A? MyCourses

CIV-E4100 - Stability of Structures L, 24.02.2020-09.04.2020

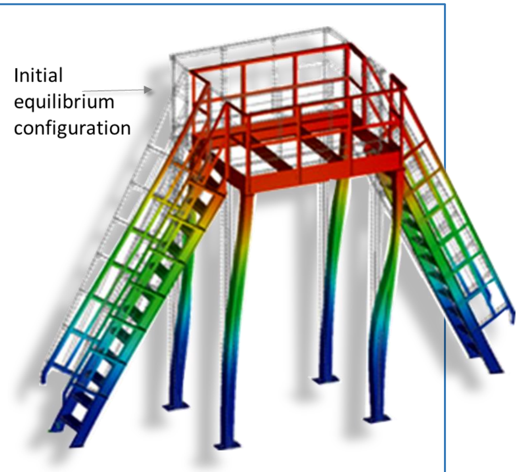
Content

0. Basic concepts
   Equilibrium, Stability
   The energy criterion of stability

Weeks #3-4 – Lectures series

1. Flexural buckling (nurjahdus)
2. Lateral-torsional buckling (kiepahdus)
3. Torsional buckling (vääntönurjahdus)
4. Buckling of thin plates
5. Buckling of shells (lommahdus)

- Lateral-torsional buckling
  kiepahdus
- Pure torsional buckling
  vääntönurjahdus
- Combined flexural-torsional buckling - avaruusnurjahdus tai yhdistetty vääntö- ja taivutusnurjahdus



Feb	24	25	26	27	28	29	1
March	2	3	4	5	6	7	8
	9	10	11	12	13	14	15
	16	17	18	19	20	21	22
	23	24	25	26	27	28	29
March	30	31	1	2	3	4	5

One topic per week

Lecturer
Djebar Baroudi, Dr.
Civil Engineering Department
Aalto University

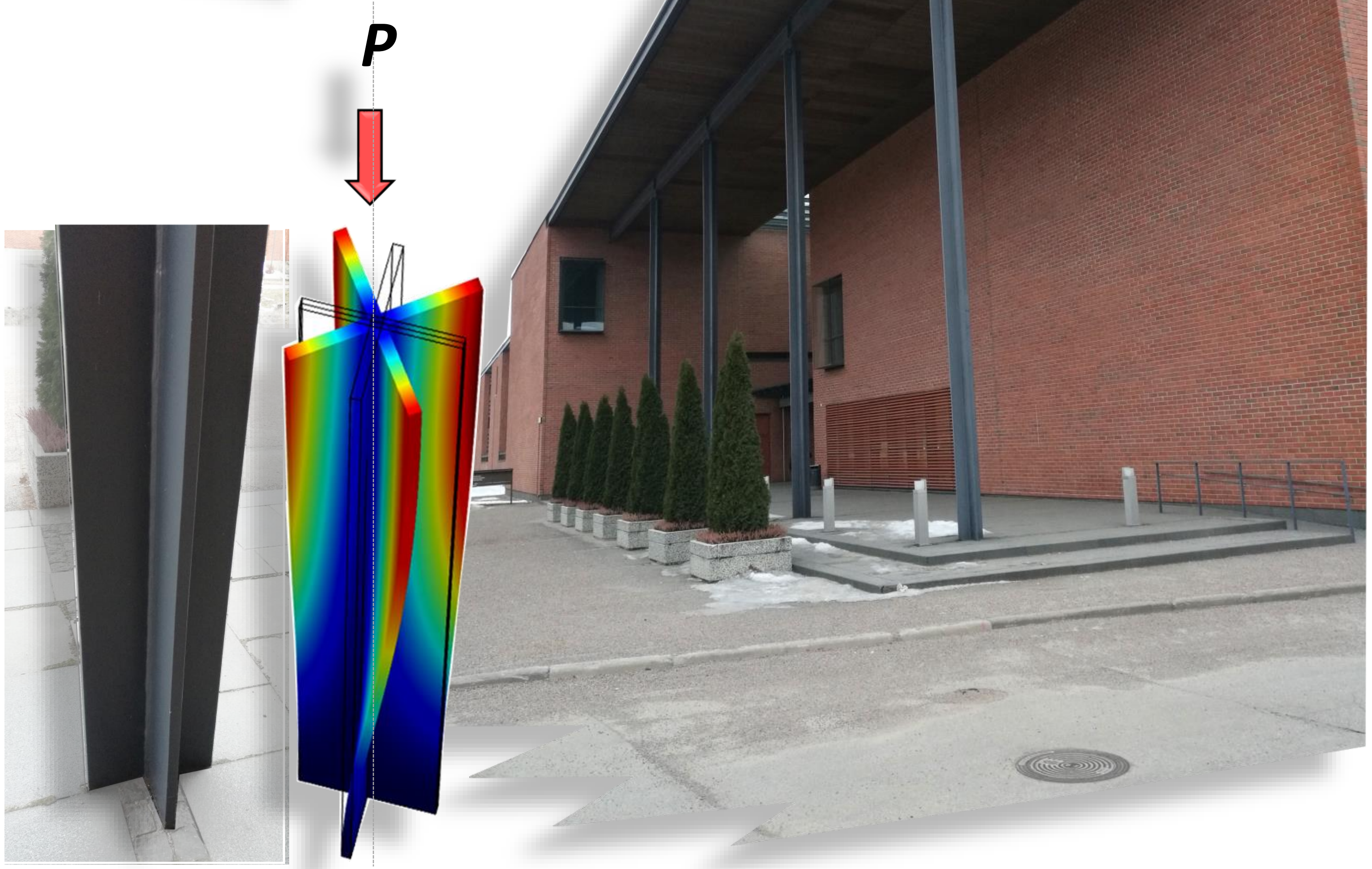
version 13.3.2020

Elastic Stability of Structures


CIV-E4100 - Stability of Structures L, 25.02.2019-11.04.2019

Weeks #3-4 – Lectures series

- Lateral-torsional buckling (kiepahdus)
- Pure torsional buckling (vääntönurjahdus)
- Combined flexural-torsional buckling (avaruusnurjahdus tai yhdistetty vääntö- ja taivutusnurjahdus)



Lecture slides for internal use only

D. Baroudi, Dr.

Version: 03.3.20120

Homework #3

Lateral torsional buckling,
Pure torsional buckling
and
Combined flexural-torsional buckling

* * *

Deadline 18.3.2020 before 23:45

March 7, 2020

Topics: *Lateral torsional, pure torsional and combined flexural-torsional buckling.*

Contents

NB: Only two exercises are compulsory. The remaining one, will be counted as extra points. Each **Question** is graded by five points and **EXTRA**, five points, respectively.

Readings

1. CHAI H. YOO & SUNG C. LE. *Stability of Structures*
   Chapter 6. *Torsional and Flexural-Torsional Buckling*
   Chapter 7. *Lateral-Torsional Buckling*
2. Lecturer's reading-supporting material pdf:
   Chapter 2: *Torsion of open thin-walled beams*
3. Lecture slides of the third week
4. Use of other sources is not prohibited but is encouraged

---

1 Exercise: Lateral torsional buckling

Use energy principles[1] and determine an approximative expression for the buckling load P_E of the simply supported elastic beam of length ℓ is centrically loaded by a compressive axial load P as shown in Figure (1). The end-rotations support is a fork-type. The buckling load should be expressed as $P_E = f(EI_y, EI_\omega, GI_t, \ell, a)$.

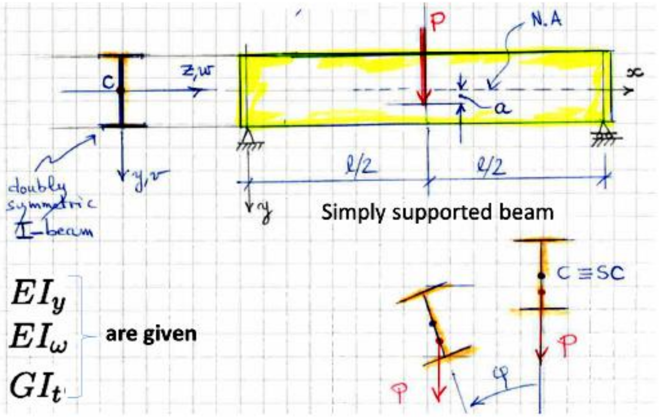


Figure 1: Simply supported beam. The support condition for end-rotations is a fork-type. The load P is at a distance a from from the neutral axis. The cross-section of the I-beam is doubly symmetric.

For comparison, the analytical exact solution is given and is

$$\frac{P_E \ell}{4M_{ref}} \approx 1.35\left[\sqrt{1+[0.54P_{E,y}a/M_{ref}]^2}+0.54P_{E,y}a/M_{ref}\right], \qquad (1)$$

where $P_{E,y} = \pi^2 EI_y/\ell^2$ and $M_{ref} = \sqrt{P_{E,y}[GI_t + \pi^2 EI_\omega/\ell^2]}$

Hints: 1) Trigonometrical trials lead to less work for the student. For instance, for rotation $\phi(x) \approx A\sin(\pi x/\ell)$ is enough. Naturally, the student is free to chose his own kinematically admissible approximation.

---

2 Exercise: Combined flexural and torsional buckling

An elastic cantilever column (Figure 2) is centrically axially loaded at its free end. The load acts on the center of gravity of the section (= centroid).

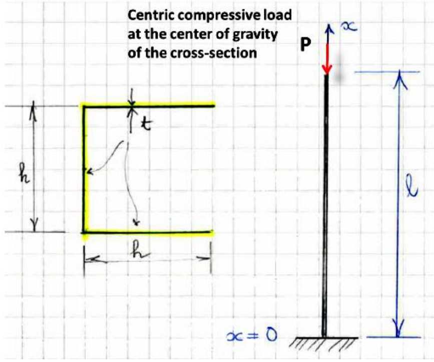


Figure 2: Axially loaded cantilever column. The geometrical parameters are such that $t = h/5$ and $\ell = 10h$.

- Determine the buckling load P_E and the corresponding mode (flexural or torsional or combined?). The location of the center of shear (SC) and the warping inertia moment I_ω can be determined using tables.

ution
cation
MPa.
what
am)

---

3 Exercise: Flexural-torsional buckling

Consider the simply supported elastic column (sub-figure a) in Fig. 3). The cross-section is in the form of a crucifix `X` or `+`. The thrust P is axially centric.

At both end we have a fork support for rotations and also warping is free to happen at both ends and thus $\phi'' = 0$ at $x = 0$ and $x = \ell$. As regard to bending both ends can be assumed, for the purpose of the exercise, freely supported.

- Determine the buckling load and the corresponding mode
- (EXTRA 5 pnts) Determine only the pure torsional buckling load for the real X-column in sub-figure b) in Fig. 3) Hint: find the column in Finland and determine its dimensions (approximative). Assume it made of steel and simply supported and the end-load being centric. Do not account for self-weight.
- (EXTRA 2 pnts) Determine the critical length ℓ_{cr} for mode transition between pure torsional and pure flexural. Draw a diagram of the critical load $P_{cr} = P_{cr}(\ell)$ as a function of ℓ for both flexural and torsional buckling. Show the buckling envelope.

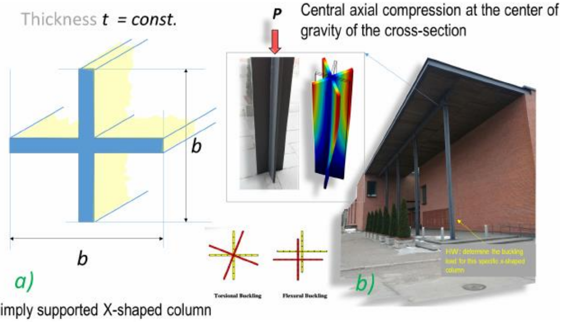


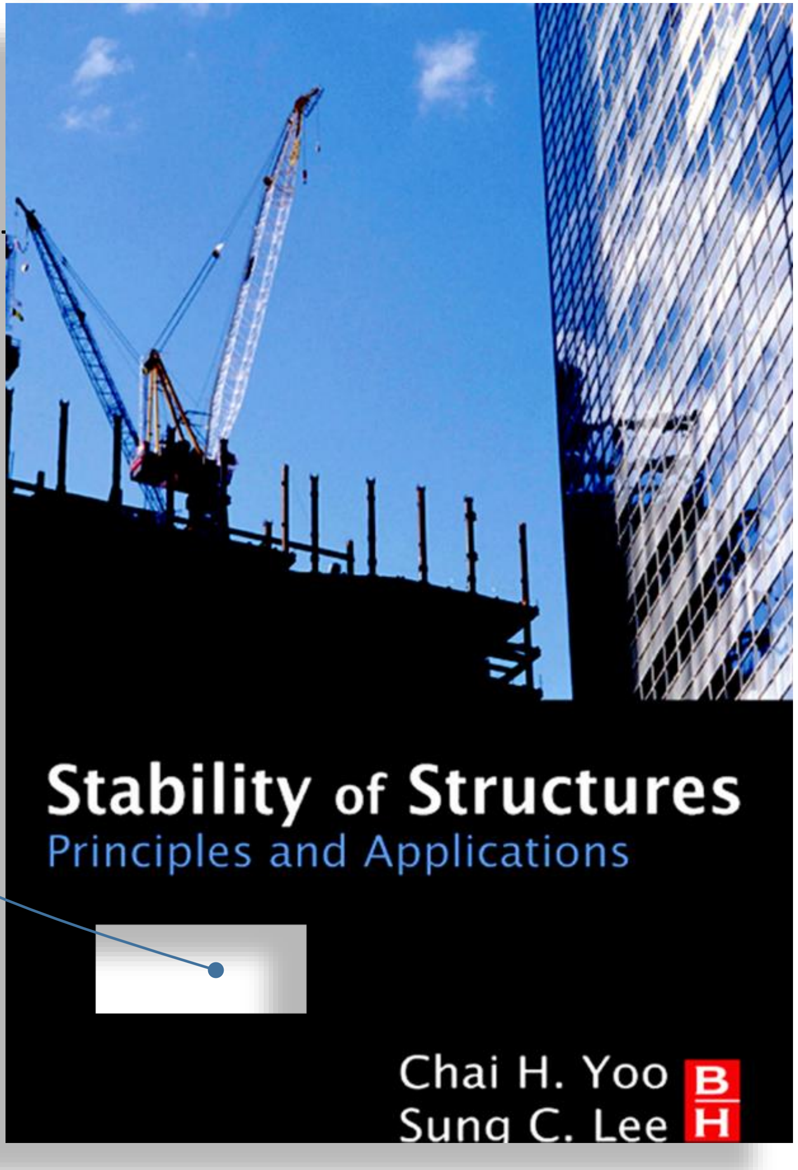
Figure 3: a) Simply supported elastic column (of length ℓ) under centric thrust P. b) X-shaped column somewhere in Finland.

