10
F
0 50 100 150 200 250 300
x_0 or X_0 (mm)

'snap-through' like motion with resisting friction

There is no critical point visible (in this numerical example, I have put too much dissipation; so friction works and resists at joints. So this numerical example is not 100%-elastic as here)

Case without dissipation

Snap-through

1
truss
Limit-point
2
limit-point
3
4

force
stable
unstable
stable stable
Snap back
deflection

1
0.5
0
-0.5
0 0.5 1 1.5 2 2.5

Fundamental questions

Here the content of this course in four points through questions that will be addressed:

1. can we predict the buckling (critical) load?

2. what happens at the bifurcation (or limit) point?
   (*i.e.*, after the buckling)

3. can we determine the post-critical branches?
   What would be their shape? Nature of stability?

4. what imperfection-sensitive is the structure under study?

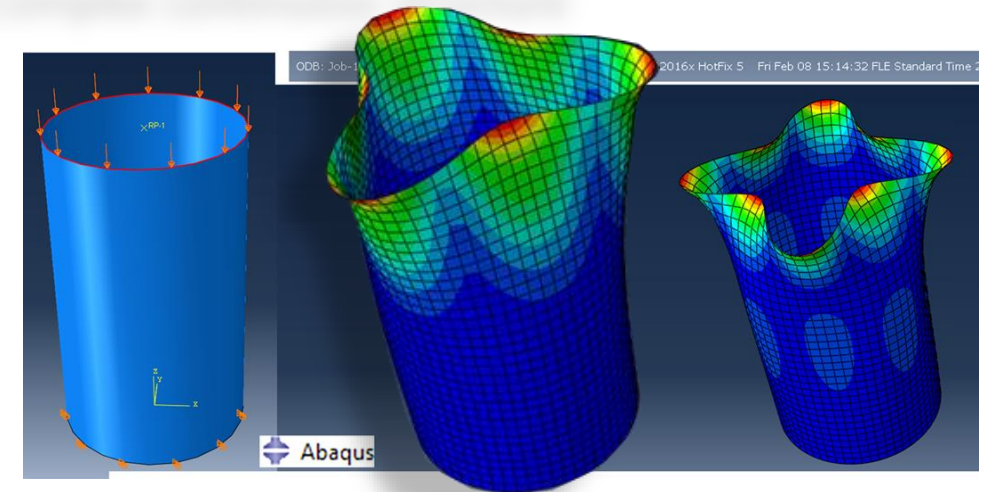## Equilibrium paths for simple rigid bar systems with springs

- Such simplified models (rigid bars-spring discrete systems) allow to grasp the fundaments or invariants of stability
  - ✓ equilibrium paths
  - ✓ critical points (bifurcation, limit-points)
  - ✓ nature of stability on primary and secondary branches
  - ✓ the principle difference between full-non-linear equations describing the post-critical state and their linearized homogeneous versions in the vicinity of critical points

- With rigid bars, all the strain energy concentrate in the springs (makes the formulation tractable by hand)

Rigid bars-spring discrete systems

Rigid bar

Elastic rotational spring
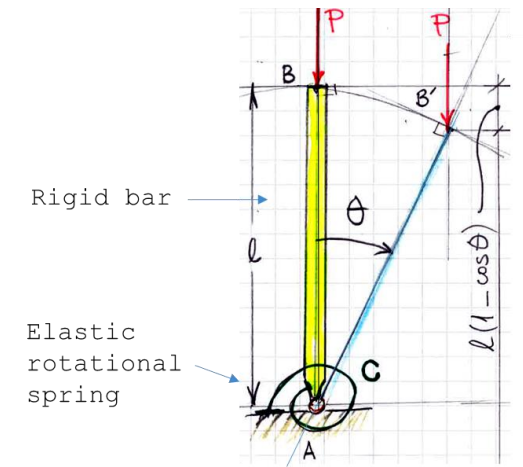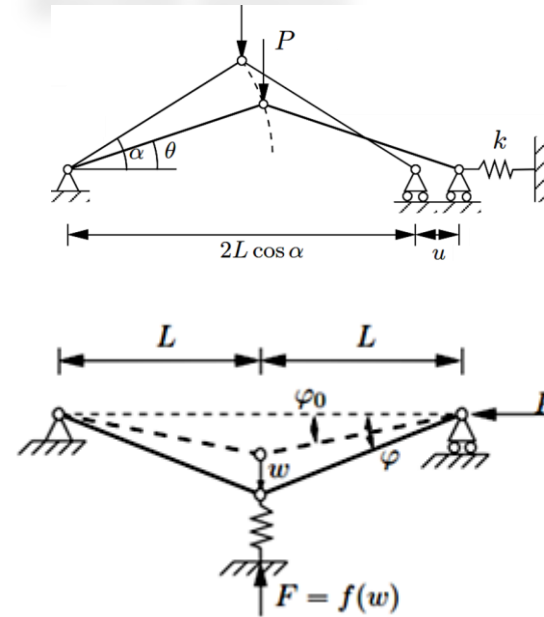
Straight equilibrium configuration (yellow), perturbed (blue).

Complex continuous structure

Equilibrium paths for simple rigid bar systems with springs

Rigid bars-spring discrete systems

Rigid bar

Elastic rotational spring

Straight equilibrium configuration (yellow), perturbed (blue).

$$F = f(w)$$

Why study rigid-springs systems? I want to directly study stability of real structures?

- Such simplified models allow to *apprehend* the *key invariant concepts* without being distracted by irrelevant for the task details rising from accounting for more complexity

- These *concepts are independent on the level of complexity* of the structure

- Do **not mix** *general concepts* and the *examples* used to illustrate them

- The concept is general and the example is particular with mean to passing the concept
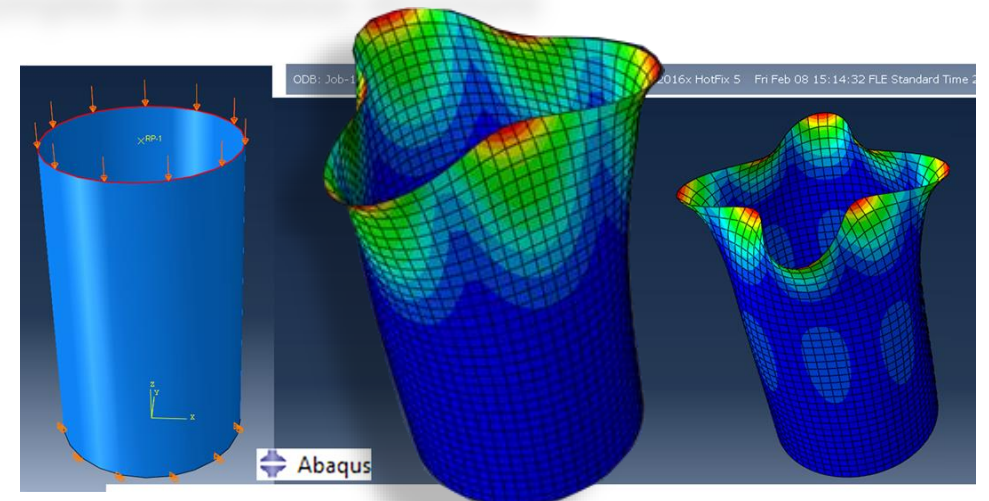
Complex continuous structure

Abaqus

Equilibrium paths for simple rigid bar systems with springs

Equilibrium paths
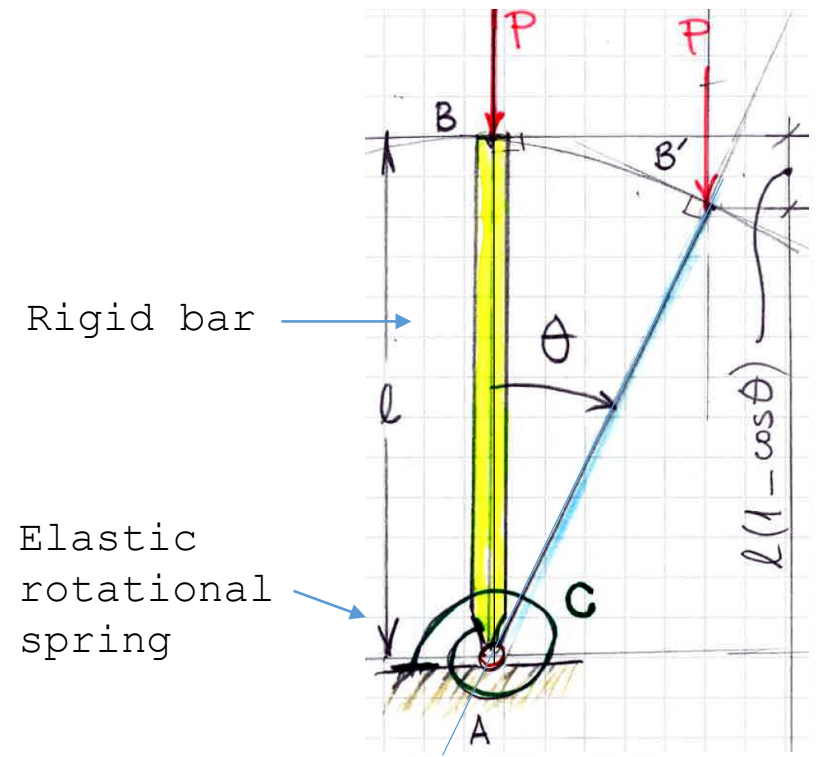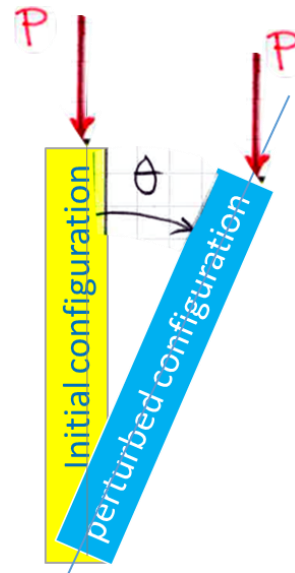
Stable-symmetric bifurcation

Tasks:
- determine all the equilibrium configurations:
  - ✓ Bifurcation (or limit) point
  - ✓ pre-buckling primary or fundamental equilibrium
  - ✓ post-buckling configurations or secondary equilibrium branches

Increment of total potential energy

between **initial** and **perturbed** equilibrium configurations

$$\Pi = \frac{1}{2}c\theta^2 - P\ell(1 - \cos\theta).$$

Initial configuration

perturbed configuration

Rigid bar

Elastic rotational spring

Straight equilibrium configuration (yellow), perturbed (blue).