Read

Chapter 6. Torsional and Flexural-Torsional Buckling

Chapter 7. Lateral-Torsional Buckling

This course textbook
e-book

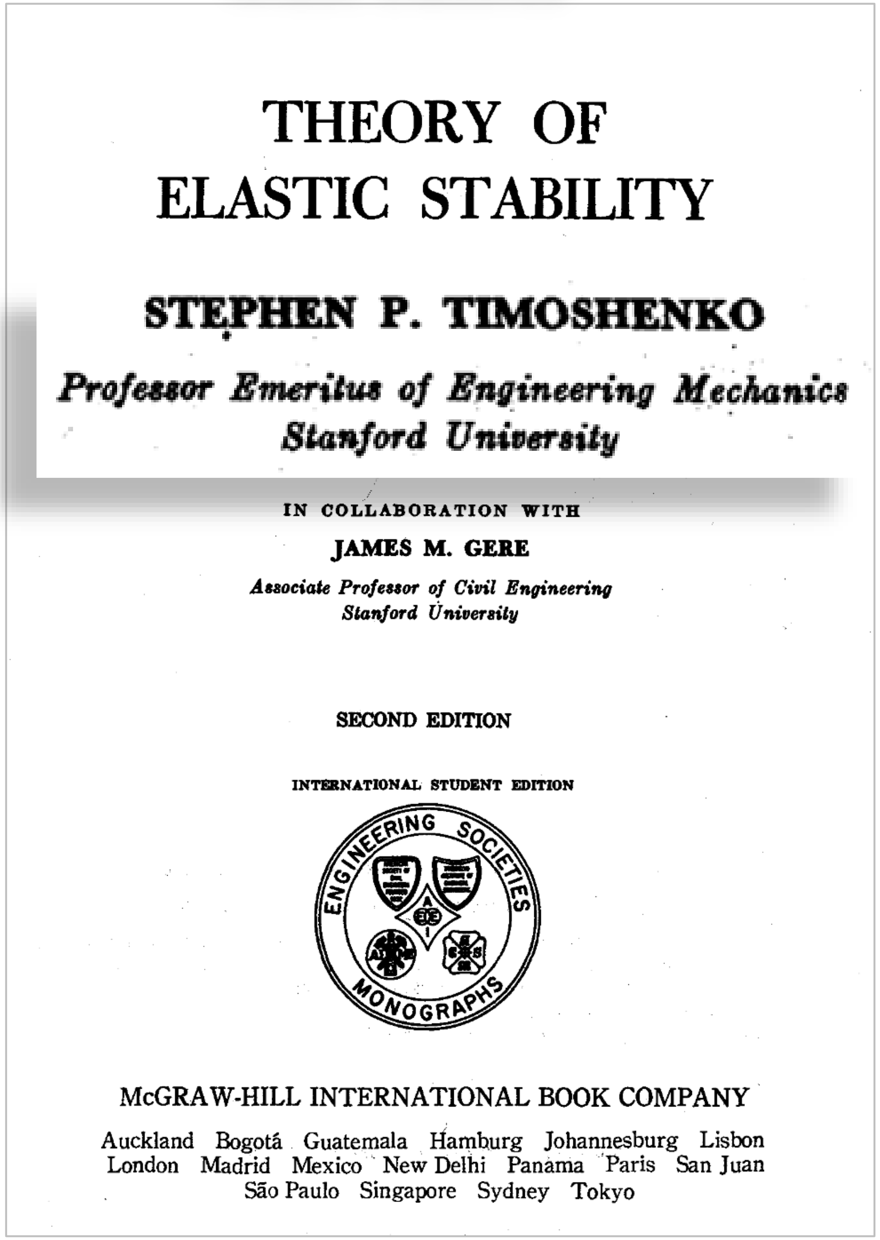


Stability of Structures
Principles and Applications

Chai H. Yoo
Sung C. Lee

BH

Must classics



THEORY OF
ELASTIC STABILITY

STEPHEN P. TIMOSHENKO
Professor Emeritus of Engineering Mechanics
Stanford University

IN COLLABORATION WITH
JAMES M. GERE
Associate Professor of Civil Engineering
Stanford University

SECOND EDITION

INTERNATIONAL STUDENT EDITION

ENGINEERING SOCIETIES MONOGRAPHS

McGRAW-HILL INTERNATIONAL BOOK COMPANY
Auckland Bogotá Guatemala Hamburg Johannesburg Lisbon
London Madrid Mexico New Delhi Panama Paris San Juan
São Paulo Singapore Sydney Tokyo

CIV-E4100 - Stability of Structures L, 25.02.2019-11.04.2019

Topics of the lectures and homework

Content

0. Basic concepts
   Equilibrium, Stability
   The energy criterion of stability

First week

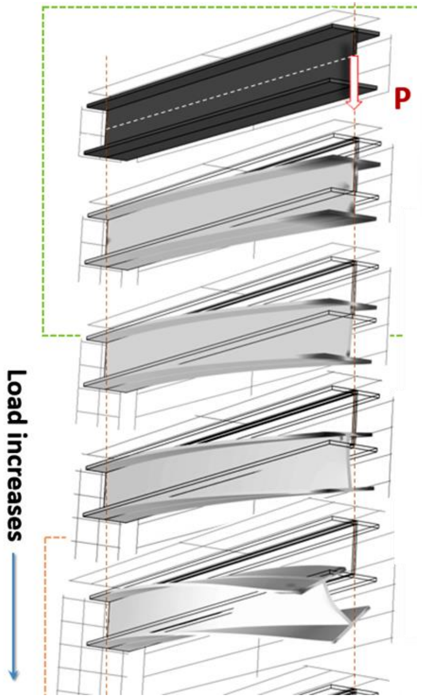
1. Flexural buckling (nurjahdus) 2nd week
2. **Lateral-torsional buckling (kiepahdus)**
3. **Torsional buckling (vääntönurjahdus)**

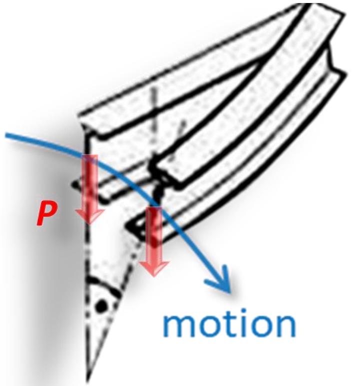
3rd week
+ 4th (1/2)

4. Buckling of thin plates
5. Buckling of shells (lommahdus)

Lecturer

Djebar BAROUDI,PhD.
Lecturer
Allto University

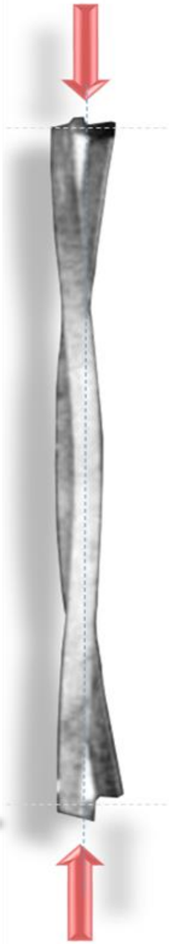


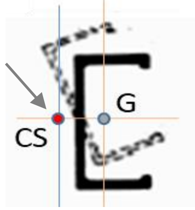


Lateral torsional buckling

Kiepahdus

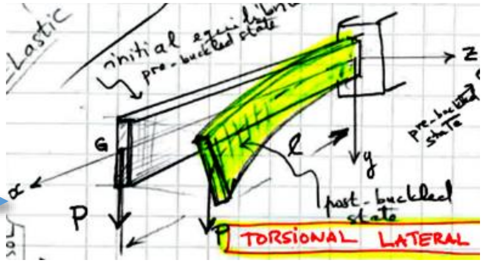
Vääntönurjahdus



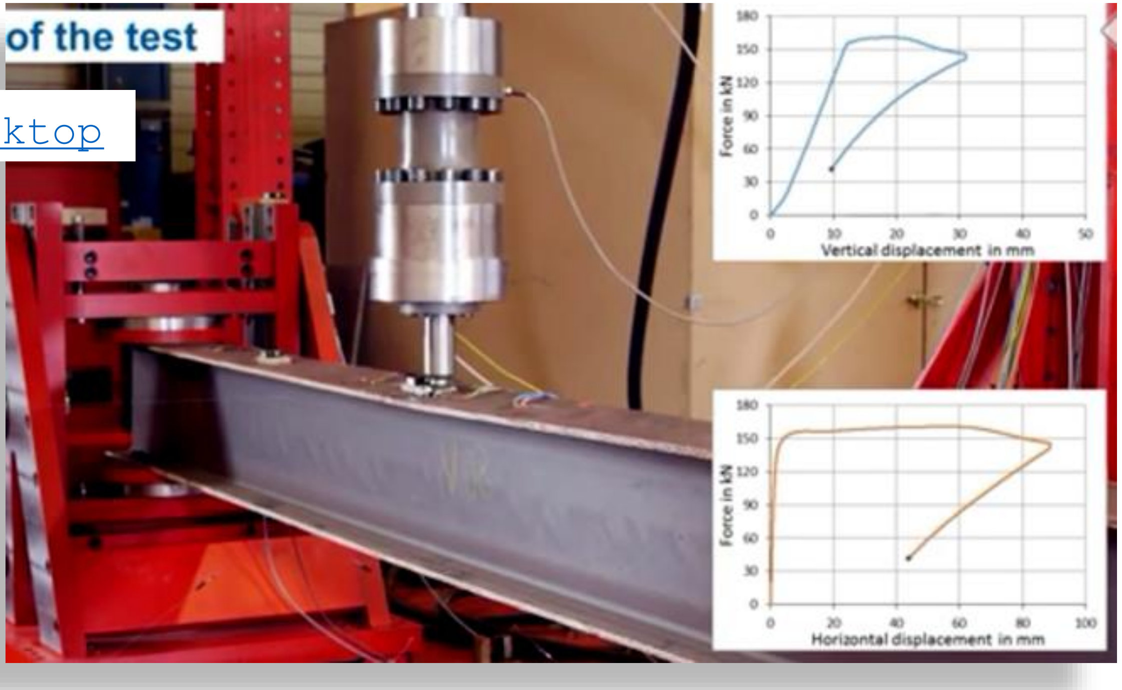


Beams having **thin-walled open cross-sections** can have **torsional** modes of **stability loss** due to their relatively low torsional rigidity .

… and for narrow cross-sections, too



Some videos on stability of structures



https://www.youtube.com/watch?v=OoORi_2Vkcg&app=desktop

(24.02.2010)

1: Lateral torsional buckling of I-beam (kiepahdus)

Comment: Good experiment with load-displacement curves

The student can clearly see the transition from bending in the vertical plane to bending in the horizontal plane and torsion

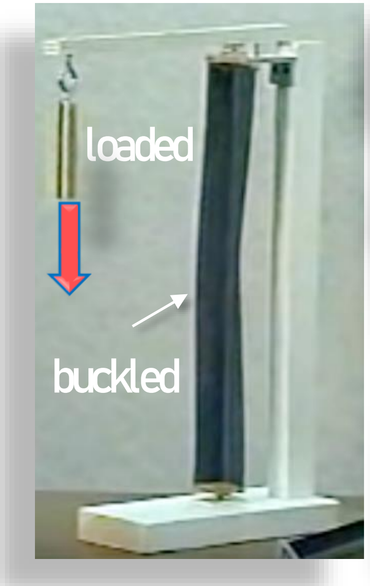
https://www.youtube.com/watch?feature=youtu.be&v=cYRicTk-Q08&app=desktop

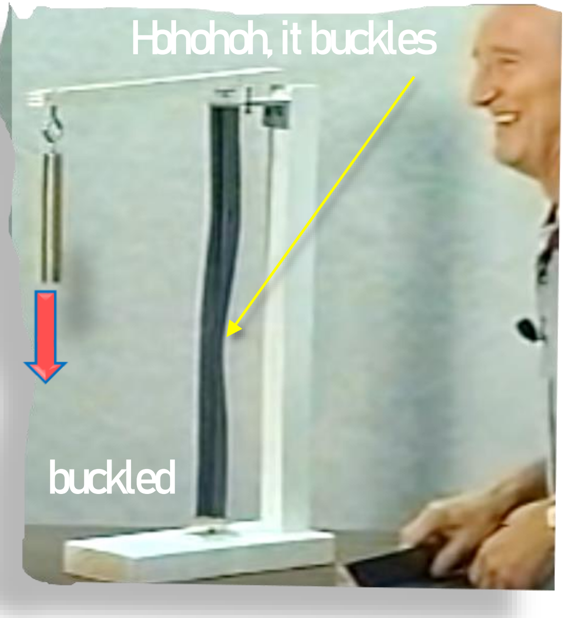
(24.02.2010)

2: Pure Torsional buckling of L-shape cross-section (angle) column (Puhdas vääntönurjahdus)

Comment: Good experiment with a funny professor.

Note that, the apparent (torsional) rigidity gets dramatically reduced close to the buckling load



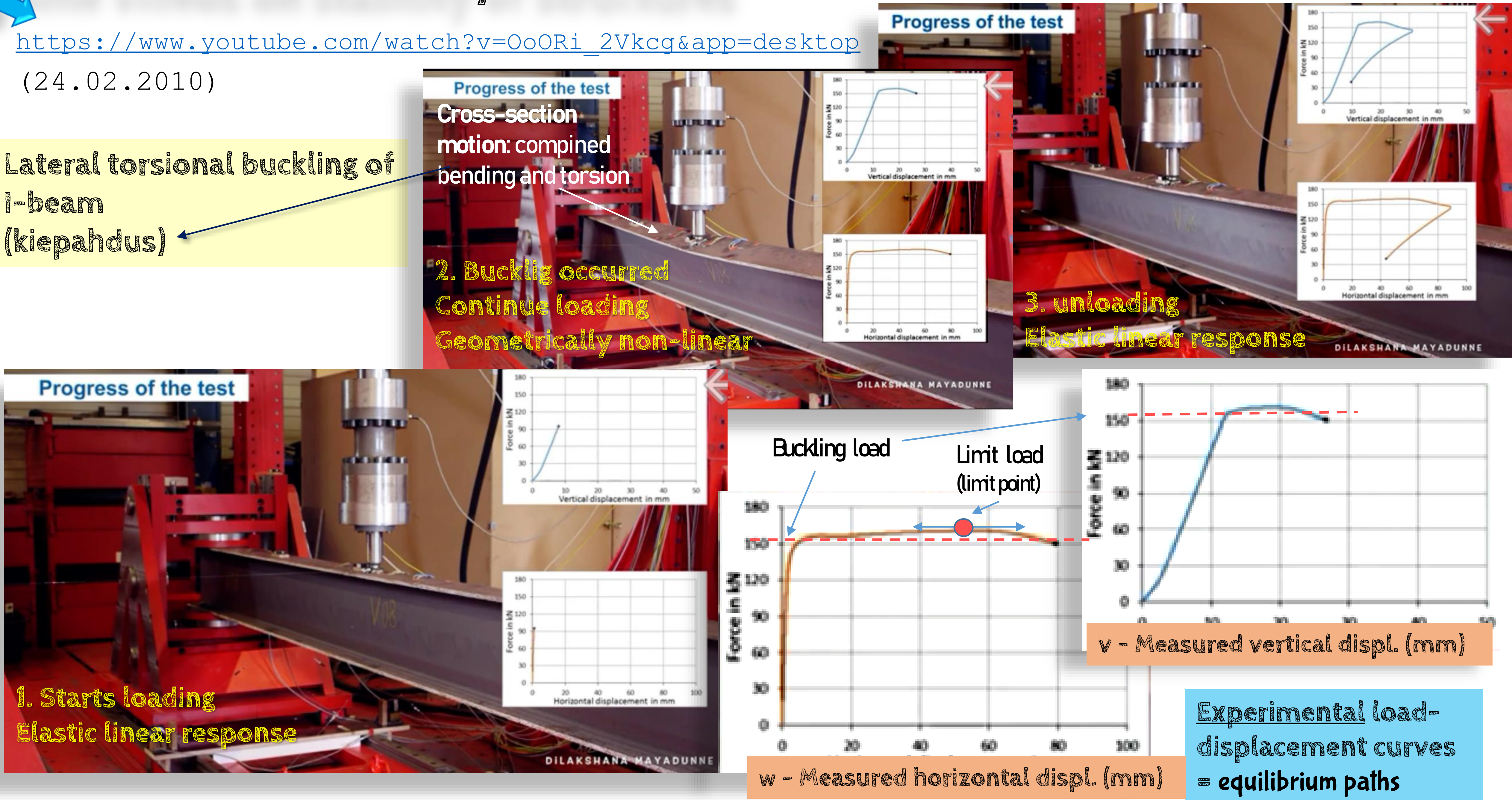


Some videos on stability of structures

https://www.youtube.com/watch?v=OoORi_2Vkcg&app=desktop

(24.02.2010)

Lateral torsional buckling of I-beam (kiepahdus)



Progress of the test

Cross-section motion: compined bending and torsion

2. Bucklig occurred

Continue loading

Geometrically non-linear

3. unloading

Elastic linear response

1. Starts loading

Elastic linear response

Buckling load

Limit load (limit point)

v – Measured vertical displ. (mm)

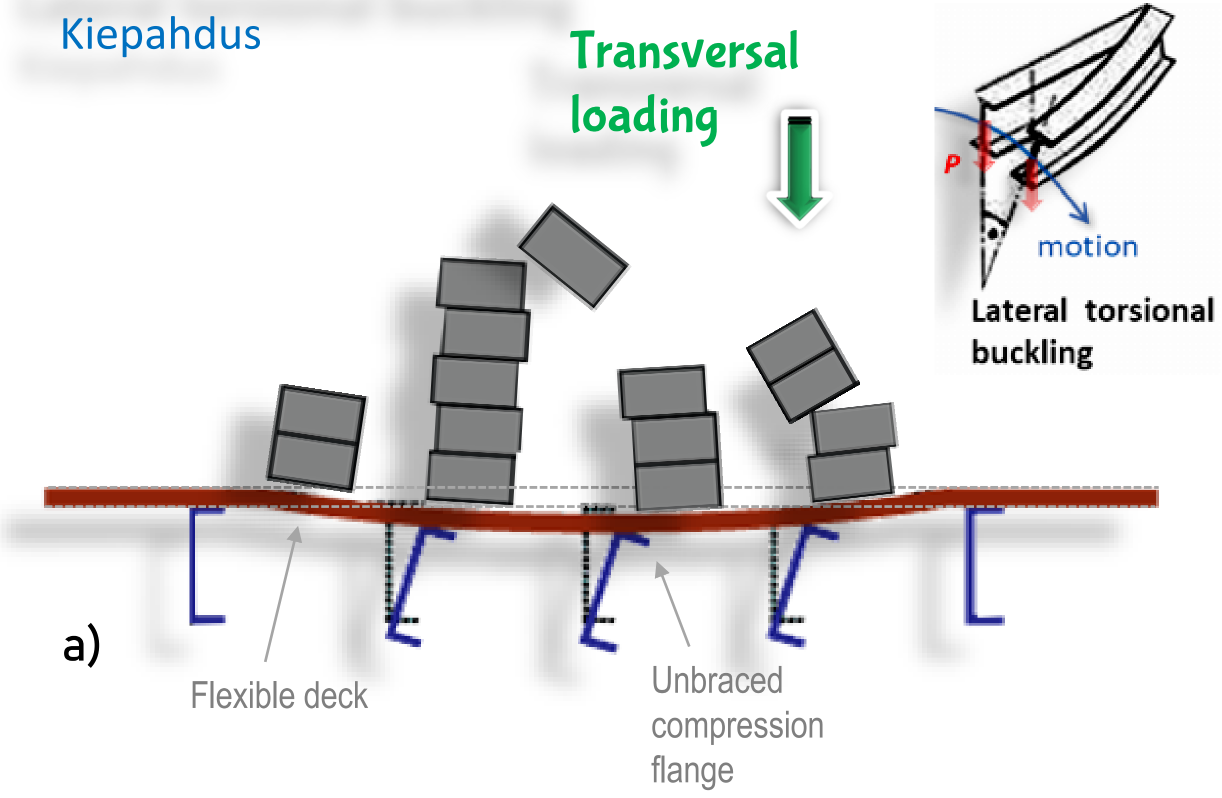
w – Measured horizontal displ. (mm)

Experimental load-displacement curves = equilibrium paths

The phenomenon

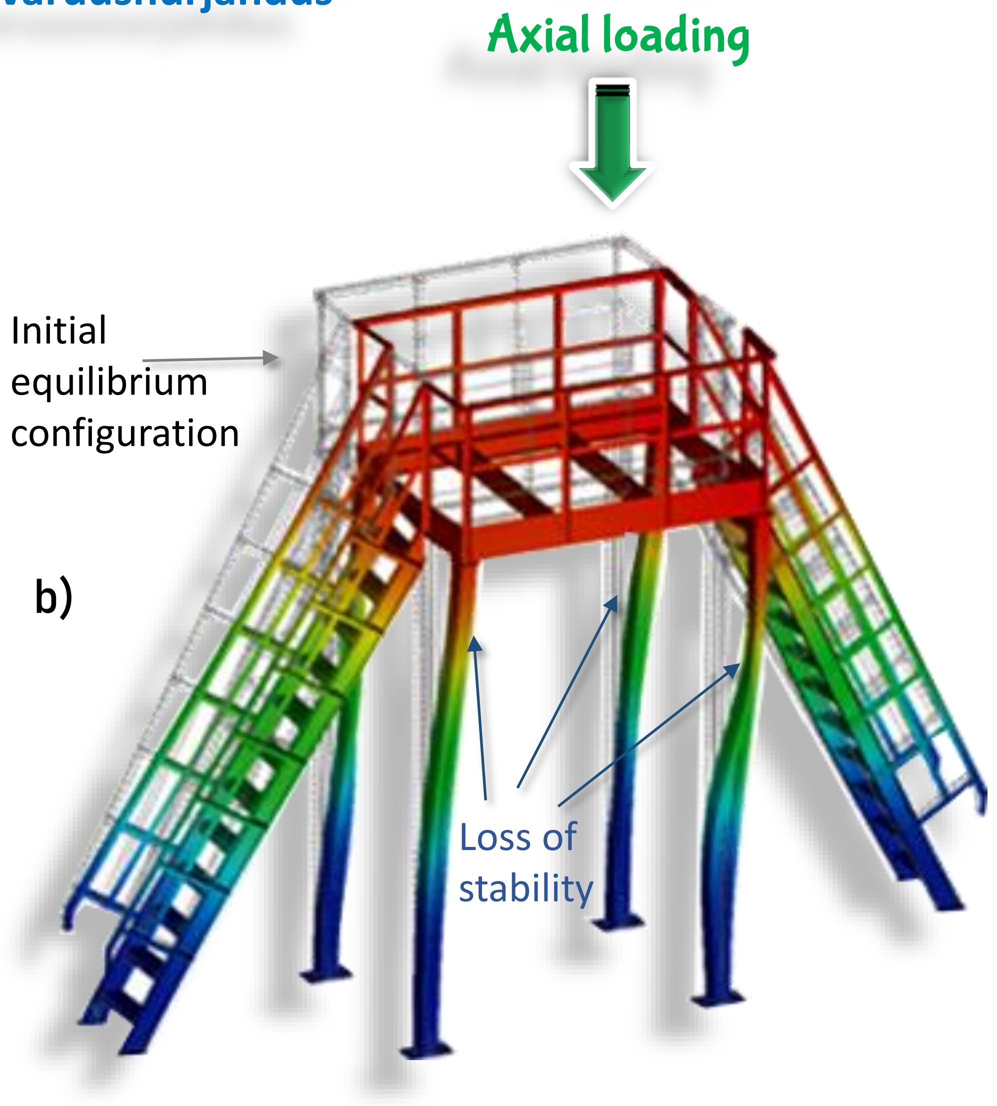
Lateral torsional buckling

Kiepahdus



Combined flexural and torsional buckling

Yhdistetty taivutus- ja vääntönurjahdus, eli **avaruusnurjahdus**

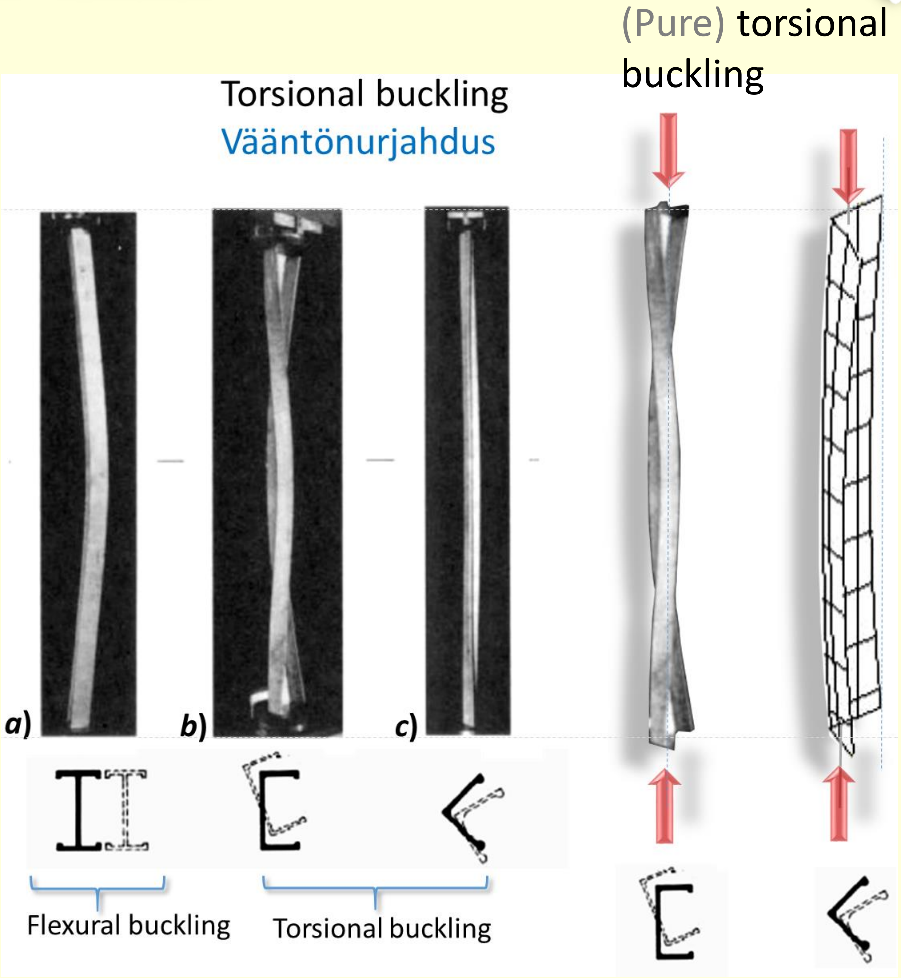


Beams with thin-walled open cross-sections can have torsional modes of stability loss due to their relatively low torsional rigidity