- **Stable symmetric**
  - ✓ Structures having this type of behavior are always *imperfection insensitive* and have consequently *a reserve of resistance*

- **Unstable symmetric**
  - ✓ This gives *imperfection sensitive structures*

- **Asymmetric or unsymmetrical**
  - ✓ This gives much *more imperfection sensitive structures* than above

Equilibrium paths

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EIv''^2\,\mathrm{d}x - P\int_0^\ell \frac{1}{2}v'^2\,\mathrm{d}x$$

Compare continuous and discrete models

Increment of total potential energy

$$\Pi = \frac{1}{2}c\theta^2 - P\ell(1 - \cos\theta).$$

Note the non-linearity of this problem

Equilibrium
(stationarity)

$$\delta\Pi = 0 \implies \frac{\mathrm{d}\Pi}{\mathrm{d}\theta} \equiv \Pi' = c\theta - P\ell\sin\theta = 0.$$
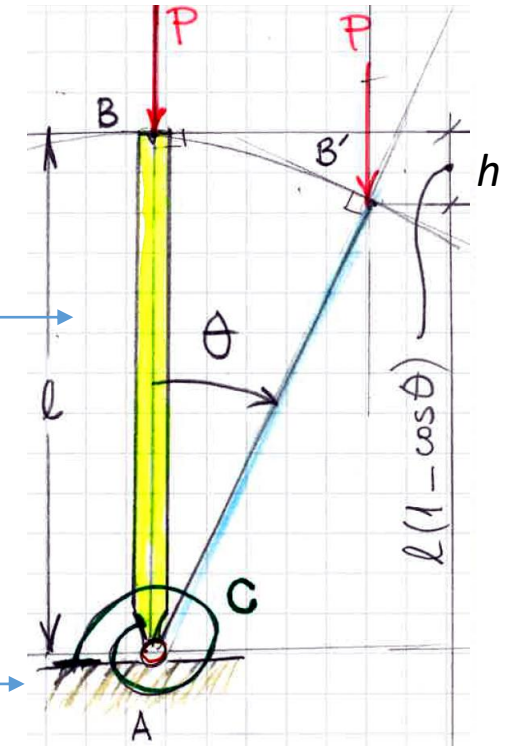
solutions

$$\theta = 0 \quad \text{or} \quad P = \frac{c}{\ell}\cdot\frac{\theta}{\sin\theta}, \quad \theta \neq 0.$$

fundamental equilibrium branch
(= pre-buckling branch)

post-buckling equilibrium branch

Rigid bar

Elastic
rotational
spring

h

θ

$\ell(1 - \cos\theta)$

Straight equilib-
rium configuration
(yellow), perturbed
(blue).

notation $\quad \dfrac{c}{\ell} \equiv P_{cr}$

Equilibrium paths

Increment of total potential energy

$$\Pi = \frac{1}{2}c\theta^2 - P\ell(1 - \cos\theta).$$

full non-linear equilibrium equation

Note the non-linearity of this problem

Equilibrium (stationarity)

$$\delta\Pi = 0 \implies \frac{d\Pi}{d\theta} \equiv \Pi' = c\theta - P\ell\sin\theta = 0.$$

Equilibrium solutions

$$\theta = 0 \quad \text{or} \quad P = \frac{c}{\ell}\cdot\frac{\theta}{\sin\theta}, \theta \neq 0.$$
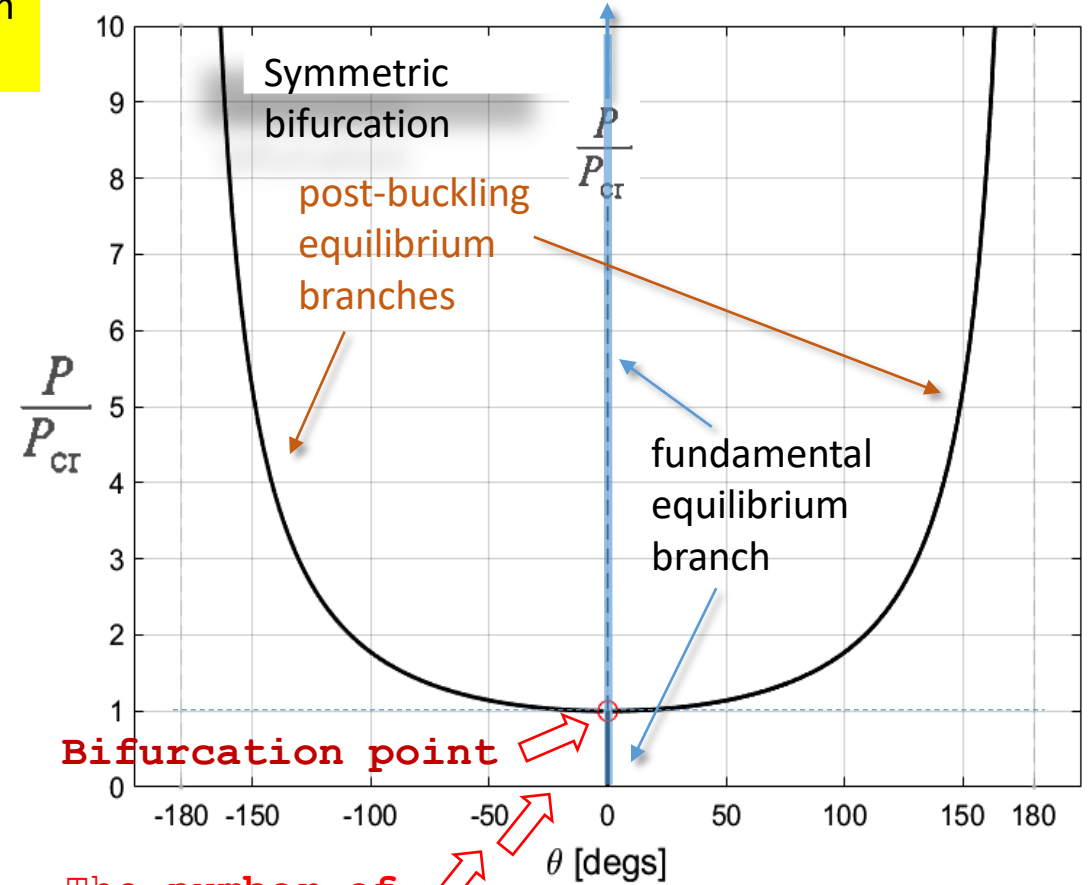
fundamental equilibrium branch
or pre-buckled branch

post-buckling equilibrium branch

The buckling load (bifurcation point) and the nature of stability of all branches will be discussed in next slides …

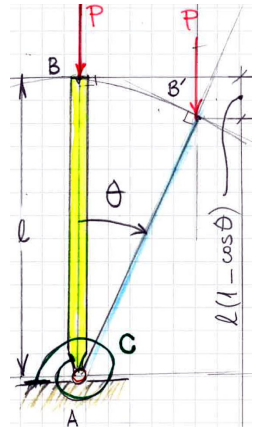$$\text{sign } \Pi''|_{\theta=0}? \qquad \text{sign } \Pi''|_{P=c/\ell\cdot\theta/\sin\theta}?$$

Symmetric bifurcation

post-buckling equilibrium branches

fundamental equilibrium branch

Bifurcation point

The **number of solutions changes** from one (primary) to two (secondary branch)

notation $\dfrac{c}{\ell} \equiv P_{cr}$

Stability of Equilibrium
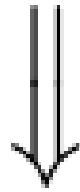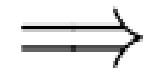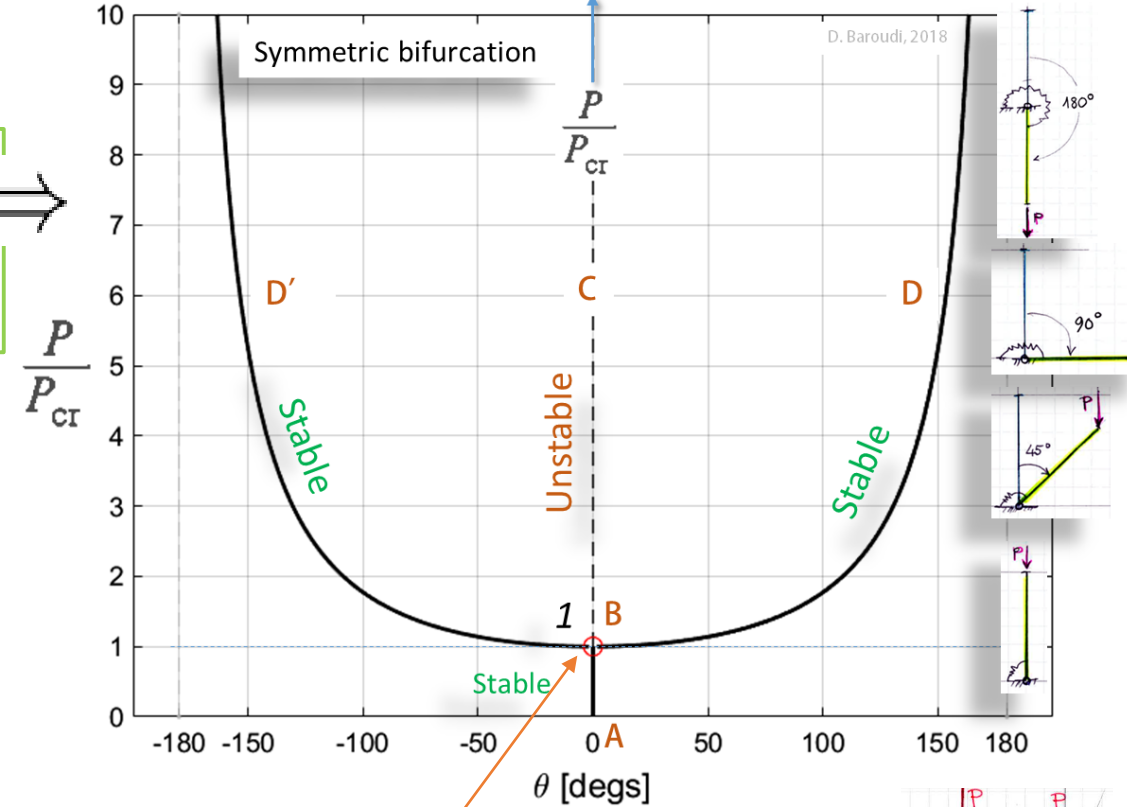
$$\text{sign } \Pi''|_{\theta=0}?$$

$$\text{sign } \Pi''|_{P=c/\ell\cdot\theta/\sin\theta}?$$

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases} \Longrightarrow$$

Stability of equilibrium

Let's study the sign of the second derivative $\Pi''(\theta) = c - P\ell\cos\theta$ (Eq. 1.31)
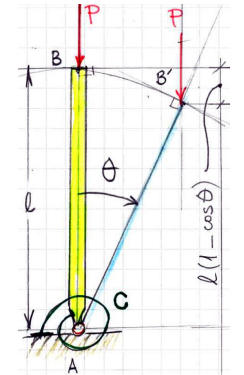
- **trivial case:** $\theta = 0$ (branches AB and BC)
  1.1) $\Pi''(\theta = 0) = c - P\ell = 0 \implies P_{cr} = c/\ell$ (we have a *bifurcation point* at B since the second derivative Π'' changes sign) (P_{cr} is called the **buckling load**).