Thin-walled open cross-sections examples

The **phenomenon** of buckling with torsion

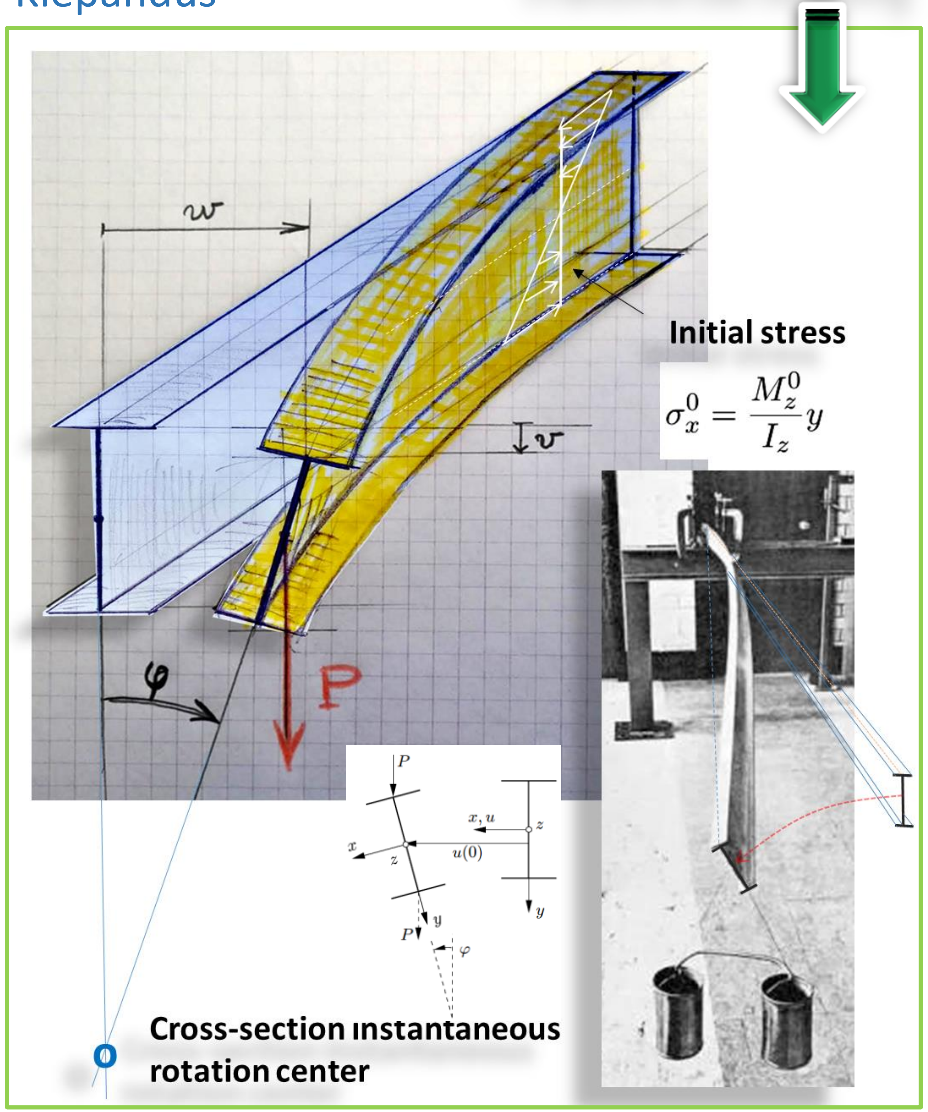
Axial loading



(Pure) **Torsional** buckling
Vääntönurjahdus

Lateral torsional buckling
Kiepahdus

Transversal loading



Initial stress

$$\sigma_x^0 = \frac{M_z^0}{I_z} y$$

Cross-section instantaneous rotation center

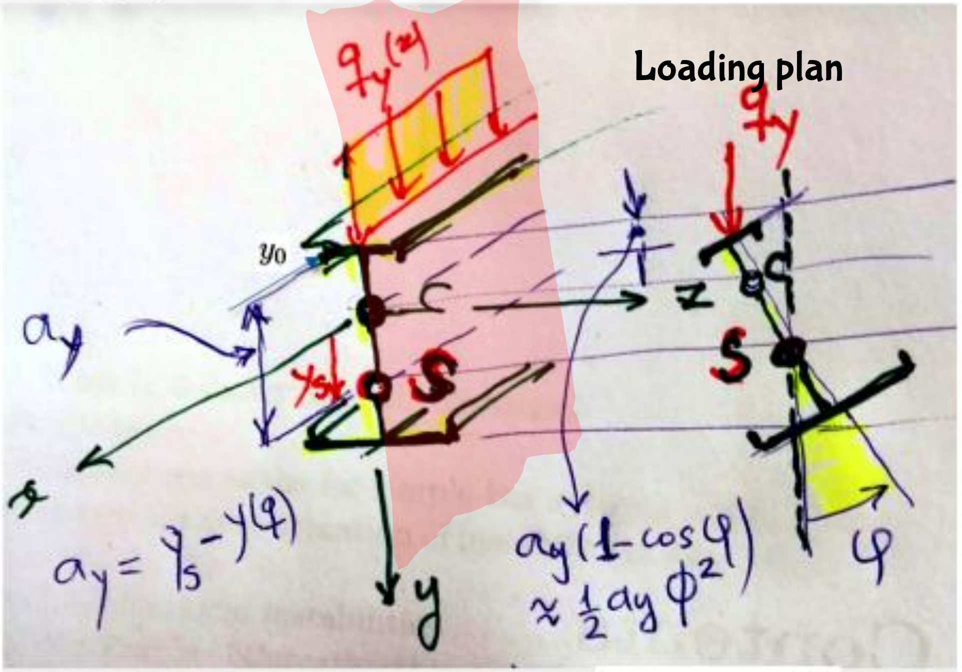
Beams having **thin-walled open cross-sections** can have **torsional** modes of **stability loss** due to their relatively low torsional rigidity .

In both cases the cross-section have a torsional motion

The phenomenon

Lateral-Torsional buckling

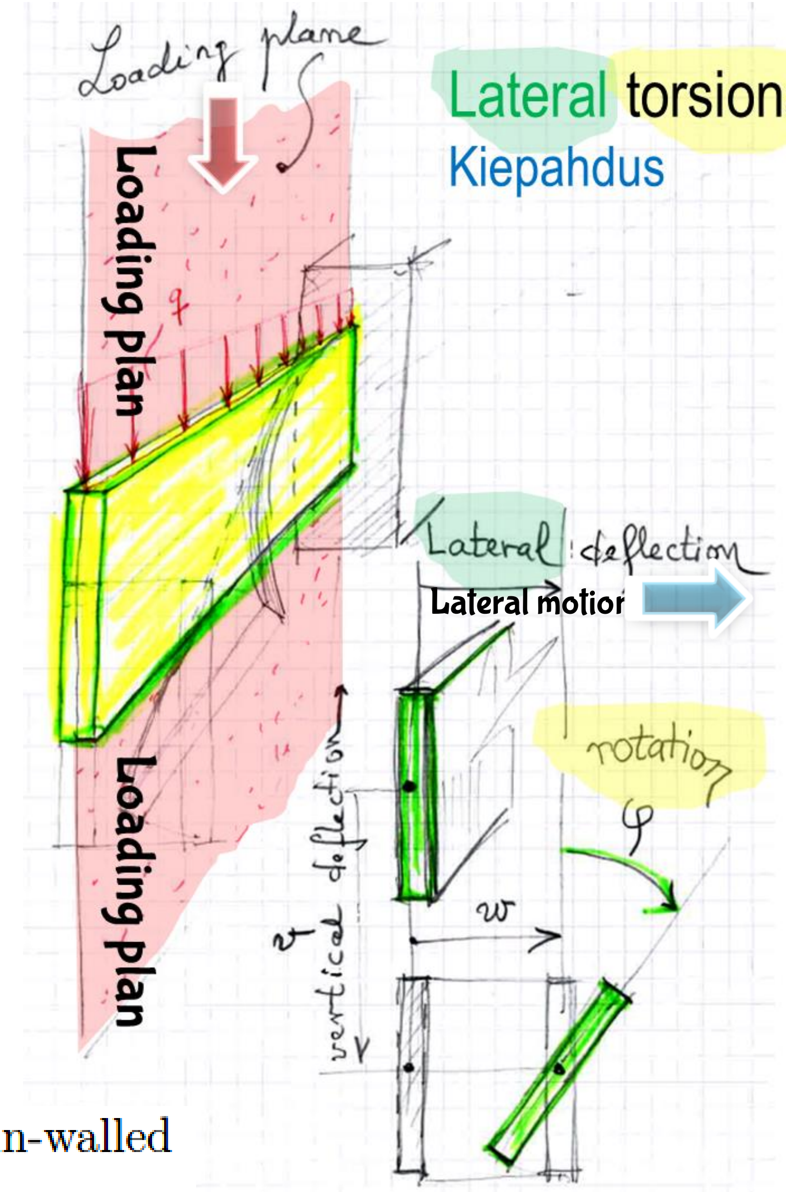
Singly symmetric cross section

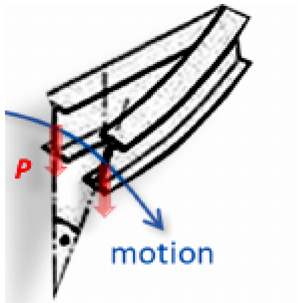


S- shear center (rotation center): $(y_s, z_s = 0)$

C- center of mass

Moment arm of external force: $a_y \equiv (y_s - y_0)$





Lateral torsional buckling

Lateral-torsional buckling of singly symmetric thin-walled open section beam under transversal load q_y.

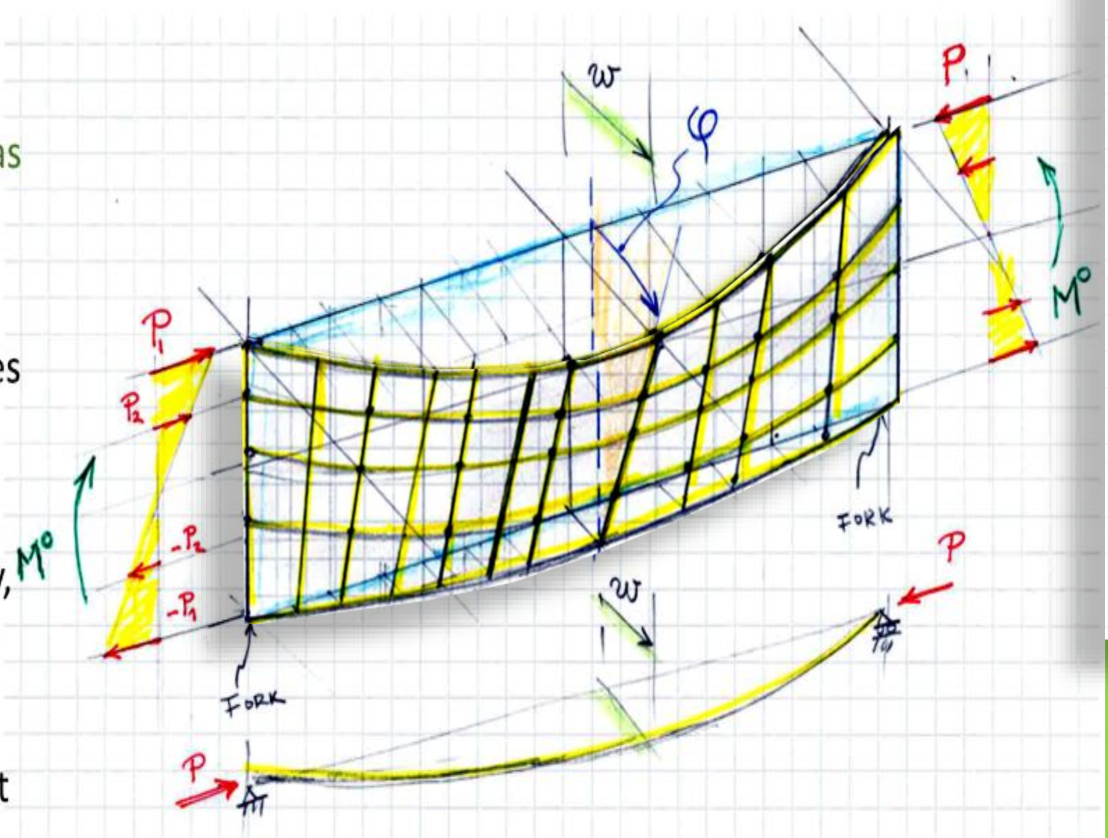
Vääntökeskiö = shear center

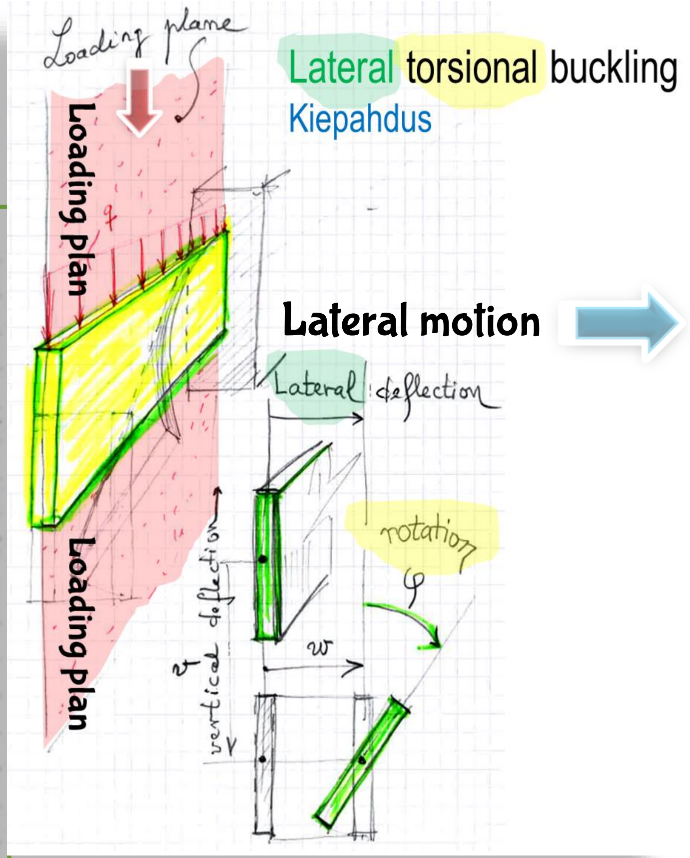
The phenomenon

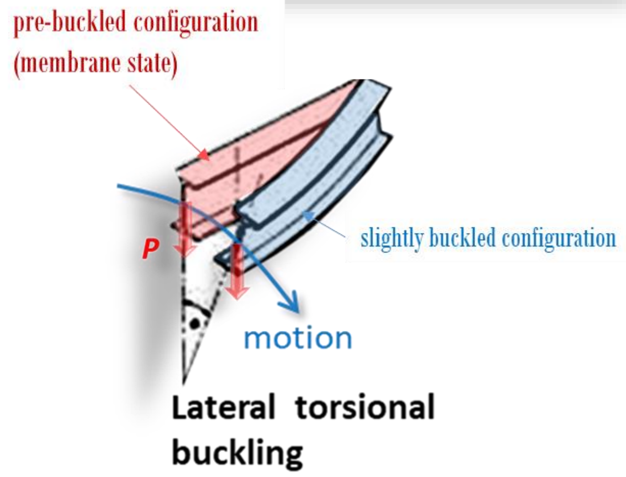
Kinematics of lateral torsional buckling

Kinematics of the lateral buckling : the flanges as thin plate being physically as a discrete grid or network of slender inter-connected thin bars in which

1. Each *compressed bar* separately buckles as simple axially compressed column, resulting in: lateral deflection
2. The vertical bars, because of continuity, **rotate**, resulting in: rotation of cross-sections
3. Bars in tension have *a stabilizing* effect







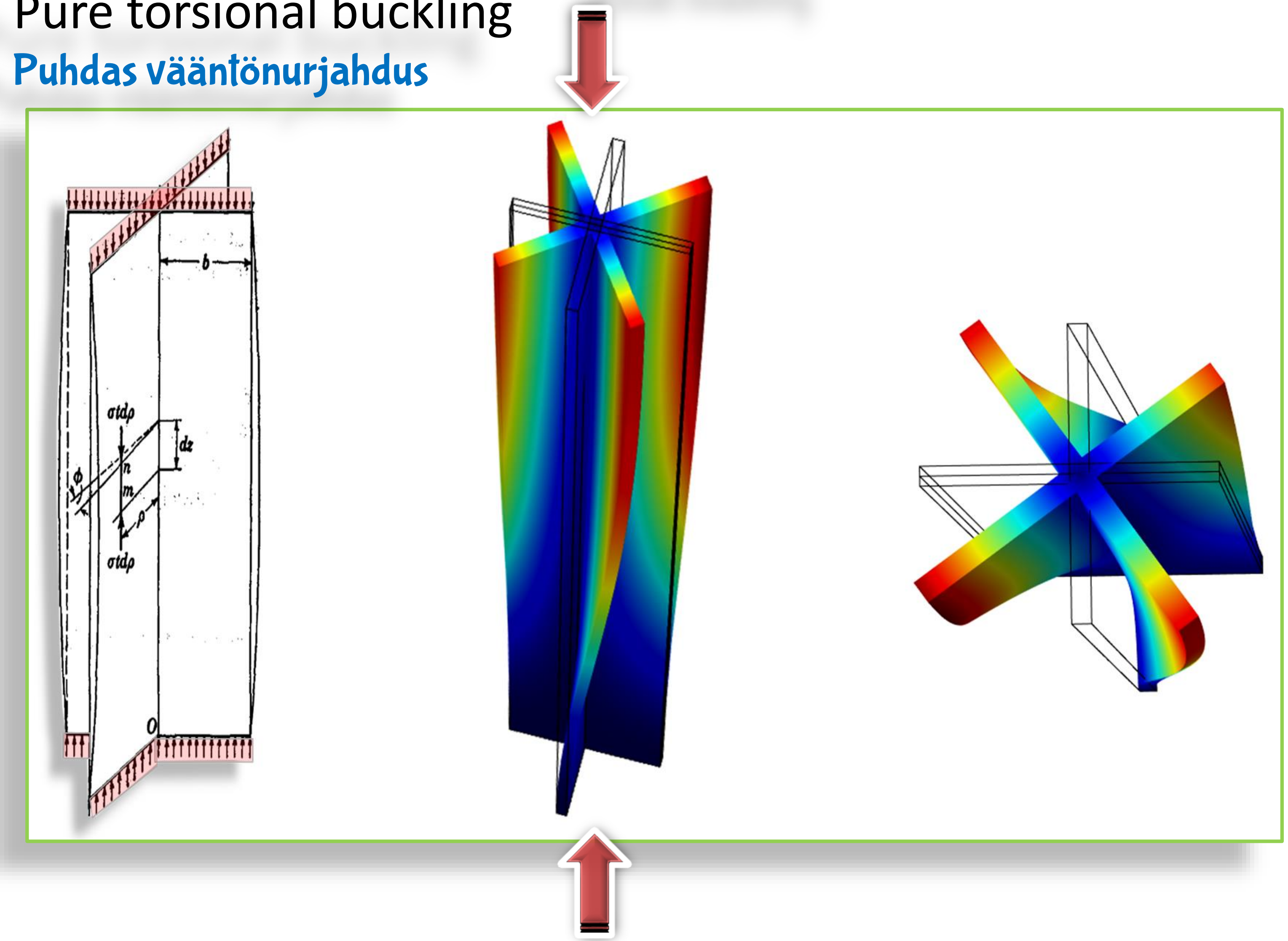
Lateral motion due to vertical loading

The phenomenon

Pure torsional buckling

Puhdas vääntönurjahdus

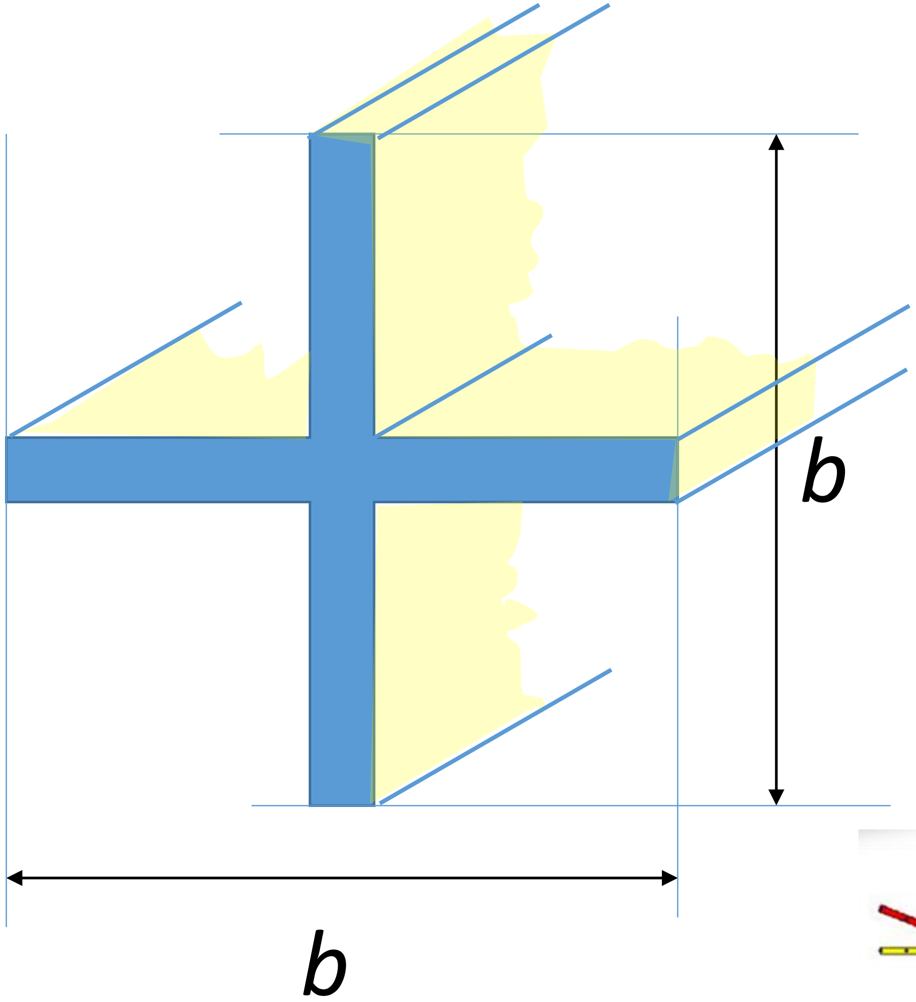
Axial loading



Axial vertical loading

Axial loading

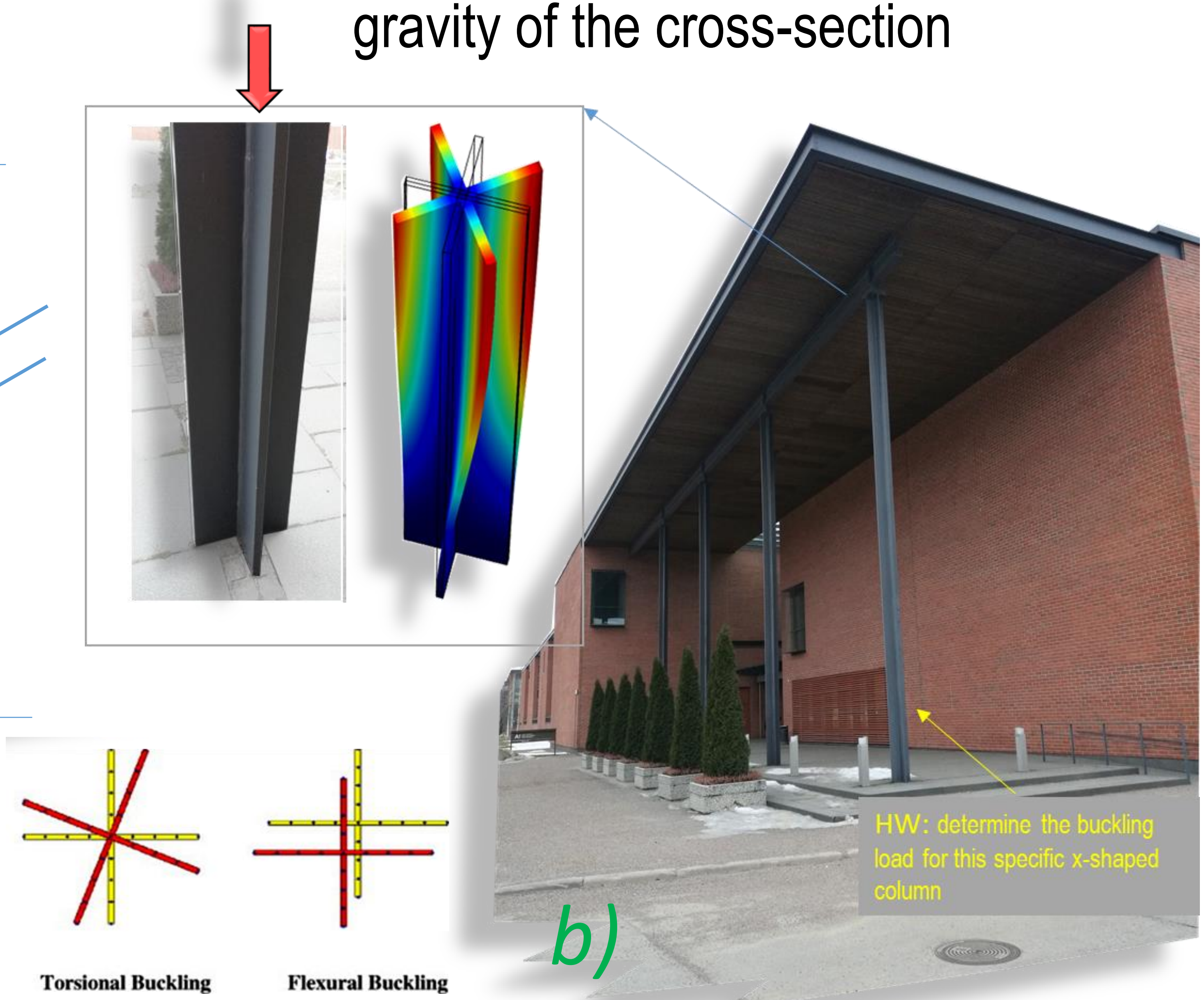
Thickness t = *const.*



a)

Simply supported X-shaped column

P Central axial compression at the center of gravity of the cross-section



HW: determine the buckling load for this specific x-shaped column

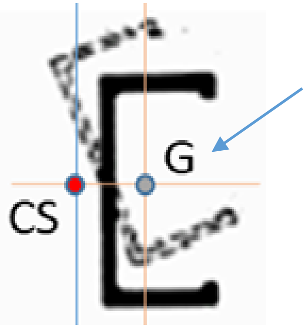
What is warping? What is the shear center?