- 1.2) $AB, \theta = 0$: $\Pi''(0) = c - P\ell > 0 \implies P < P_{cr}$ (AB *stable*)

- 1.3) $BC, \theta = 0$: $\Pi''(0) = c - P\ell < 0 \implies P > P_{cr}$ (BC *unstable*)

- **post-bucked case:** $\theta \neq 0$ (branches BD and BD')
  $\rightarrow$ so, the second derivative should be evaluated on the secondary branch $P = \frac{c}{\ell} \cdot \theta/\sin\theta$:

  Eq. (1.31) $\rightarrow \Pi''(\theta) = c - P\ell\cos\theta > 0?$ $\implies$ the sign of $\Pi'' = c - \ell\cos\theta = c(1 - \theta\cos\theta/\sin\theta) > 0$, where $c > 0$, so we have always $\Pi'' > 0$ for $\theta[\in 0, \pi]$ (see Fig. 1.31)

  So $\rightarrow BD$ and BD' *stable*). $\qquad \Pi'' > 0, \quad \text{stable,}$

Equilibrium paths
Stable-symmetric bifurcation

Symmetric bifurcation
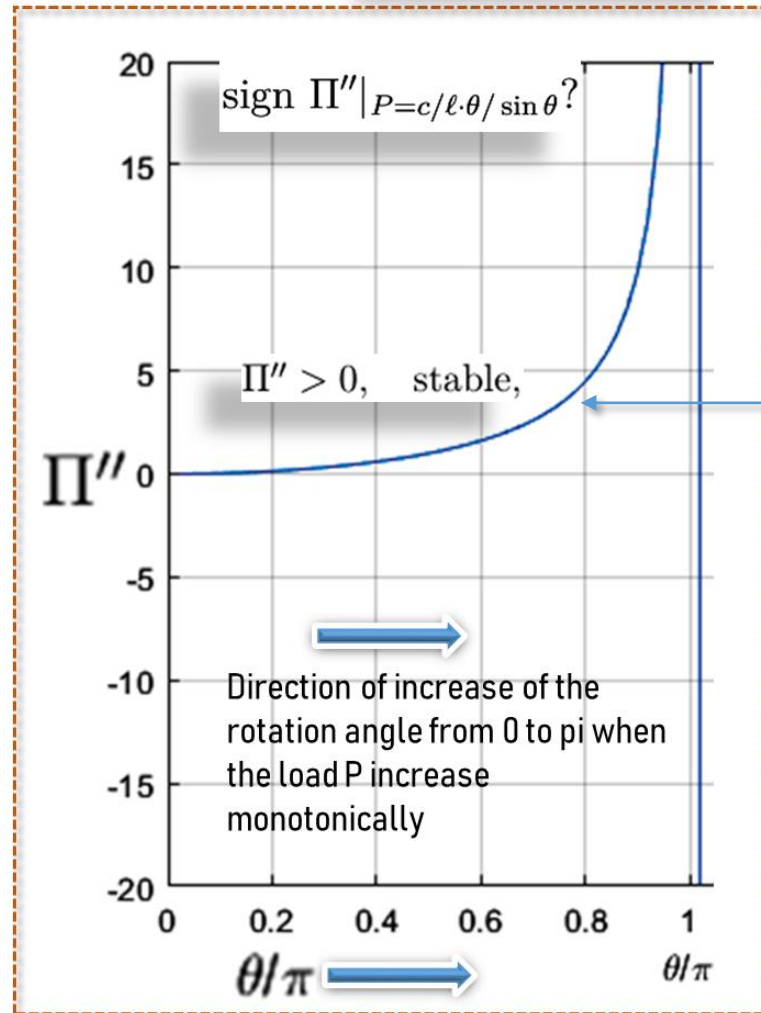
D. Baroudi, 2018

Bifurcation point

$$\text{notation} \quad \frac{c}{\ell} \equiv P_{cr}$$
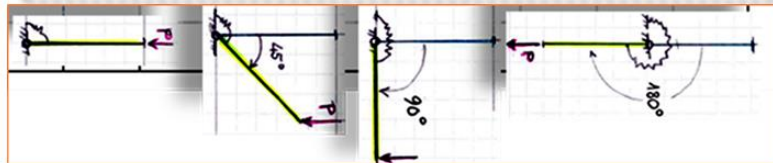
$$\Pi''(\theta) = c - P\ell\cos\theta > 0$$

Stability of Equilibrium on the
secondary path: $\mathrm{sign}\ \Pi''|_{P=c/\ell\cdot\theta/\sin\theta}?$

$\Pi'' = c - \ell\cos\theta = c(1 - \theta\cos\theta/\sin\theta) > 0$, where $c > 0$

- **post-bucked case:** $\theta \neq 0$ (branches BD and BD')
  $\rightarrow$ so, the second derivative should be evaluated on the secondary
  branch $P = \frac{c}{\ell} \cdot \theta/\sin\theta$:

  Eq. (1.31) $\rightarrow \Pi''(\theta) = c - P\ell\cos\theta > 0?$ $\implies$ the sign of $\Pi'' = c - \ell\cos\theta = c(1 - \theta\cos\theta/\sin\theta) > 0$, where $c > 0$, so we have always $\Pi'' > 0$ for $\theta[\in 0, \pi]$ (see Fig. 1.31)

  So $\rightarrow BD$ and BD' stable).

In-figure text:
$\mathrm{sign}\ \Pi''|_{P=c/\ell\cdot\theta/\sin\theta}?$

$\Pi'' > 0,$ stable,

Π''

Direction of increase of the
rotation angle from 0 to pi when
the load P increase
monotonically

θ/π θ/π

Stability of Equilibrium

Stability of Equilibrium on the secondary path: $\operatorname{sign} \Pi''|_{P=c/\ell \cdot \theta / \sin \theta}$?

Equilibrium paths
Stable-symmetric bifurcation

$\Pi'' > 0,$ stable,

$\Pi'' < 0,$ unstable

Direction of increase of the rotation angle from 0 to pi

Symmetric bifurcation

D. Baroudi, 2018

$\dfrac{P}{P_{\mathrm{cr}}}$

D′   C   D

Stable   Unstable   Stable

1 B

Stable

θ [degs]

Bifurcation point

notation $\quad \dfrac{c}{\ell} \equiv P_{cr}$

Equilibrium paths for simple rigid bar systems with springs

Equilibrium paths
Stable-symmetric bifurcation
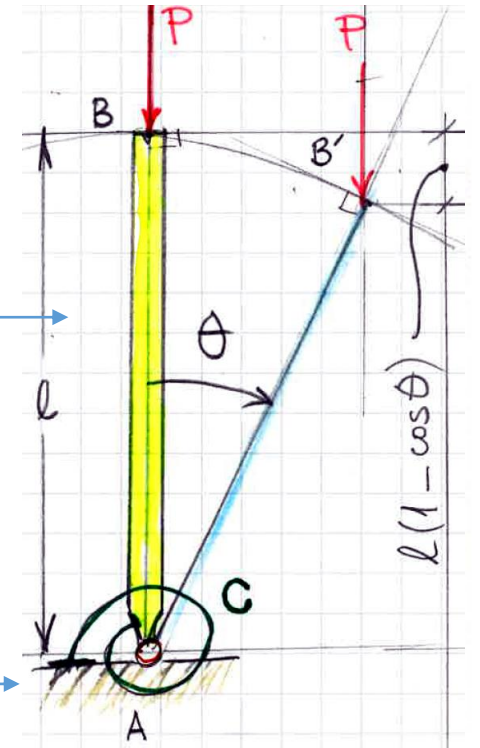
Increment of total potential energy
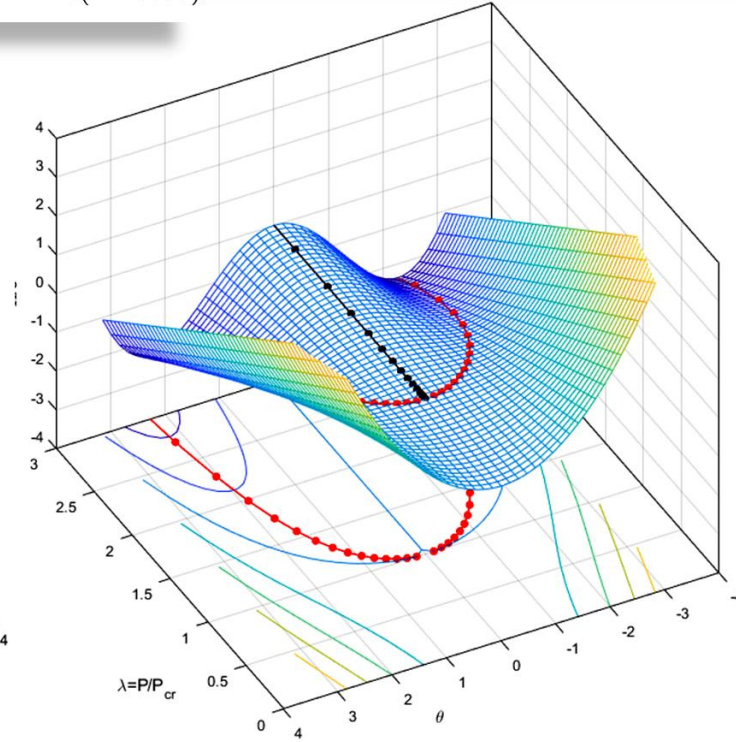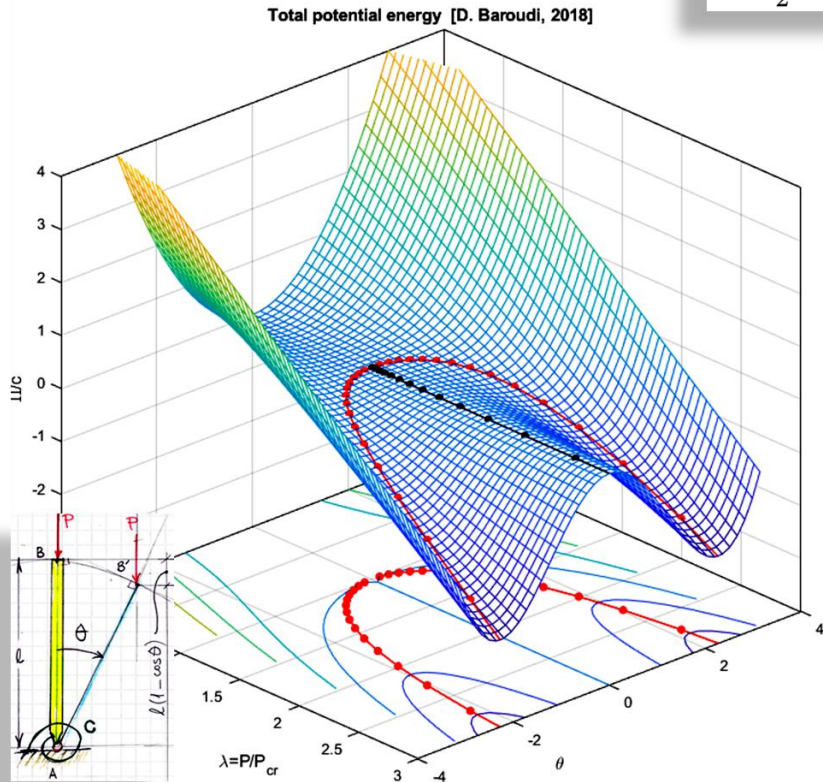
$$\Pi = \frac{1}{2}c\theta^2 - P\ell(1 - \cos\theta).$$

Rigid bar ⟶

Elastic rotational spring ⟶

Straight equilibrium configuration (yellow), perturbed (blue).

$$\Pi = \frac{1}{2}c\theta^2 - P\ell(1 - \cos\theta)$$

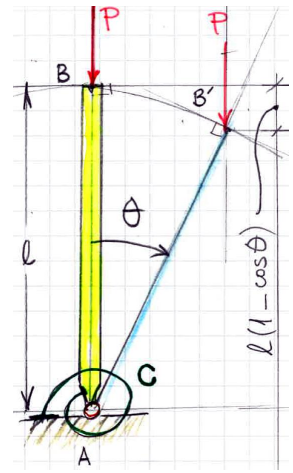Total potential energy [D. Baroudi, 2018]