The student should refer to the additional reading material and textbook for details on warping

For the purpose of this course: use tables to find SC and Iw

PROPERTIES OF SHEAR CENTER

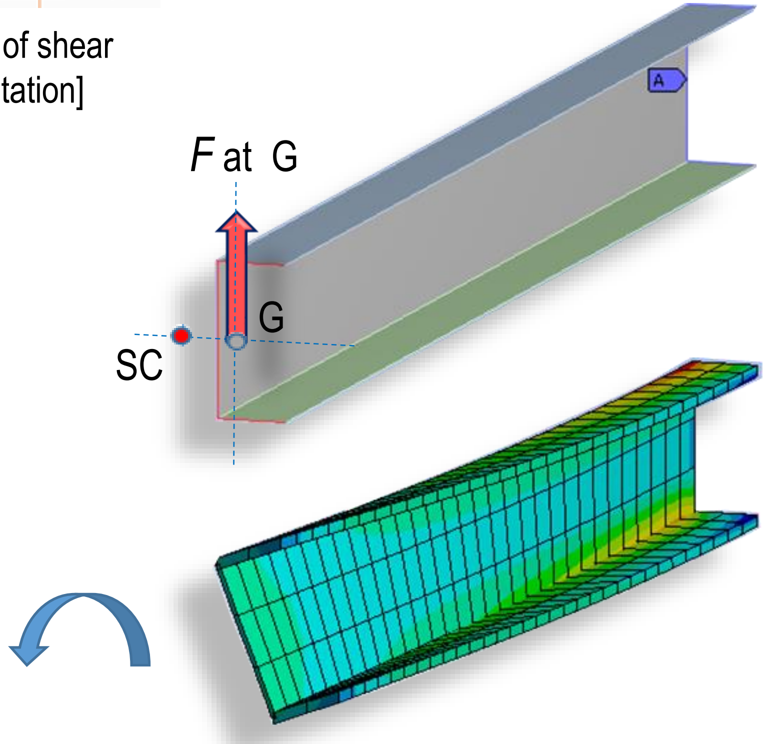
1. When transverse force applied at shear center it does not lead to torsion
2. The shear center (SC) is the center of rotation for a thin-walled section of beam subjected to pure torsion
3. The shear center is a location of shear flows resultant force when the thin-walled beam is subjected to pure shear



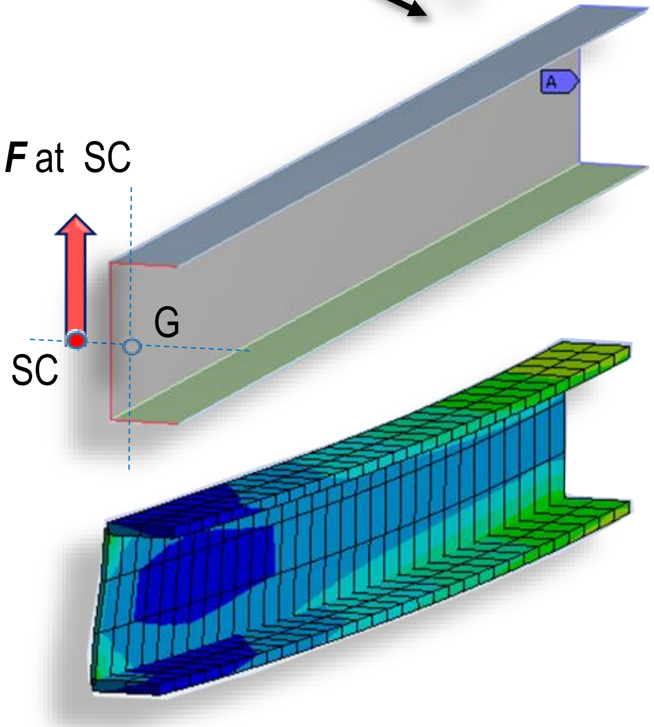
(G) center of gravity [= area center (C)]

(SC) center of shear [center of rotation]

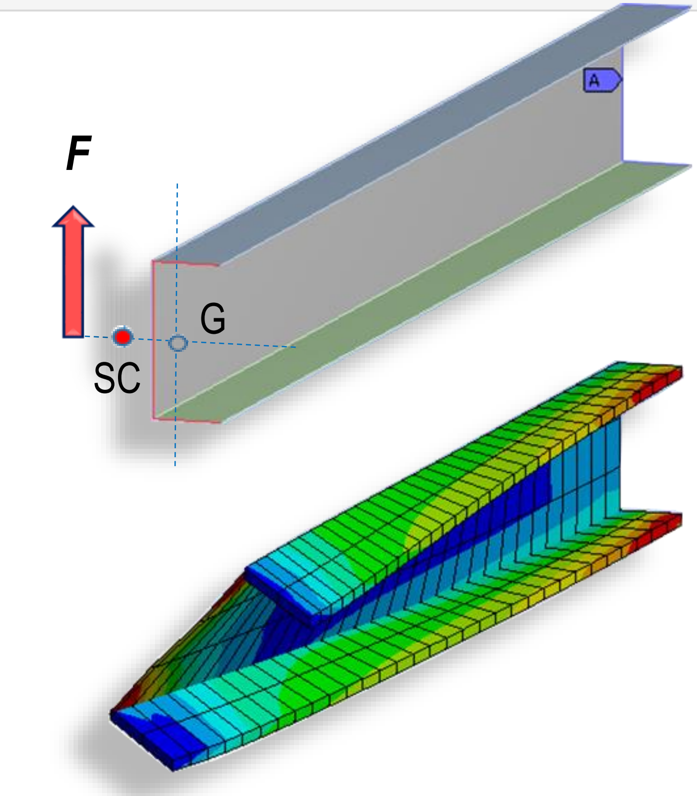
Clamped boundary



Combined and torsion



Only bending in transversal plane



Combined and torsion

Distortional modes in some thin-walled cross sections

NB. **In addition to the modes shown in previous slide,**

Local distortional buckling modes for beams (or beam-shells) with a very thin-walled cross-section are possible → the cross-section geometry is distorted

For such very-thin walled beams it becomes impossible and not practical to put stiffeners to keep the cross-section undistorted

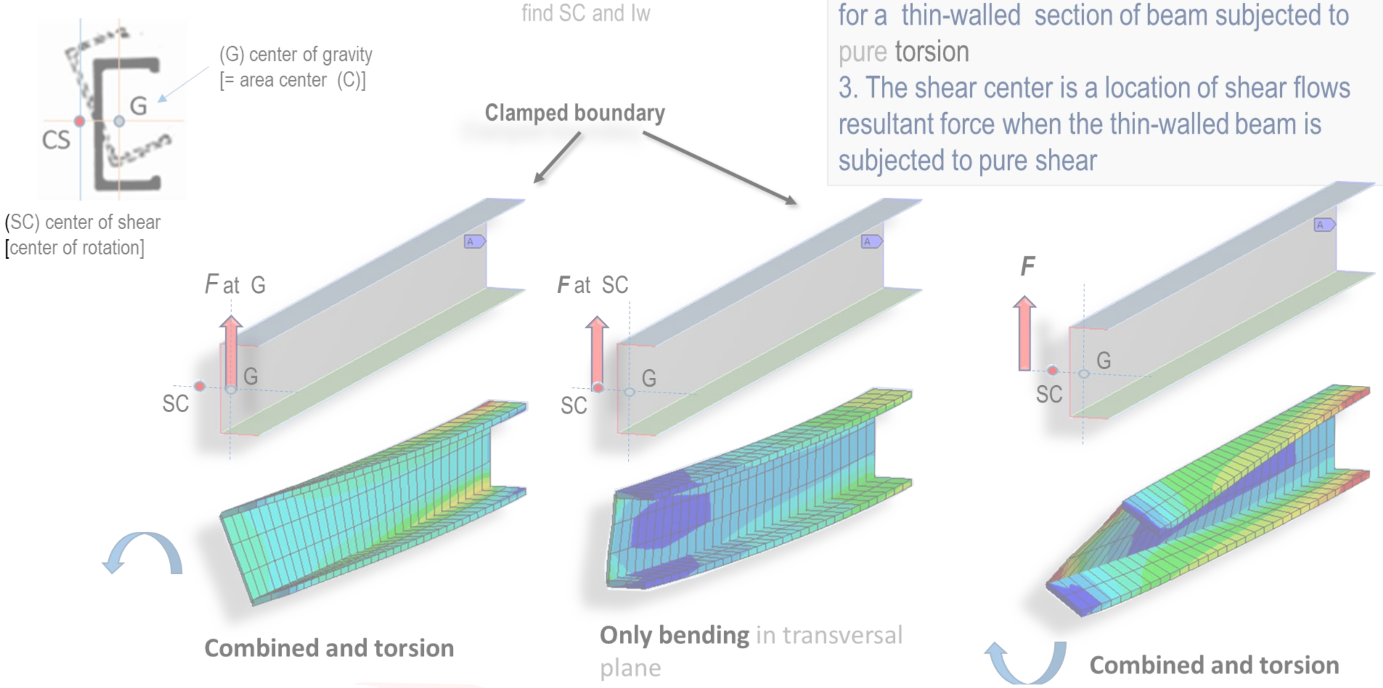
Distortional local buckling. *Vlasov* theory does not account for such deformation mode.

What is warping? What is the shear center?

The student should refer to the additional reading material and textbook for details on warping

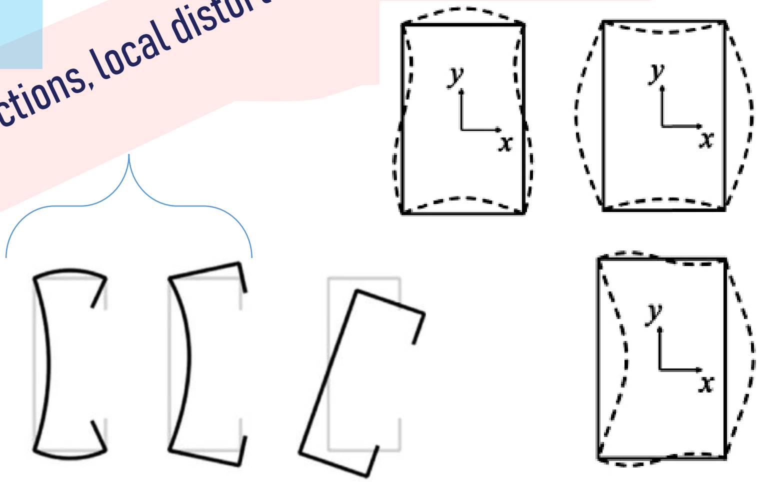
For the purpose of this course: use tables to find SC and Iw

1. When transverse force applied at shear center it does not lead to torsion
2. The shear center (SC) is the center of rotation for a thin-walled section of beam subjected to pure torsion
3. The shear center is a location of shear flows resultant force when the thin-walled beam is subjected to pure shear



The cross-section shape is assumed not distorting in Vlassov's theory

For very thin-walled cross sections, local distortional deformation modes exits



Mechanics of thin-walled beams with open cross-sections

In lateral-torsional and torsional buckling we should consider warping to obtain the **correct strain energy** change due to these modes of deformation

...

In order to derive the correct **stability (loss) equation**

Lateral torsional buckling equation:

Kiepahdus

Warping effects Pure torsion

$$EI_\omega \varphi^{(4)} - GI_{t}\varphi'' - \frac{(M_x^0)^2}{EI_y}\varphi = 0,$$

Torsion problem (no buckling)

Thin-walled open cross-section

$$EI_\omega \frac{d^4\varphi}{dz^4} = m$$

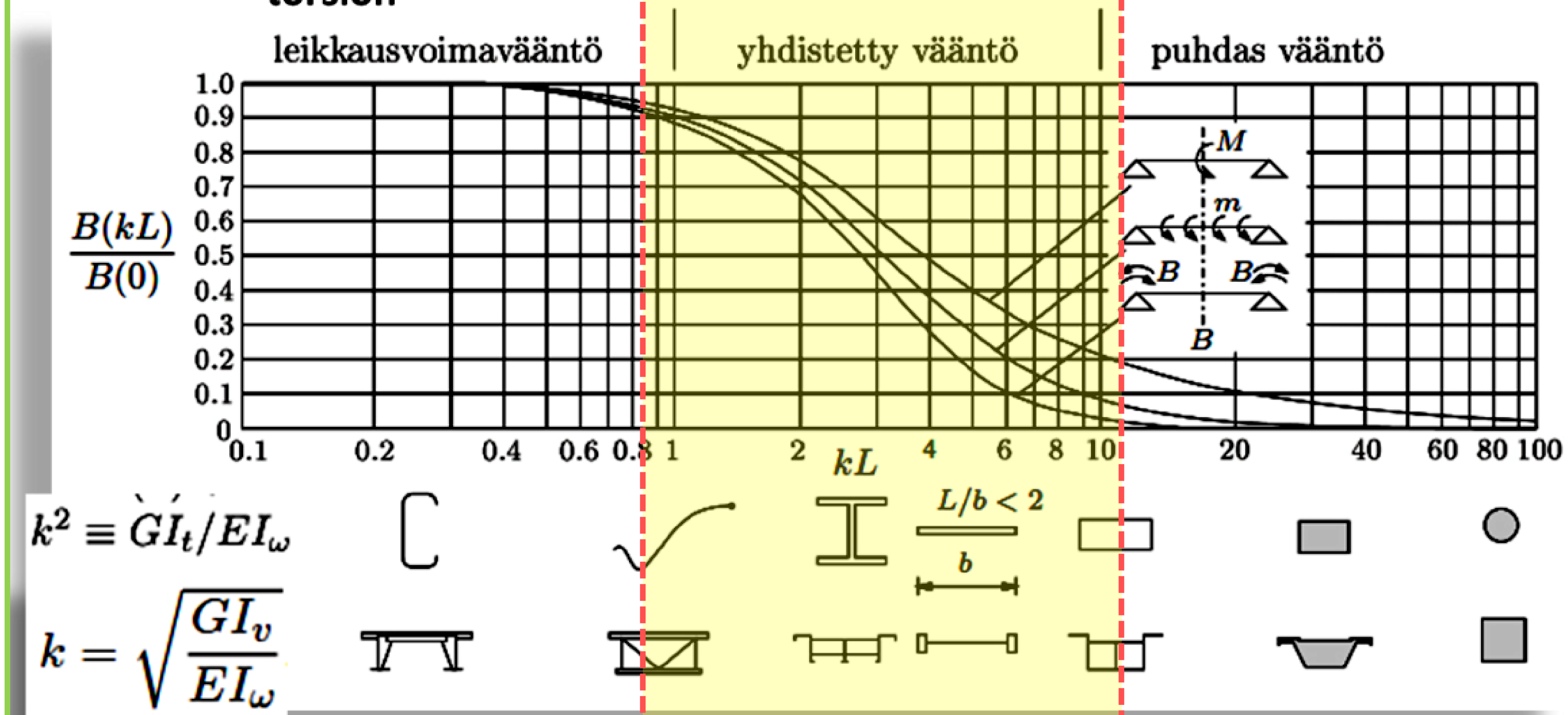
Fully restrained torsion

$$\frac{d^4\varphi}{dz^4} - k^2\frac{d^2\varphi}{dz^2} = \frac{m}{EI_\omega}$$

Combined torsion

$$-GI_v\frac{d^2\varphi}{dz^2} = m$$

Free torsion

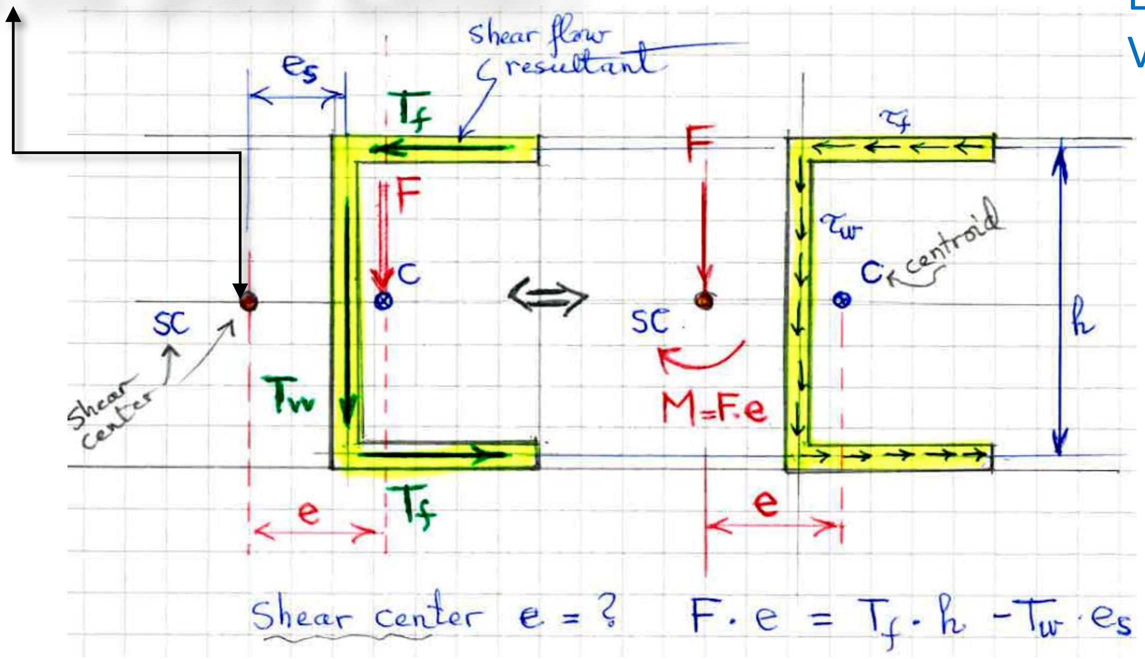


$$k^2 \equiv GI_t/EI_\omega$$

$$k = \sqrt{\frac{GI_v}{EI_\omega}}$$

Ref. Rakenteiden mekaniikan jatkokurssi, lecture-notes, *Emeritus prof*. Markku Tuomala

What is the Shear Center?



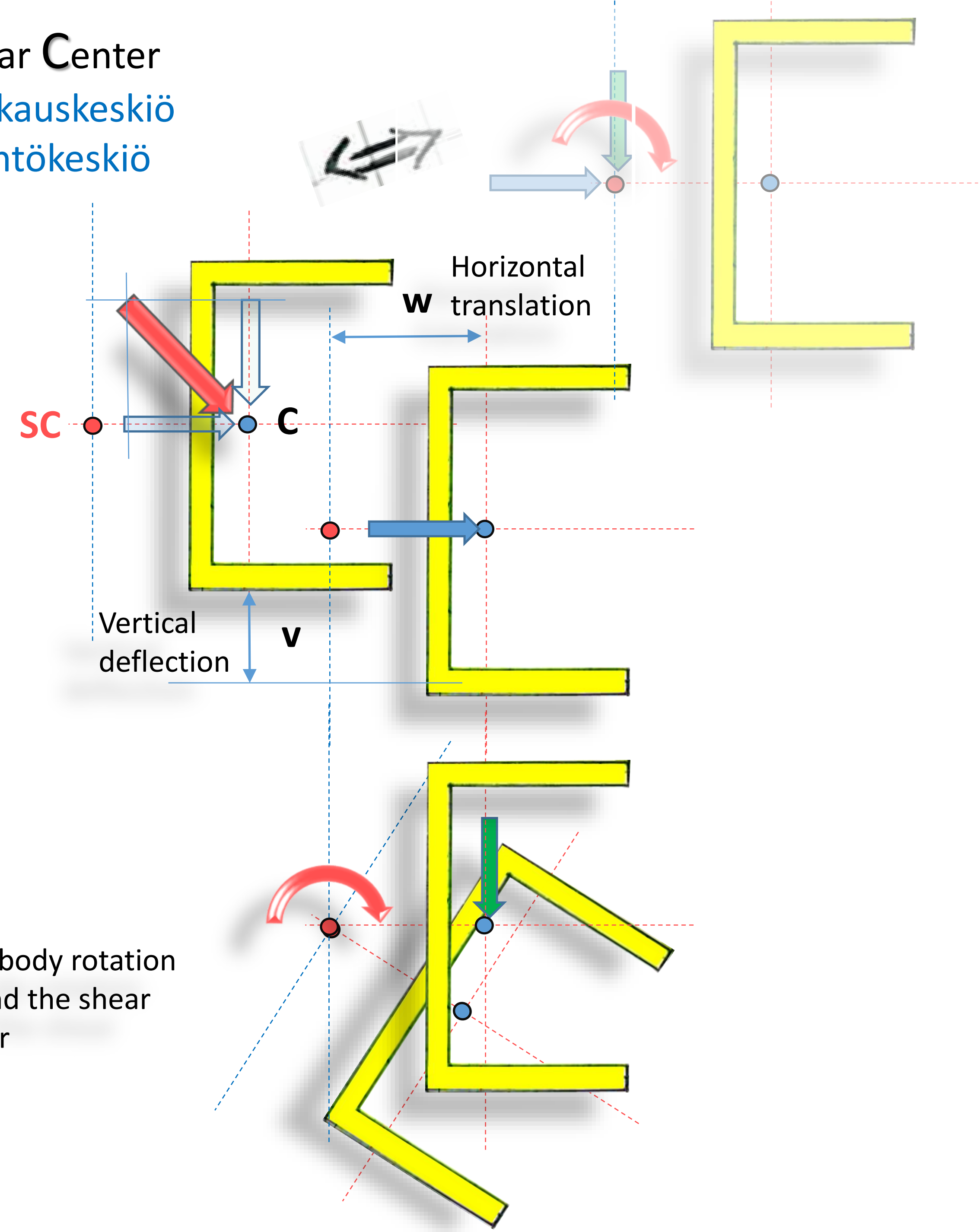
Shear center e = ?

$F \cdot e = T_f \cdot h - T_w \cdot e_s$

Shear Center

Leikkauskeskiö

Vääntökeskiö



Horizontal translation

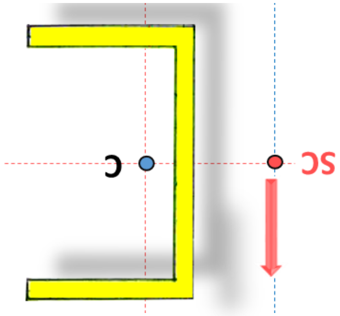
Vertical deflection

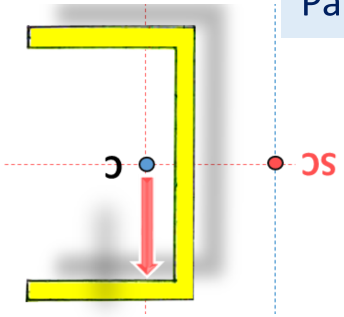
Rigid body rotation around the shear center

Let’s repeat this experiment in class

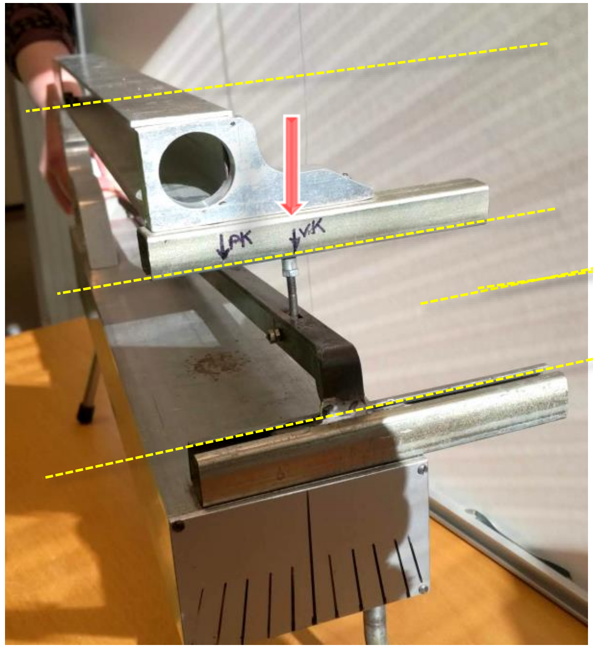
What is this high-tech device?

Let's repeat this experiment in class

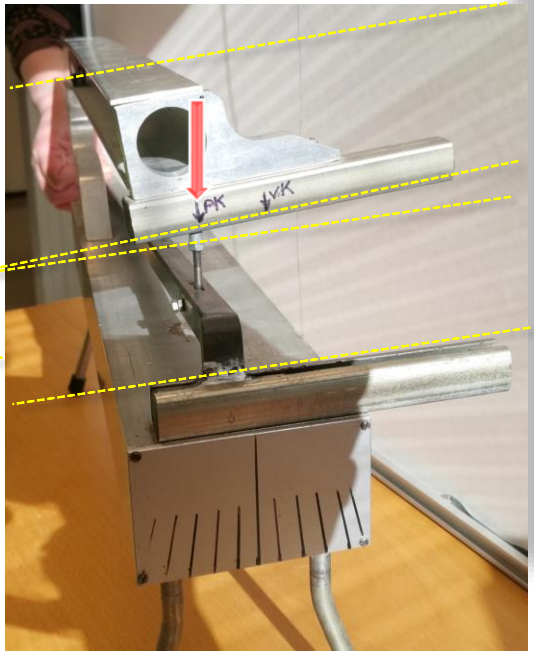




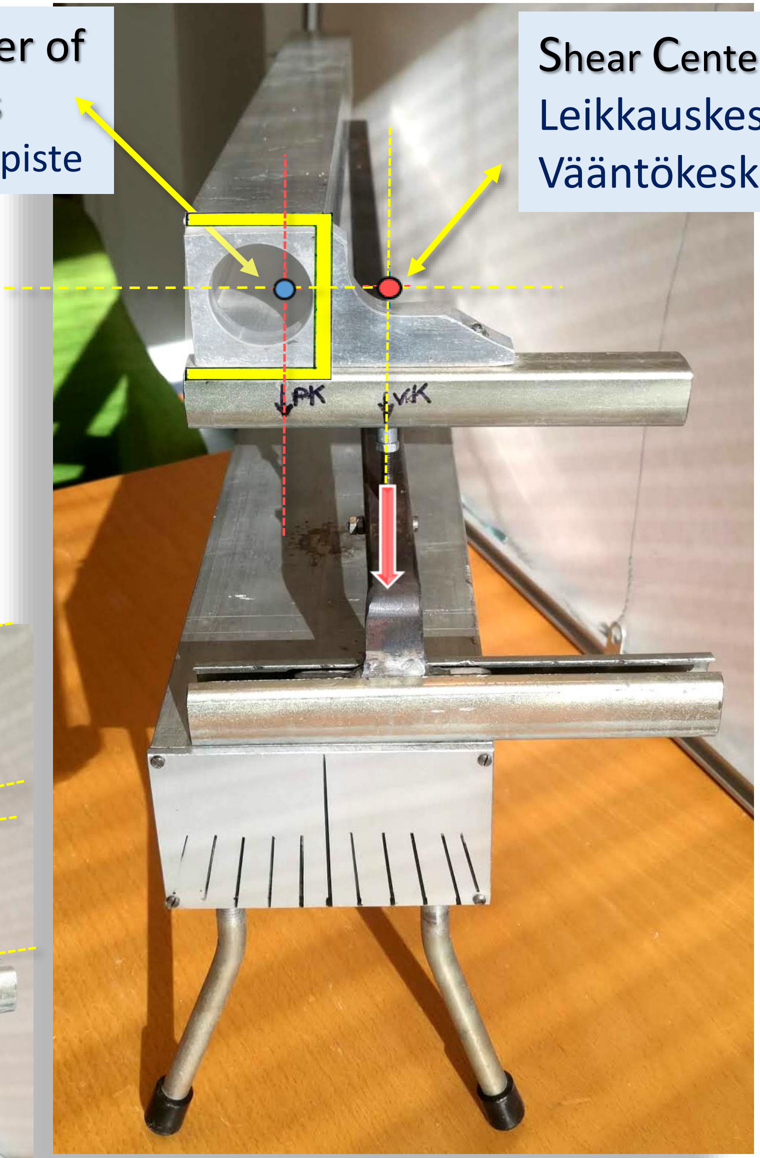
Bending only



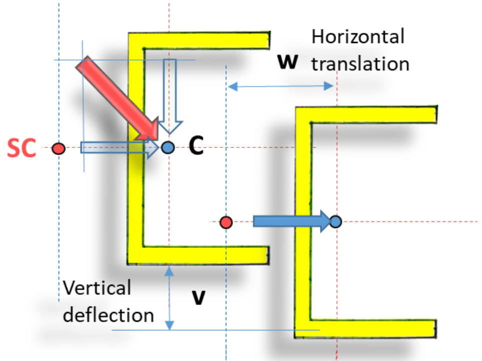
Bending & rotation

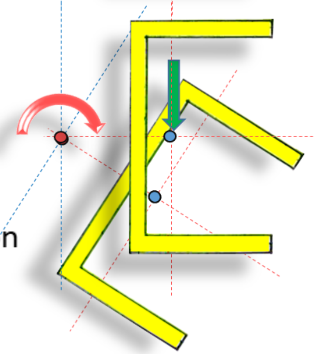


Center of mass
Painopiste



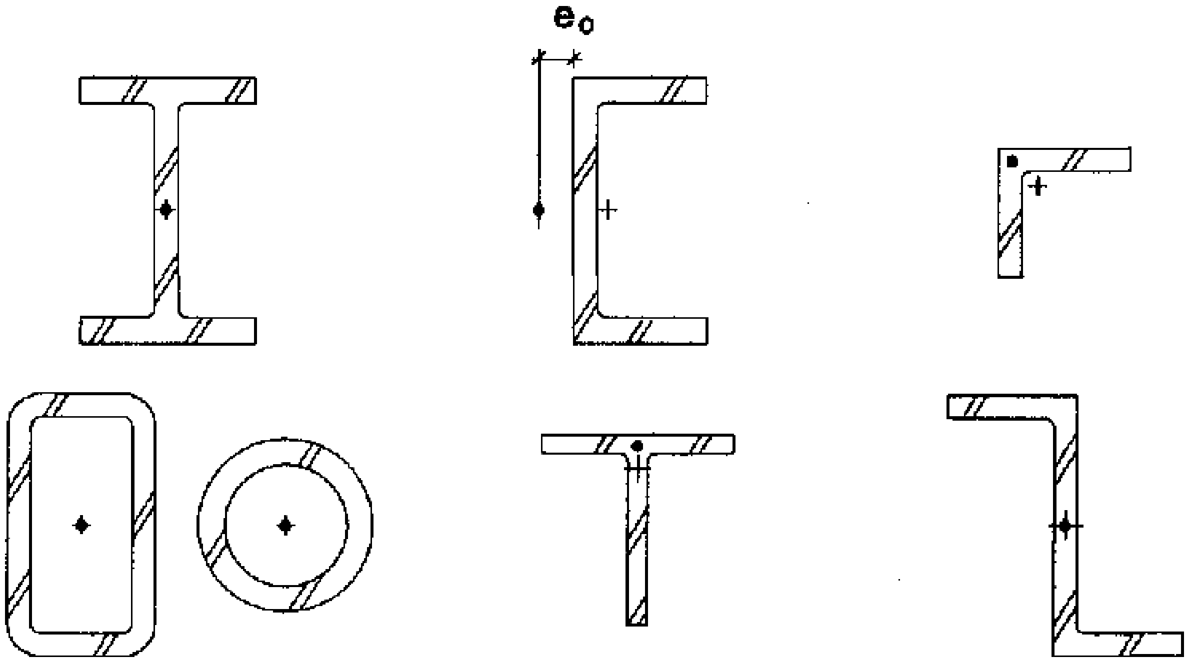
Shear Center
Leikkauskeskiö
Vääntökeskiö





Rigid body rotation around the shear center

What is warping? What is the shear center?