Linearised model   = Linear buckling analysis

- Expanding total potential energy in Taylor's series around the neutral equilibrium position $\theta = 0$ and keep up-to quadratic terms

? What is the fundamental difference as regarded to the full **non-linear analysis** performed just above where **all equilibrium branches were** completely **determined**.

$$\Pi(\theta; P) = \frac{1}{2}c\theta^2 - P\ell(1 - \cos\theta)$$

$$\approx \frac{1}{2}c\theta^2 - P\ell \cdot \frac{\theta^2}{2}$$

- **E**quilibrium:

$$\delta(\Delta\Pi) = 0 \implies$$

stability loss problem:

$$\boxed{\Pi' = c\theta - P\ell\theta = (c - P\ell) \cdot \theta = 0}$$
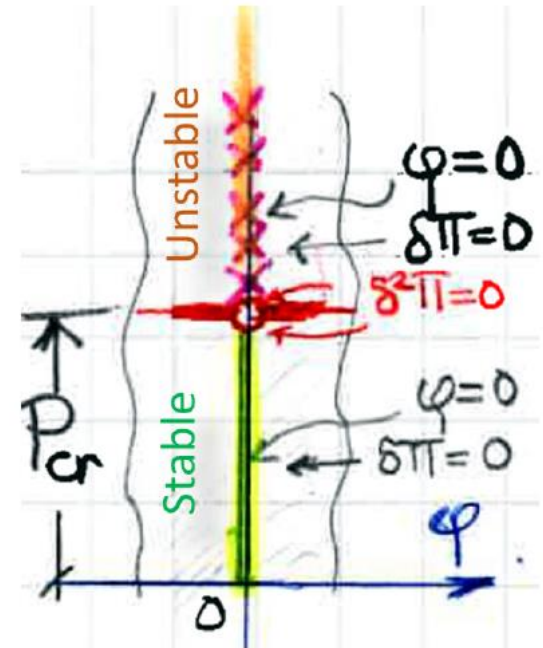
- **C**riticality:

$$\Pi'' = c - P\ell = 0.$$

This is a set of *homogeneous linear equations* which forms an **EIGEN-VALUE problem**

- **no buckling**: $\theta = 0$, is a solution (trivial initial straight $A - B - C$)

- **buckling**: $\theta \neq 0 \implies c - P\ell = 0 \implies P_{cr} = c/\ell$ (buckling load)

$$\implies$$

Linear buckling analysis provides ONLY:
- the **buckling load**
- **buckling modes** up-to a multiplicative coefficient

Linearised criterion

Linearised model

= Linear buckling analysis

- **E**quilibrium: $\boxed{\Pi' = c\theta - P\ell\theta = (c - P\ell) \cdot \theta = 0}$

$$\Pi(\theta; P) = \frac{1}{2}c\theta^2 - P\ell(1 - \cos\theta)$$

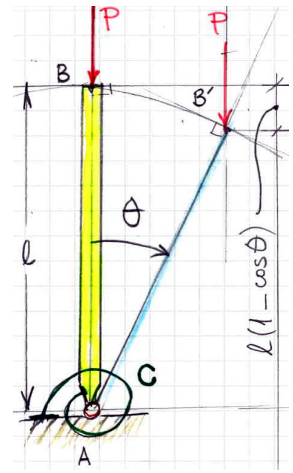$$\approx \frac{1}{2}c\theta^2 - P\ell \cdot \frac{\theta^2}{2}$$

- **C**riticality: $\Pi'' = c - P\ell = 0.$

Linear buckling analysis provides ONLY:
- the **buckling load**
- **buckling modes** up-to a multiplicative coefficient

$\Longrightarrow$

- **What is the nature of stability at the bifurcation point?**
- **Can we answer with the linearized model?** **Answer: NO**

$$\Pi''(\theta = 0; P = P_{cr}) = c - P_{cr}\ell = c - \frac{c}{\ell} \cdot \ell = 0.$$

One may wrongly or too fast, conclude that the equilibrium is *indifferent*. However, this is not true and this result is an artefact of the linearisation. We should take higher order[41] (than quadratic) terms in the expansion of $W_{ext.}(\theta; P) = -P\ell(1 - \cos\theta)$ with respect to θ, in order to decide (the sign) of the stability at the bifurcation point. For instance, the expansion $\cos\theta \approx 1 - \theta^2/2 + \theta^4/4!$ can solve the sign problem. This, physically, means that we use a *asymptotic expansion of non-linear equations* and capture the moderate rotations and displacements around $\theta = 0.$[42]

$\nearrow$

$\Longrightarrow$
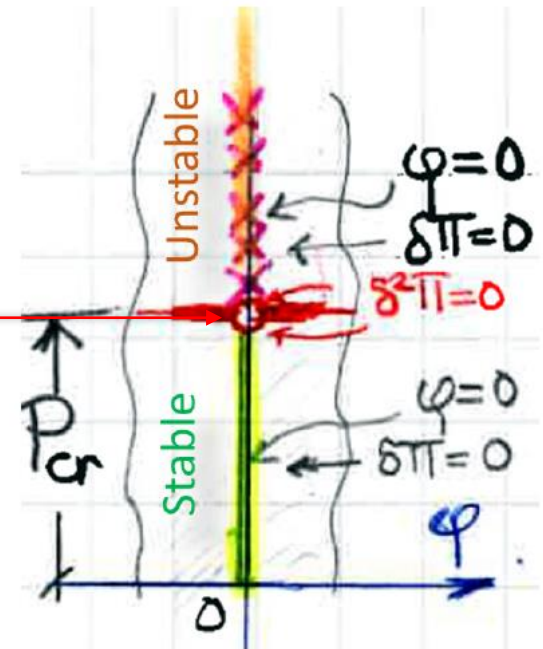
Linearised criterion

Post-buckling analysis - asymptotic non-linear approach

- **N.B.** *Linearized buckling analysis* cannot provide information about the post-buckling behavior

- So, needs *post-buckling analysis* **(full non-linear formulation)**

$$\Pi(\theta; P) = \frac{1}{2}c\theta^2 - P\ell(1 - \cos\theta)$$

$\Longrightarrow$ Asymptotic post-buckling analysis

Expand up-to fourth-order terms the total potential energy

$$\cos\theta \approx 1 - \theta^2/2 + \theta^4/4!$$

$$W_{ext.}(\theta; P) = -P\ell(1 - \cos\theta)$$

$$\Longrightarrow \quad \Pi = 1/2c\theta^2 - P\ell\theta^2/2 + P\ell\theta^4/4!$$

$$d\Pi/d\theta = (c - P\ell)\theta + P\ell\theta^3/3! = 0.$$

$$\theta = 0, \forall P$$

pre-buckled branch

$$P = \frac{P_{cr}}{1 - \theta^2/3!}, \quad P_{cr} = c/\ell, \ \theta \neq 0.$$

post-buckling branch

Note that now, for loading values greater than the buckling load, we obtain the corresponding value for the rotations
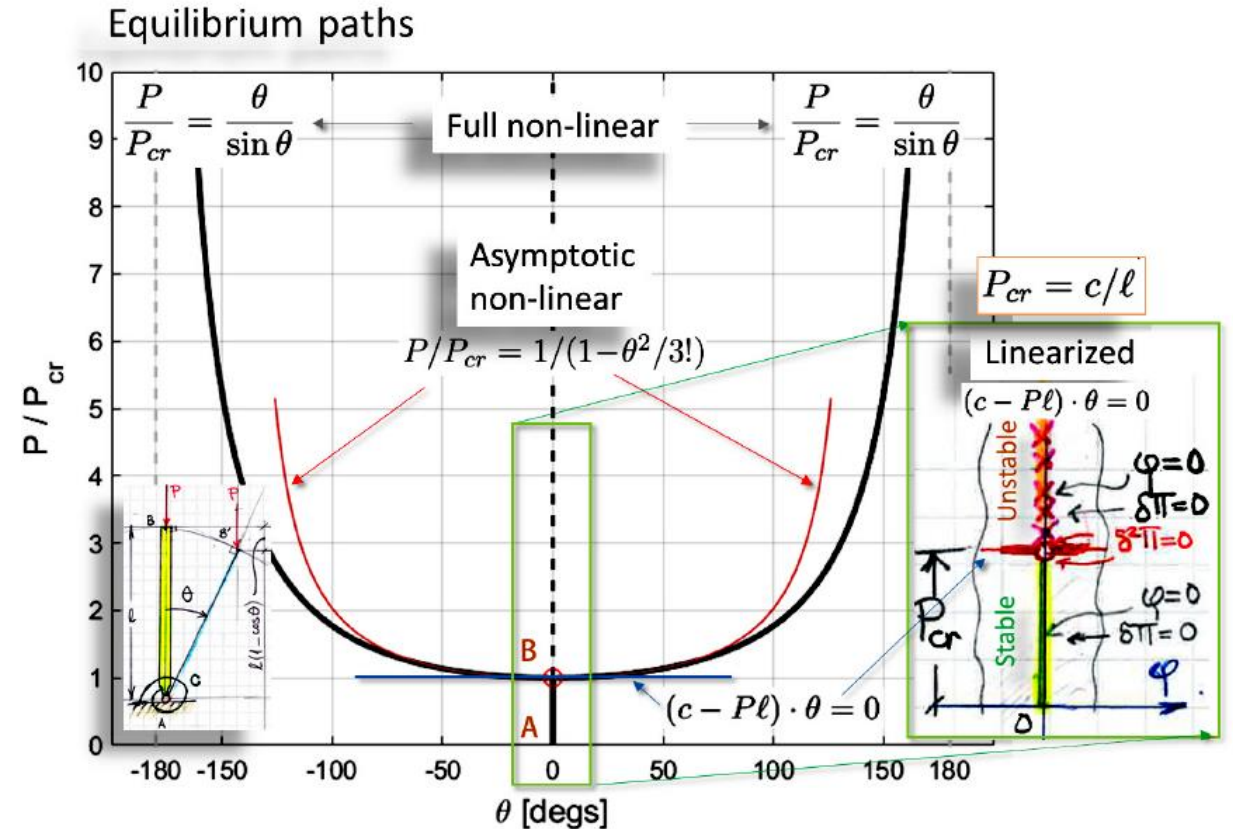
Equilibrium paths

Figure 3.47: Equilibrium paths. Full non-linear model (black), asymptotic non-linear model (orange) and the linearised model (right side).

Fundaments of Stability Theory: Example on imperfections

§34.3 THE IMPERFECT HINGED CANTILEVER

The imperfection parameter is the initial tilt angle

FIGURE 34.1. The imp... | tilt angle ε.

NB. we can have also such 'bassins' if we can access them

One key difficulty is that

We begin the study of the effect of imperfections through several simple yet instructive one-degree-of-freedom (DOF) examples.

§34.3. The Imperfect Hinged Cantilever

We take up again the critical-point analysis of the hinged cantilever already studied in the previous Chapter. But we assume that this system is *geometrically imperfect* in the sense that the rotational spring is unstrained when the rigid bar "tilts" by a small angle ε with the vertical. By varying ε we effectively generate a *family* of imperfect systems that degenerate to the perfect system when ε → 0.

Denoting again the total rotation from the vertical by θ as shown in Figure 34.1, the strain energy of the imperfect system can be written

$$U(\theta, \epsilon) = \tfrac{1}{2}k(\theta - \epsilon)^2. \tag{34.1}$$

The potential energy of the imperfect system is

$$\Pi(\theta, \lambda, \epsilon) = U - W = \tfrac{1}{2}k(\theta - \epsilon)^2 - fL(1 - \cos\theta) = k\left[\tfrac{1}{2}(\theta - \epsilon)^2 - \lambda(1 - \cos\theta)\right]. \tag{34.2}$$

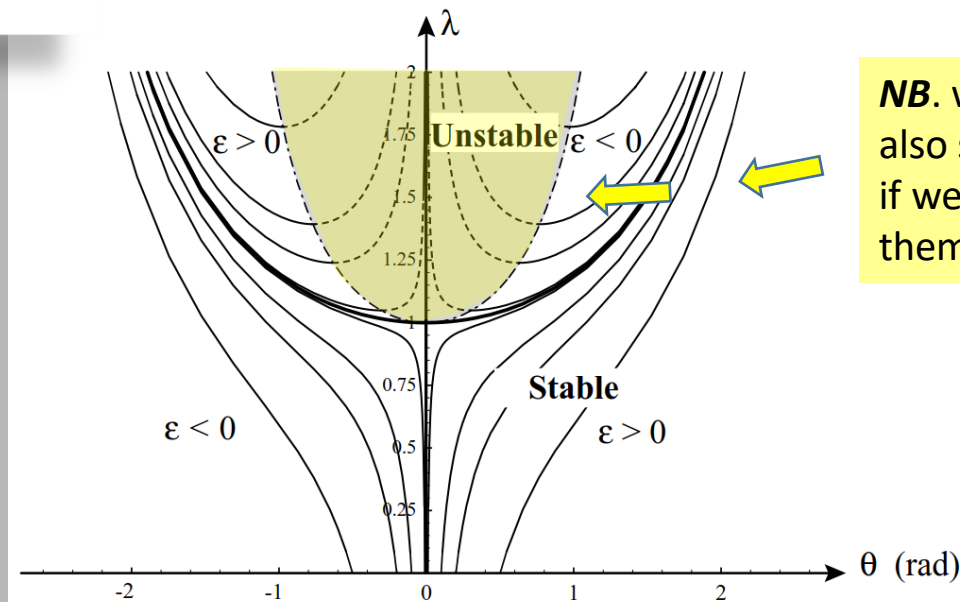in which as before we take $\lambda = fL/k$ as dimensionless control parameter.

§34.3.1. Equilibrium Analysis

The equilibrium equation in terms of the angle θ as degree of freedom is

$$r = \frac{\partial\Pi}{\partial\theta} = k(\theta - \epsilon - \lambda\sin\theta) = 0. \tag{34.3}$$

Therefore, the equilibrium path equation of an imperfect system is

$$\lambda = \frac{\theta - \epsilon}{\sin\theta}. \tag{34.4}$$

$$K = \frac{\partial r}{\partial\theta} = k(1 - \lambda\cos\theta), \qquad q = \frac{\partial r}{\partial\theta} = k\sin\theta. \tag{34.6}$$

We have stability if $K > 0$, that is

$$1 - \lambda\cos\theta > 0, \tag{34.7}$$

and instability if $K < 0$, that is

$$1 - \lambda\cos\theta < 0. \tag{34.8}$$

Critical points are characterized by $K(\lambda_{cr}) = 1 - \lambda_{cr}\cos\theta = 0$, or

$$\lambda_{cr} = \frac{1}{\cos\theta}. \tag{34.9}$$

On equating this value of λ with that given by the equilibrium solution (32.4) we obtain

$$\theta - \epsilon = \tan\theta. \tag{34.10}$$

This relation characterizes the *locus of critical points* as ε is varied. It is not difficult to show that these critical points are limit points if ε ≠ 0 (imperfect systems) and a bifurcation point if and only if ε = 0 (perfect system).

§34.3.3. Discussion

The response of this family of imperfect systems is displayed in Figure 34.2.

In this Figure heavy lines represent the response of the perfect system whereas light lines represent the responses of imperfect systems for fixed values of ε. Furthermore continuous lines identify stable equilibrium

$$\Pi(\theta, \lambda, \epsilon) = U - W = \tfrac{1}{2}k(\theta - \epsilon)^2 - fL(1 - \cos\theta) = k\left[\tfrac{1}{2}(\theta - \epsilon)^2 - \lambda(1 - \cos\theta)\right],$$

$$\delta\Pi = 0 \Rightarrow : \frac{\partial\Pi}{\partial\theta} = k(\theta - \epsilon - \lambda\sin\theta) = 0. \quad\Rightarrow\quad \lambda = \frac{\theta - \epsilon}{\sin\theta}$$
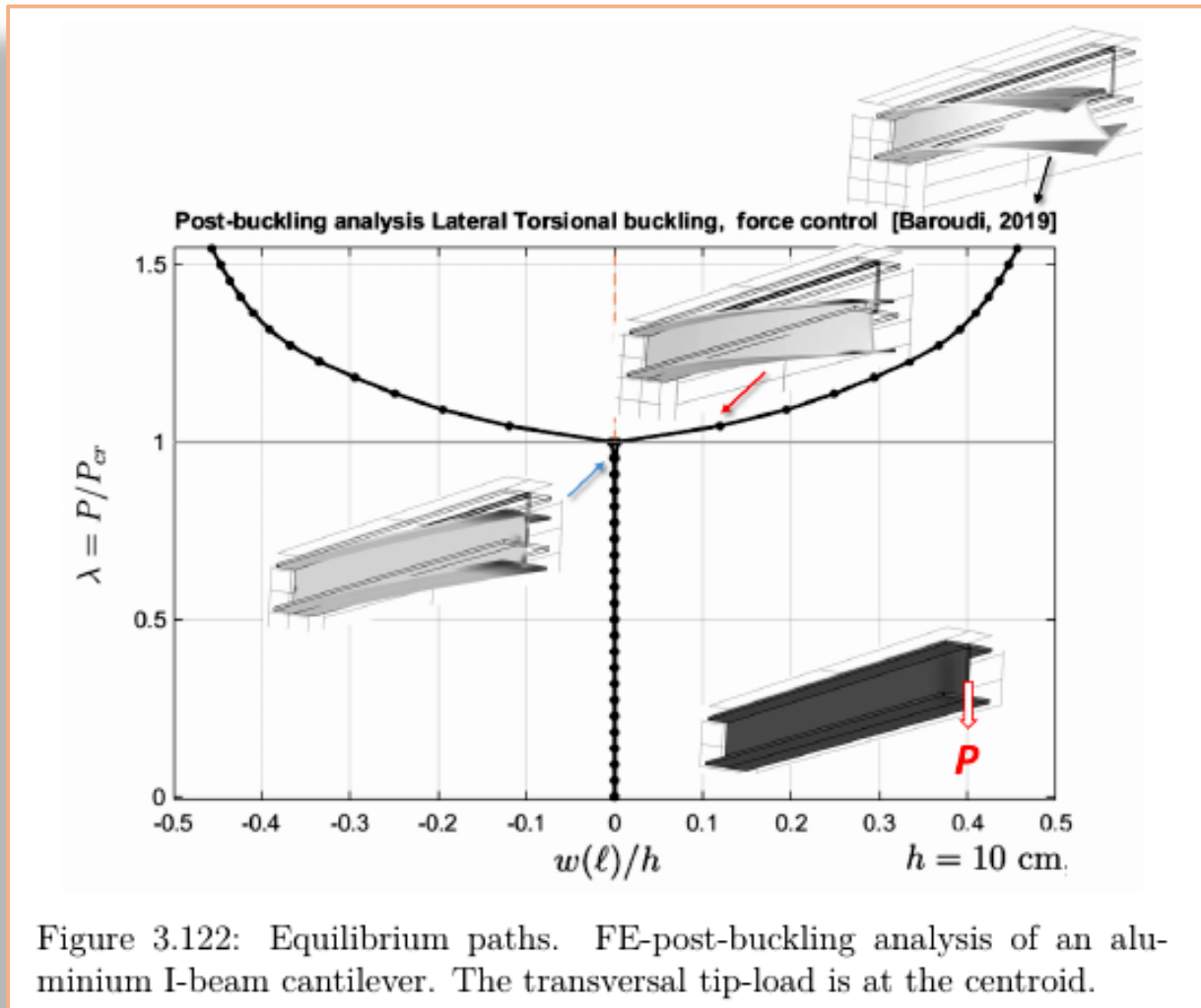
$$\delta^2\Pi(\theta) \Rightarrow \text{Pos.def?} \begin{cases} \dfrac{\partial r}{\partial\theta} = k\sin\theta \\[2mm] \dfrac{\partial r}{\partial\theta} = k(1 - \lambda\cos\theta), \end{cases}$$

REF: https://www.colorado.edu/engineering/cas/courses.d/NFEM.d/Home.html

Example of FE-based post-buckling analysis

For real problems, one should rely on **computational** technology and **experimental** approach

Post-buckling analysis Lateral Torsional buckling, force control [Baroudi, 2019]

Figure 3.122: Equilibrium paths. FE-post-buckling analysis of an aluminium I-beam cantilever. The transversal tip-load is at the centroid.

- **Validate computational models** you are using before making predictions

- There can be physics that computational models do not see (not or cannot be accounted for), which effects can be accessed only **experimentally**
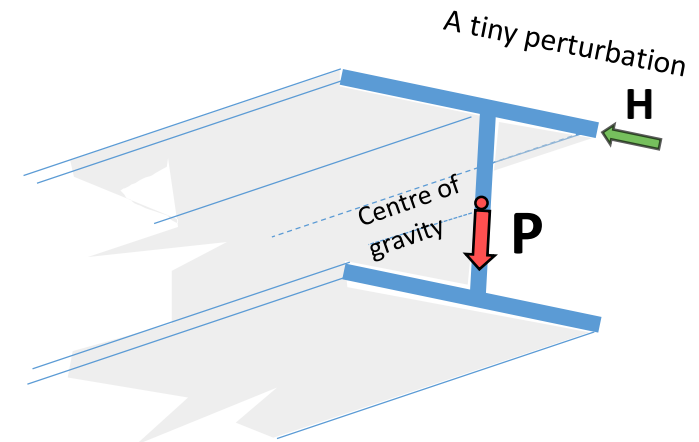
Effect of imperfections

All **real structural** systems are **imperfect**

- ✓ in form,
- ✓ in material properties,
- ✓ in the sense of residual stresses
- ✓ in the way the loads are applied

[12] *It may be safely said that all real structural systems are imperfect in form, imperfect in material properties, imperfect in the sense of residual stresses and imperfect in the way the loads are applied.* **Roorda** (1980)

Effects of imperfections
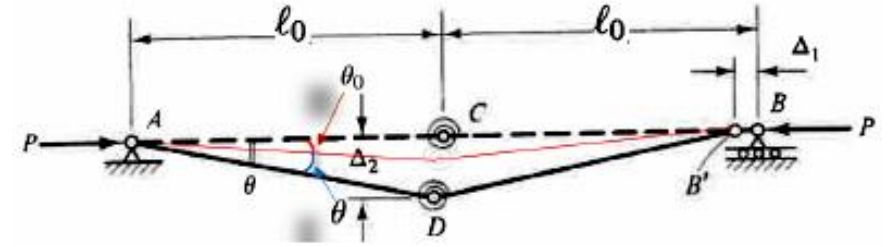
Symmetric Stable bifurcation

- Axially loaded structure with imperfections

  - ✓ imperfection in horizontality (or verticality for a column)

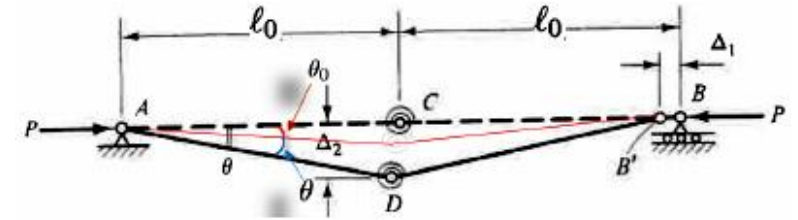$$\Pi = \frac{1}{2}\, c[2(\theta - \theta_0)]^2 - 2P\ell_0(\cos\theta - \cos\theta_0)$$

$$\frac{d\Pi}{d\theta} = 0 \implies \frac{P}{P_{cr}} = \frac{\theta - \theta_0}{\sin\theta}, \qquad P_{cr} = \frac{2c}{a}$$

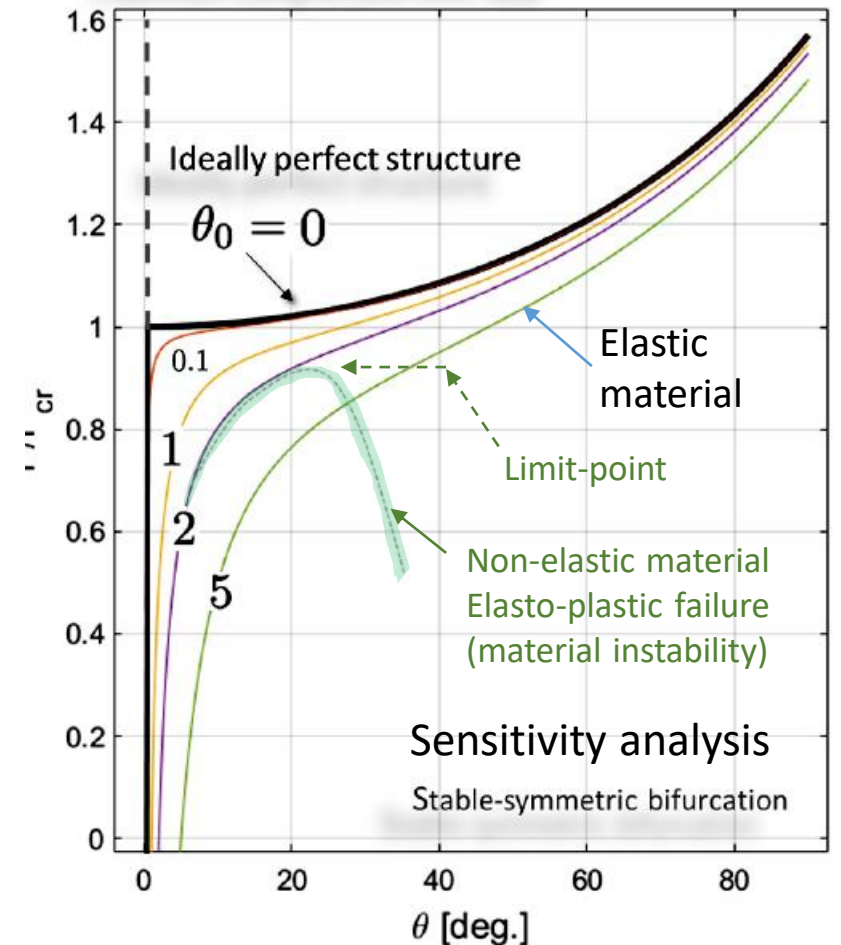- **Stable symmetric**
  - ✓ Structures having this type of behavior are always *imperfection insensitive* and have consequently *a reserve of resistance*

- **Unstable symmetric**
  - ✓ This gives *imperfection sensitive structures*

- **Asymmetric or unsymmetrical**
  - ✓ This gives much *more imperfection sensitive structures* than above

Effects of imperfections

Symmetric Stable bifurcation

- Axially loaded structure with imperfections

    ✓ imperfection in horizontality (or verticality
       for a column)

Effect of initial shape imperfection on the
maximum compressive limit load

$$\Pi = \frac{1}{2} c[2(\theta - \theta_0)]^2 - 2P\ell_0(\cos\theta - \cos\theta_0)$$

$$\frac{d\Pi}{d\theta} = 0 \implies \frac{P}{P_{cr}} = \frac{\theta - \theta_0}{\sin\theta}, \qquad P_{cr} = \frac{2c}{a}$$

⇓

Ideally perfect structure
$\theta_0 = 0$

Elastic material

Limit-point

Non-elastic material
Elasto-plastic failure
(material instability)

Sensitivity analysis

Stable-symmetric bifurcation

θ [deg.]

- **Stable symmetric**
    ✓ Structures having this type of behavior are
       always *imperfection insensitive* and have
       consequently *a reserve of resistance*

Unstable-symmetric bifurcation model

Axially loaded perfect structure:

$$\Pi = \frac{1}{2}kv^2 - P\ell_0(1 - \cos\theta)$$

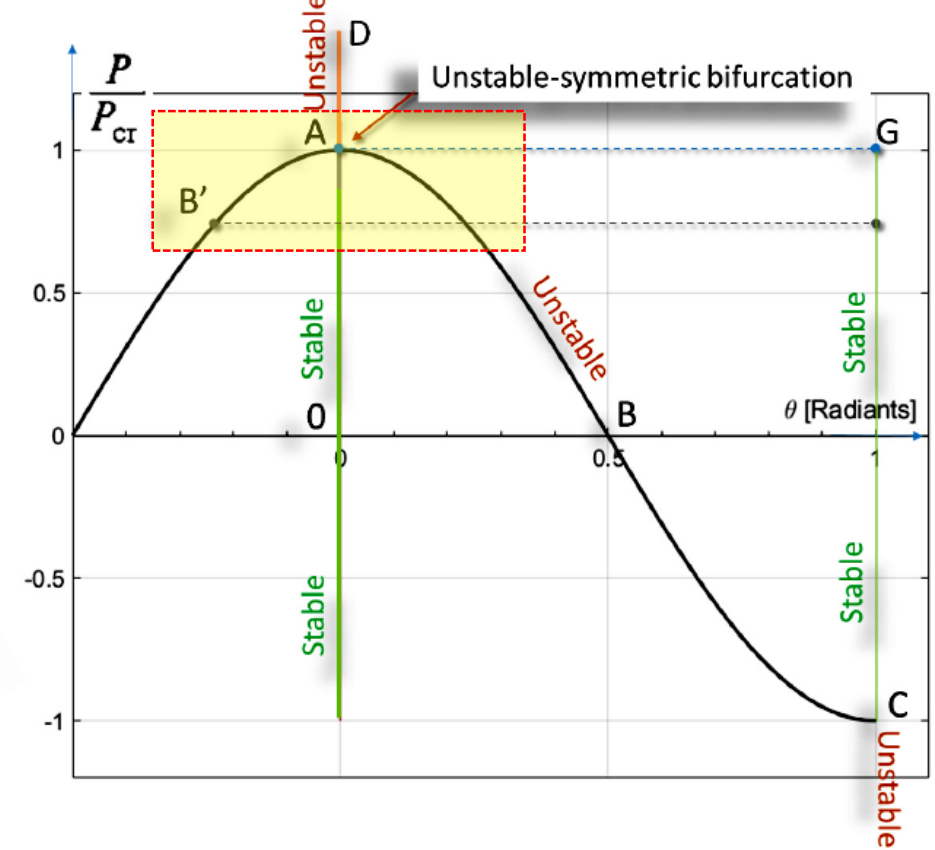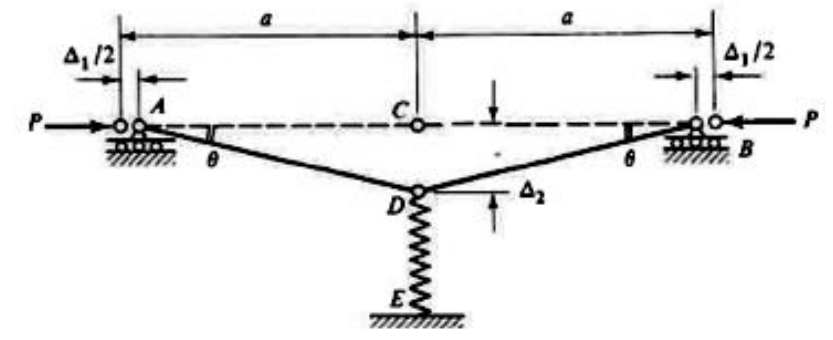$$= \frac{1}{2}k\ell_0^2 \sin^2\theta - 2P\ell_0(1 - \cos\theta)$$

The equilibrium paths

$$\frac{d\Pi}{d\theta} = k\ell_0^2 \sin\theta\cos\theta - 2P\ell_0.\sin\theta = 0$$

$$P = \frac{k\ell_0}{2}\cos\theta = P_{cr}\cos\theta,$$

where $P_{cr} \equiv \frac{k\ell_0}{2}$.

Stability is determined by studying the sign $\quad d(d\Pi/d\theta) = d^2\Pi/d\theta^2$

If the second derivative vanishes, then one should take higher derivatives till non-zero[45] value is achieved, for this case, for the sign of $\delta(\Delta\Pi)$.

Equilibrium path. Unstable-symmetric.

Unstable-symmetric bifurcation model

$$\Pi = \frac{1}{2}k\ell_0^2(\sin\theta - \sin\theta_0)^2 - 2P\ell_0(\cos\theta_0 - \cos\theta).$$
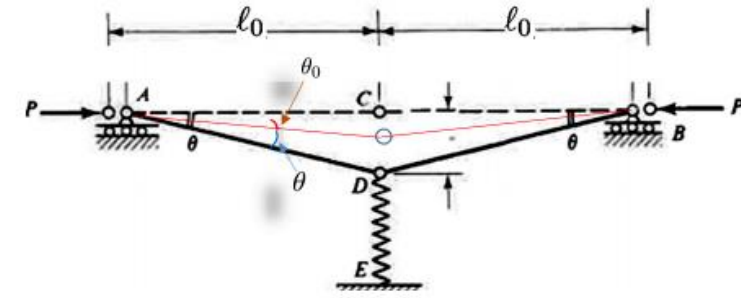

: Axially loaded structure with initial imperfection
The spring stiffness is k.

The equilibrium paths

$$\frac{P}{P_{cr}} = \left(1 - \frac{\sin\theta_0}{\sin\theta}\right)\cos\theta, \quad P_{cr} = k\ell_0/2.$$

⇓

Stability is determined by studying the sign $\mathrm{d}(\mathrm{d}\Pi/\mathrm{d}\theta) = \mathrm{d}^2\Pi/\mathrm{d}\theta^2$

If the second derivative vanishes, then one should take higher derivatives till non-zero[45] value is achieved, for this case, for the sign of $\delta(\Delta\Pi)$.

Unstable-symmetric bifurcation model

Axially loaded structure with imperfections: θ_0

$$\Pi = \frac{1}{2}k\ell_0^2(\sin\theta - \sin\theta_0)^2 - 2P\ell_0(\cos\theta_0 - \cos\theta).$$

The equilibrium paths

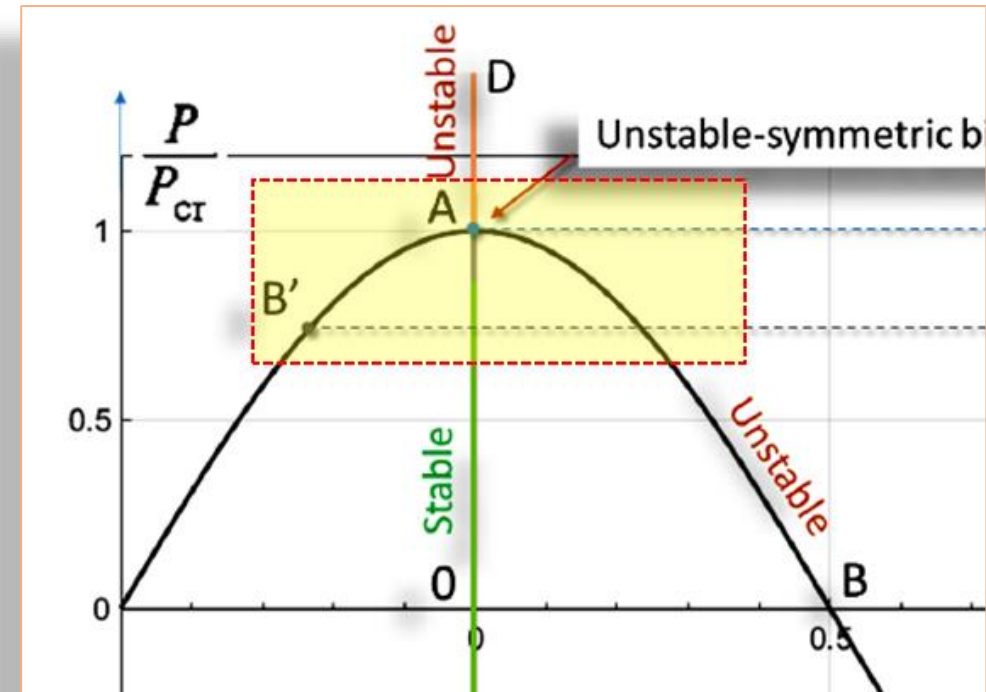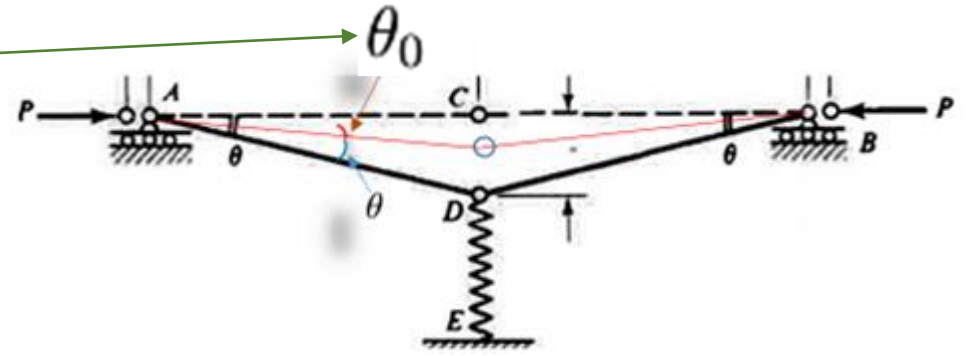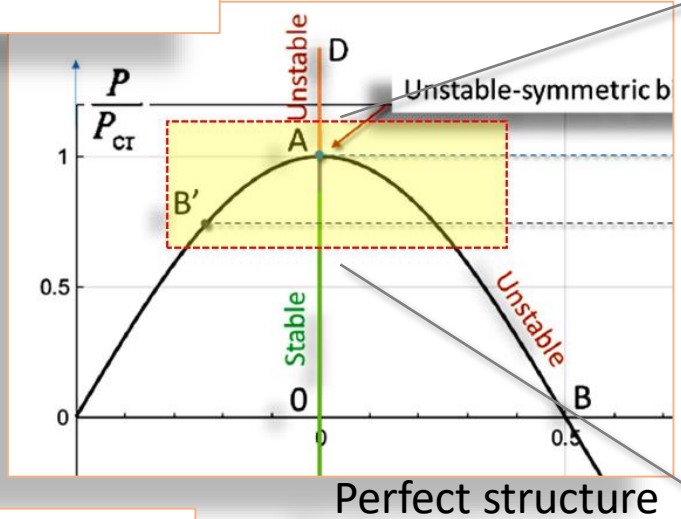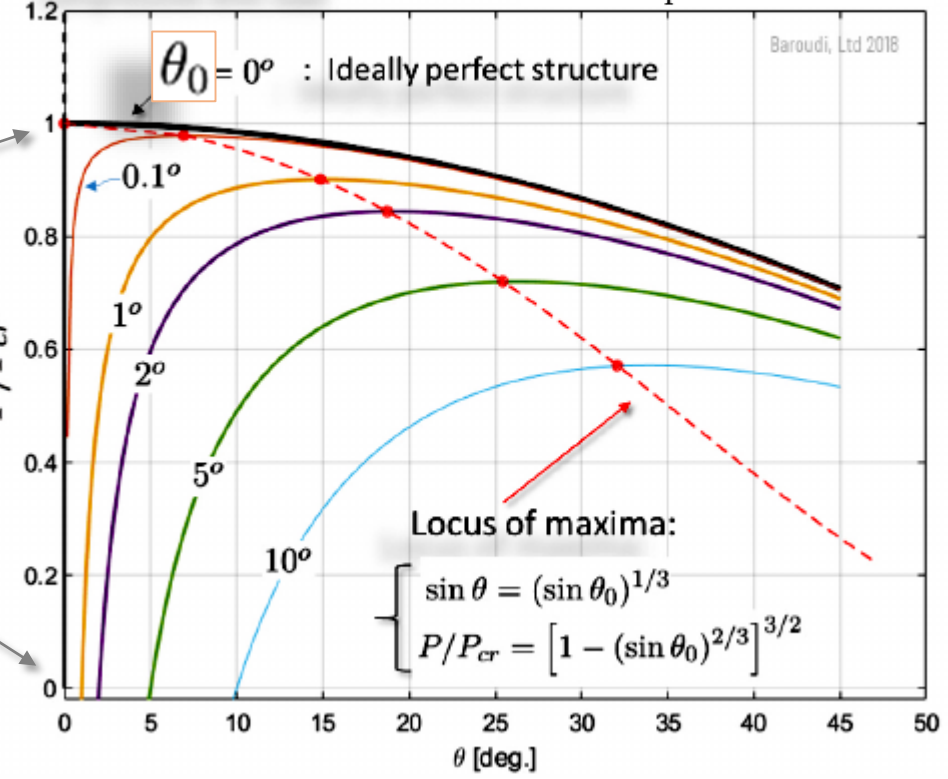$$\frac{P}{P_{cr}} = \left(1 - \frac{\sin\theta_0}{\sin\theta}\right)\cos\theta, \quad P_{cr} = k\ell_0/2.$$

- **Unstable symmetric**
  - ✓ This gives *imperfection sensitive structures*

Perfect structure

Imperfect structure

Effect of initial shape imperfection on the maximum compressive limit load

Load-displacement curve

$\theta_0 = 0^o$: Ideally perfect structure

0.1^o
1°
2°
5°
10°

Locus of maxima:
$$\begin{cases} \sin\theta = (\sin\theta_0)^{1/3} \\ P/P_{cr} = \left[1 - (\sin\theta_0)^{2/3}\right]^{3/2} \end{cases}$$

Baroudi, Ltd 2018

Unstable-symmetric bifurcation model

Axially loaded structure with imperfections: θ_0

- **Unstable symmetric**
  - ✓ T his gives *imperfection sensitive structures*

$\Longrightarrow$

$\Longleftarrow$

Effect of initial shape imperfection on the maximum compressive limit load

$\theta_0 = 0^o$: Ideally perfect structure

Locus of maxima:
$$\begin{cases} \sin \theta = (\sin \theta_0)^{1/3} \\ P/P_{cr} = \left[1 - (\sin \theta_0)^{2/3}\right]^{3/2} \end{cases}$$

Imperfect structure

Maximum axial force reduction with respect to the amplitude of initial imperfection. P_{cr} is the collapse or buckling load of the perfect structure.

Asymmetric bifurcation model

Snap-through model

Simplified example to illustrate the concept of ***limit-load***

- Rigid truss: two straight rigid bars of equal length connected to each other by a hinge. One support allows free lateral movement restrained by a elastic spring k

- Load P is kept increasing quasi-statically and we want to solve the force-displacement curve (equilibrium paths)

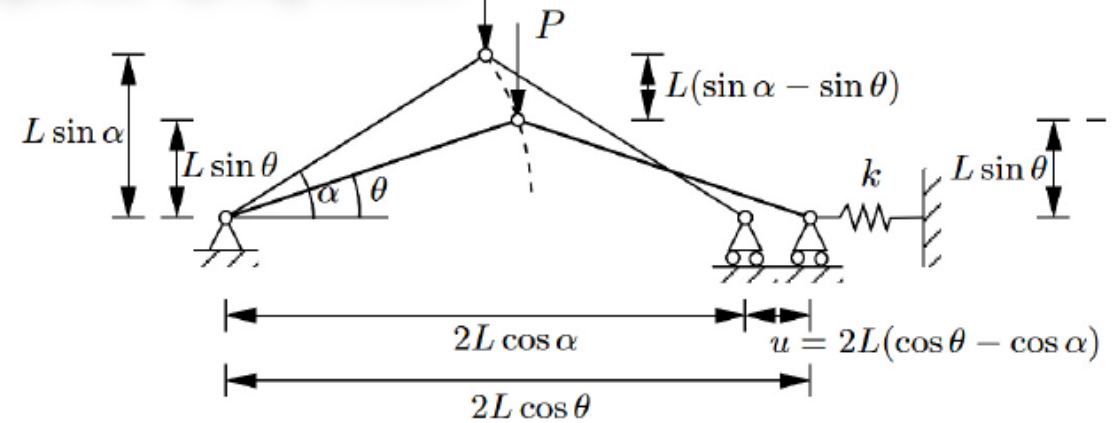$$\Pi = \frac{1}{2}k(2L)^2(\cos\theta - \cos\alpha)^2 - PL(\sin\alpha - \sin\theta).$$

2) Equilibrium paths

The equilibrium $d\Pi/d\theta = 0$

⇓ load-displacement 'curve' = equilibrium paths

$$\frac{P}{4kL} = \sin\theta - \tan\theta\cos\alpha = \sin\theta(1 - \cos\alpha/\cos\theta)$$

Rigid-bar-spring-model

P

$L\sin\alpha$

$L\sin\theta$

$L(\sin\alpha - \sin\theta)$

k

$L\sin\theta$

α θ

$2L\cos\alpha$

$u = 2L(\cos\theta - \cos\alpha)$

$2L\cos\theta$

2) Stability of equilibrium

⇓

$$d^2\Pi/d\theta^2 = 4kL^2(\cos\alpha/\cos\theta - \cos^2\theta)$$

STABLE ⟸ $\Pi'' > 0$, when $\theta < 0$ and $\theta > \theta_B$,

UNSTABLE ⟸ $\Pi'' < 0$, when $-\theta_B < \theta < \theta_B$.

The zeros of the second derivative are

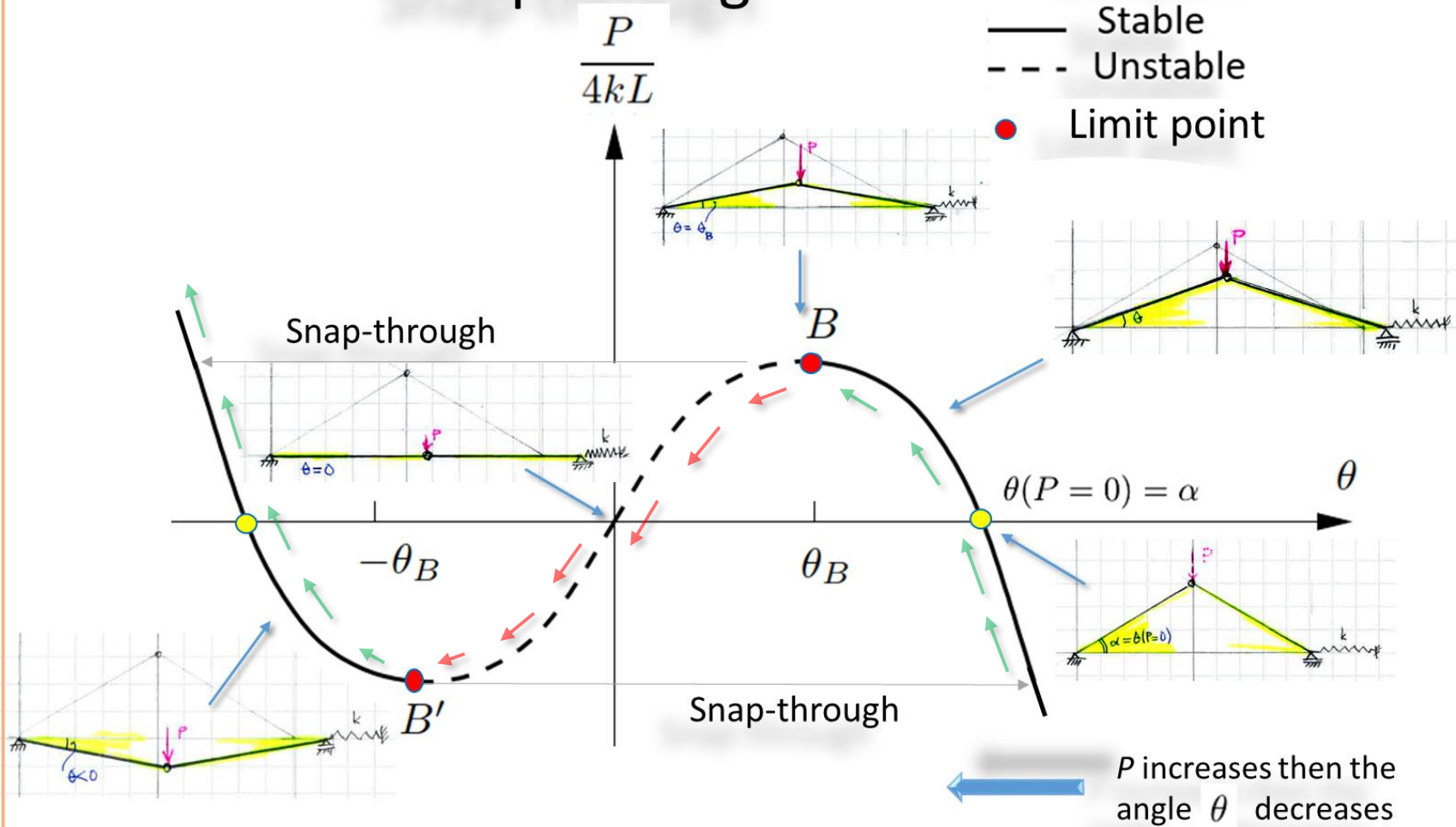$$\theta = \pm\arccos[(\cos\alpha)^{1/3}] \equiv \pm\theta_B$$

To fix the ideas,
for an initial angle $\alpha = 30^o \implies \theta_B = \pm17.6^o$

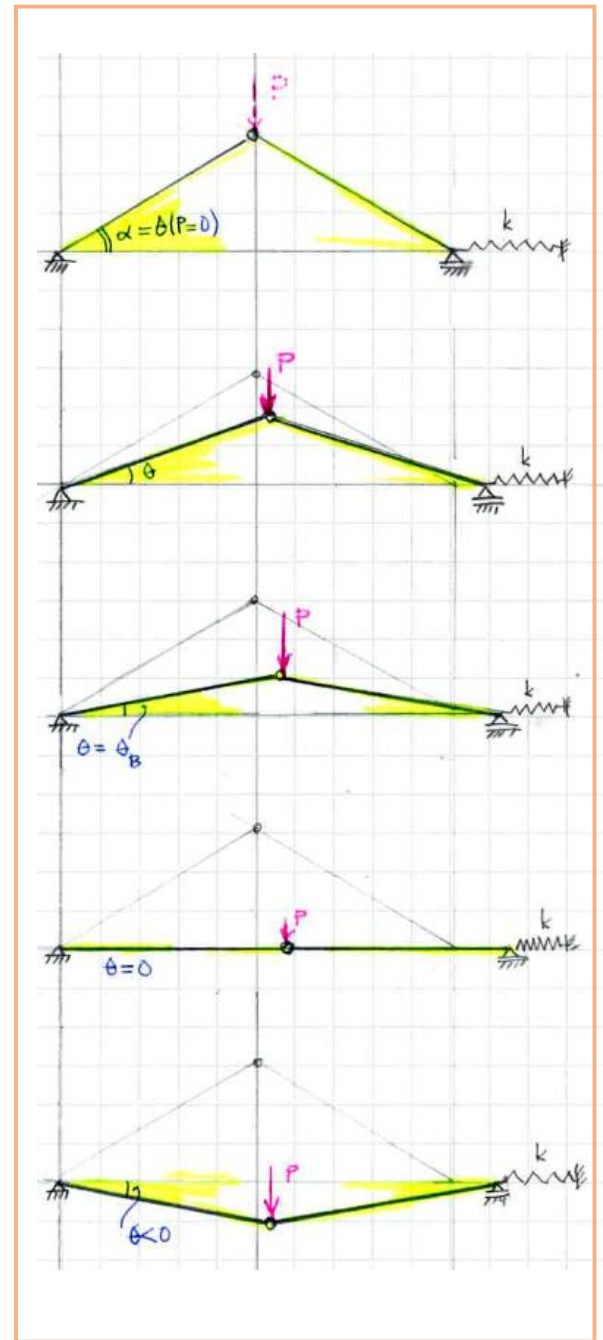$$P = \pm0.028 \cdot 4kL.$$

Snap-through

Stable
Unstable
Limit point

Snap-through

$\dfrac{P}{4kL}$

B

$\theta(P=0) = \alpha$

θ

$-\theta_B$

θ_B

B'

Snap-through

P increases then the angle θ decreases

To fix the ideas,
for an initial angle $\alpha = 30^o$ $\implies$

$\theta_B = \pm 17.6^o$

$P = \pm 0.028 \cdot 4kL$

Asymmetric bifurcation model

Limit-load, raja-kuorma

Snap-through model

load-displacement 'curve'

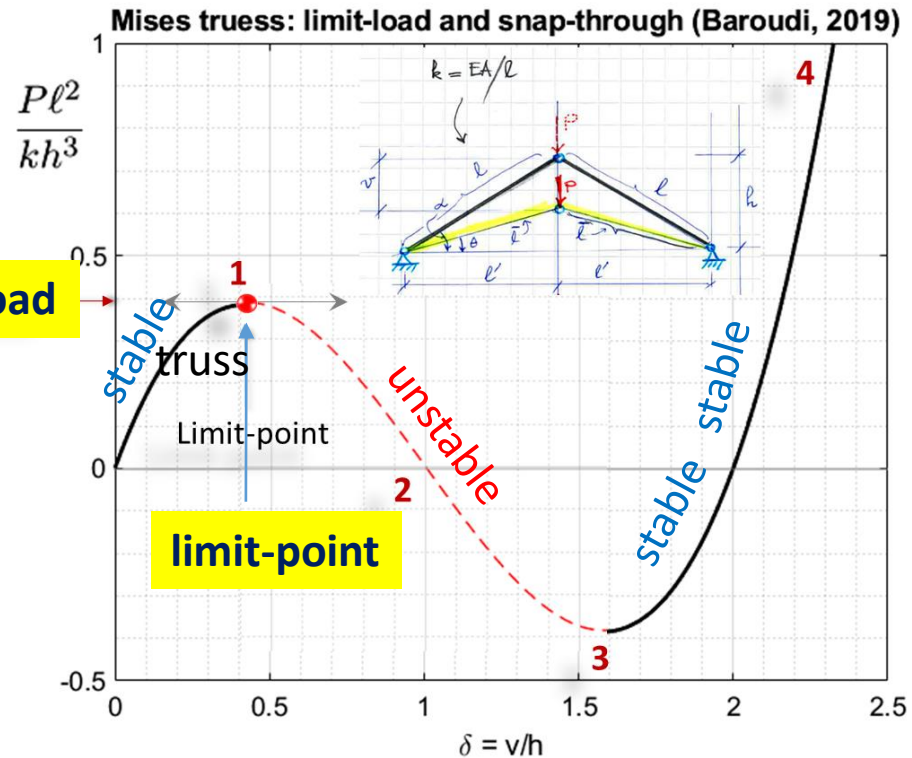$$P\frac{\sqrt{\ell^2 + v^2 - 2vh}}{2k(h-v)} = \ell - \sqrt{\ell^2 + v^2 - 2vh}$$

The following example illustrate the concept of **limit-load**

- **Mises** truss: two straight elastic bars of equal length connected to each other by a hinge to fixed supports allowing free rotations only

- Load P is kept increasing quasi-statically and we want to solve the force-displacement curve (equilibrium paths)

- The truss so shallow that no buckling of separate bars occurs: only snap-through (consequently, only vertical component of the tip-displacement occurs. If truss enough high then one should consider the horizontal component as well.)

Mises truess: limit-load and snap-through (Baroudi, 2019)
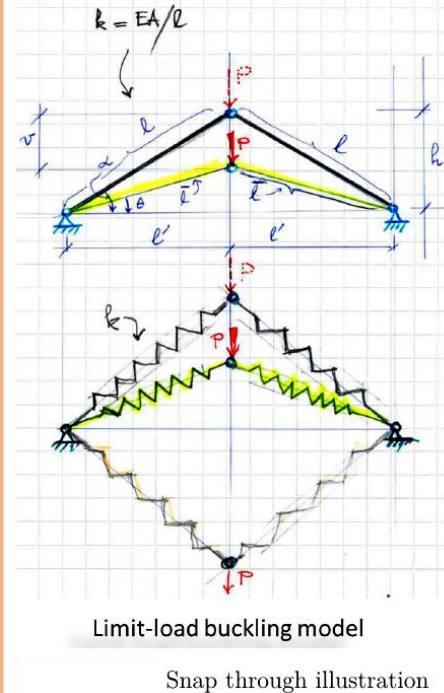
limit-load

limit-point

A shallow truss

load-displacement 'curve'
for *shallow trusses*
$h \ll \ell$

$$\frac{P\ell^2}{kh^3} = 2\delta - 3\delta^2 + \delta^3, \qquad \delta = v/h.$$