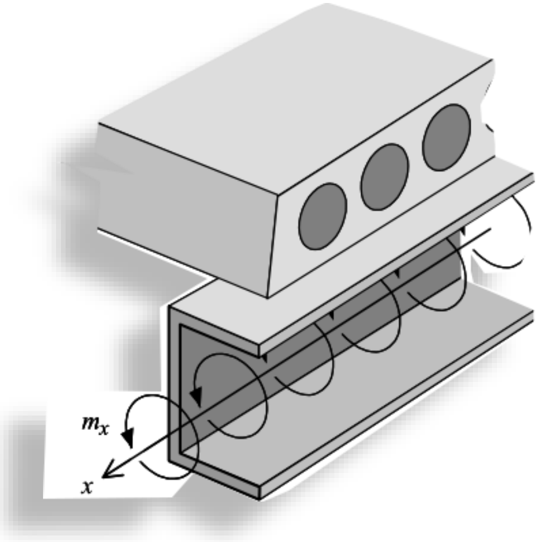


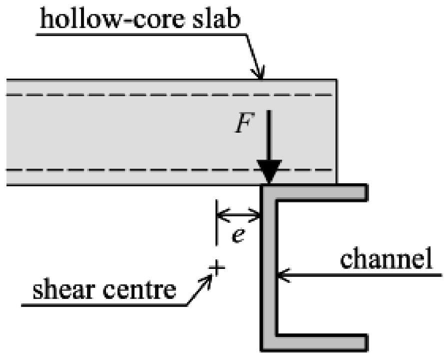
(a) doubly symmetric shapes, shear center and centroid coincide

(b) singly symmetric shapes

(c) unsymmetric shapes

+ centroid • shear center





$$m_x = -F\,e$$

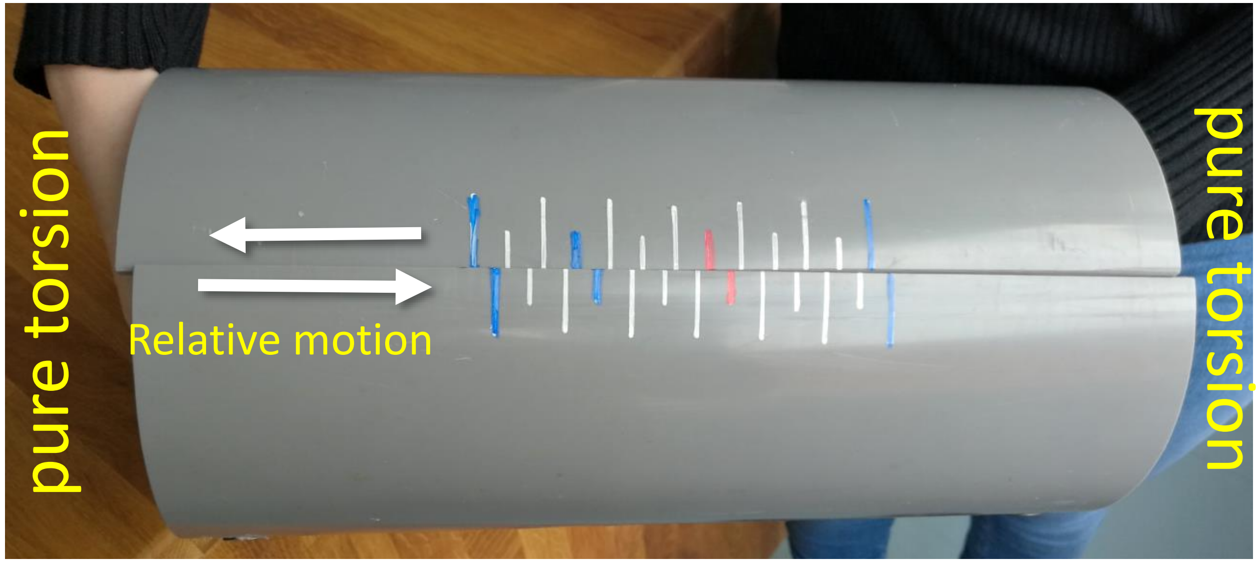
What is warping?

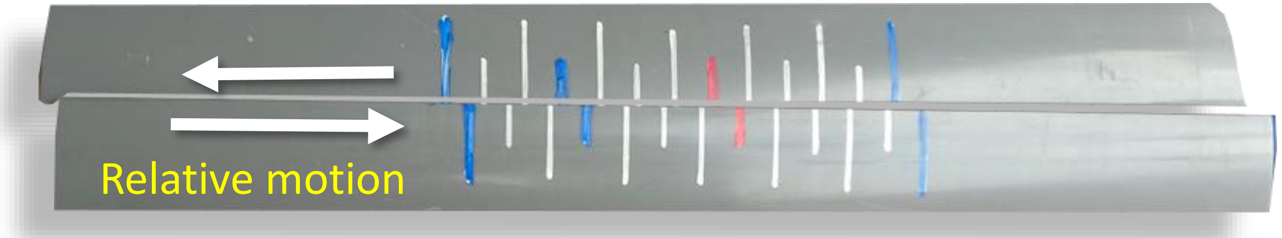
Poikkileikkauksen käyristyminen

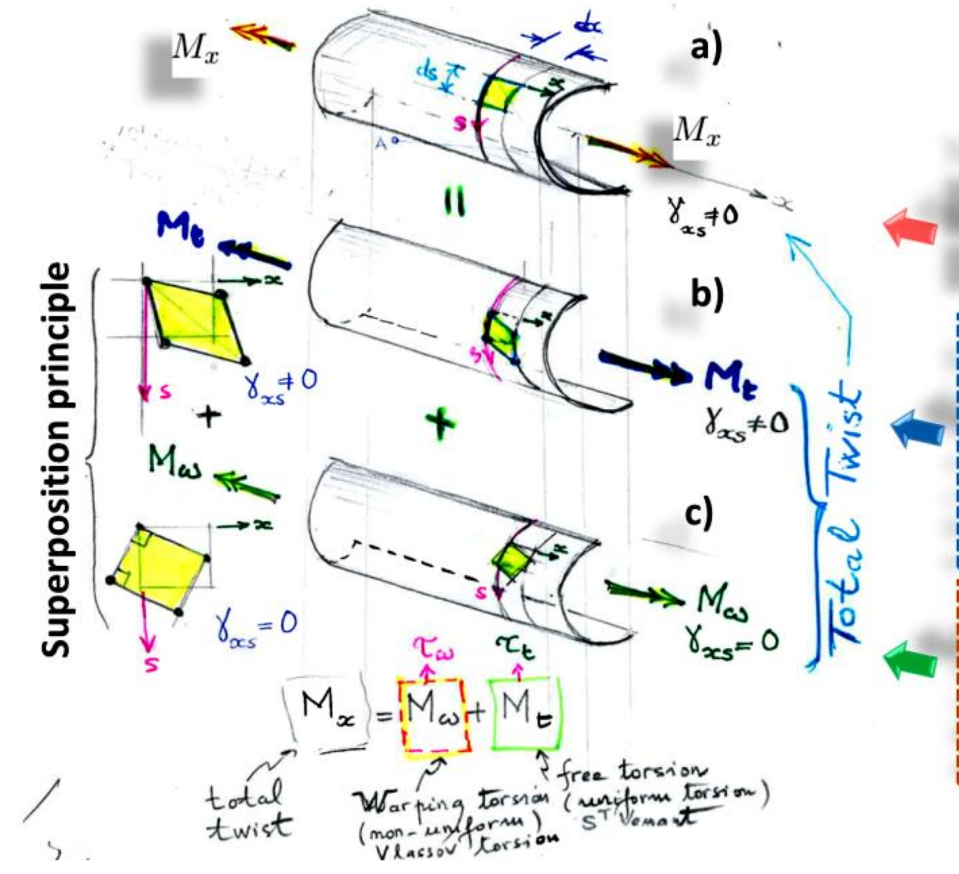
Warping is a **displacement** called deplanation which is an axial motion of points on a cross-section occurring **perpendicularly to this cross-section** and resulting from pure torsion

warping

Axial normal stresses result from restraining the warping







Total torsion

$$M_x = M_t + M_\omega$$

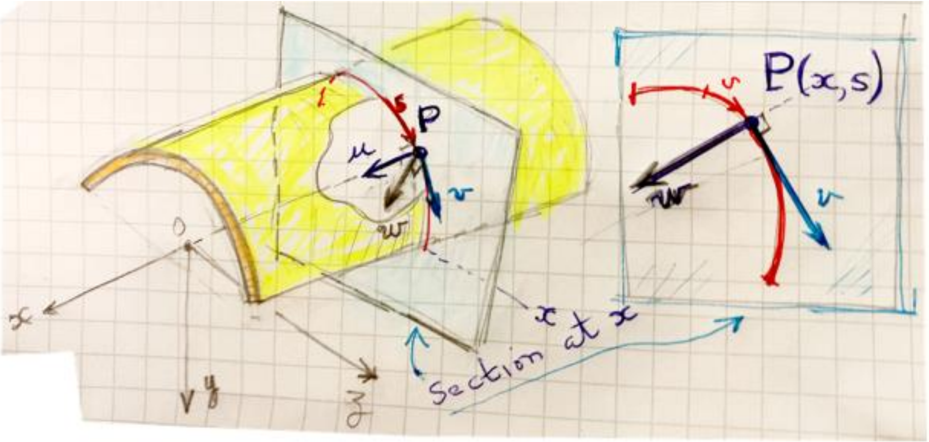
$$\rightarrow \tau = \tau_t + \tau_\omega$$

Saint Venant torsion
(uniform torsion, free torsion, out of plane warping uniform and not constrained)
Shearing – distortion of midplane

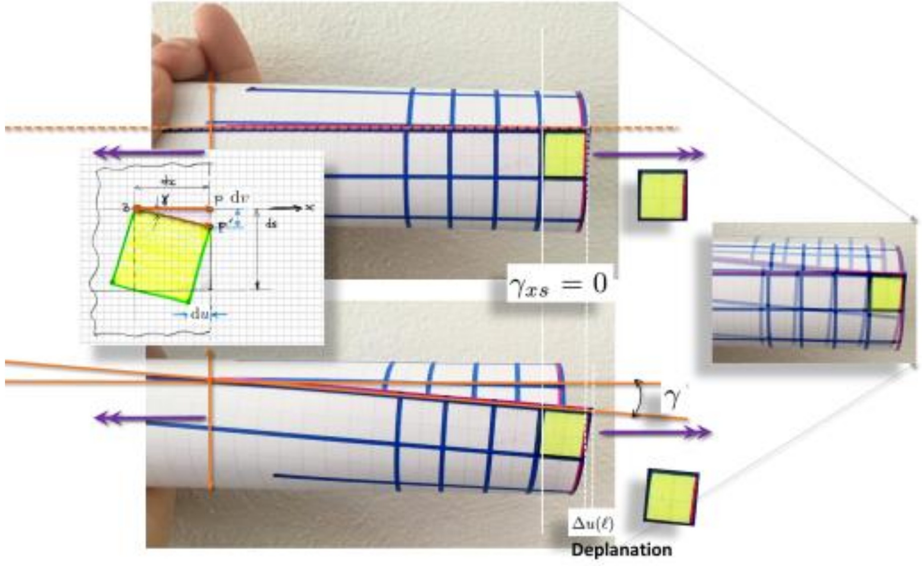
Vlassov torsion
(non-uniform torsion, warping torsion)
No shearing – no distortion

A schematic for the total torsion problem. Assume a twist moment M_x is acting at the ends of the shell-beam. By superposition we decompose the total twist moment as $M_x = M_t + M_\omega$. *Nota bene* that in this presentation of *Vlassov theory*, we consider only the contribution of torsion moment M_ω leading to *zero distortion of the mid-plane xs*.

Geometry of the motion of points on the cross-section



The two coordinate systems: global and local.



Zero shearing of the mid-plane (**Vlassov**'s kinematic hypothesis) - experimental evidence.

Geometry of the motion of points on the cross-section

$$\mathrm{d}\vec{\theta}(x) = [\mathrm{d}\theta_x, \quad 0, \quad 0]^{\mathrm{T}} \equiv \mathrm{d}\theta(x)\vec{i},$$

$$\mathrm{d}\vec{w} = \vec{PP'} = [\mathrm{d}\theta(x)\vec{i}] \times \vec{\rho}(s),$$

$$\vec{\rho}(s) = (y - y_A)\vec{j} + (z - z_A)\vec{k}.$$

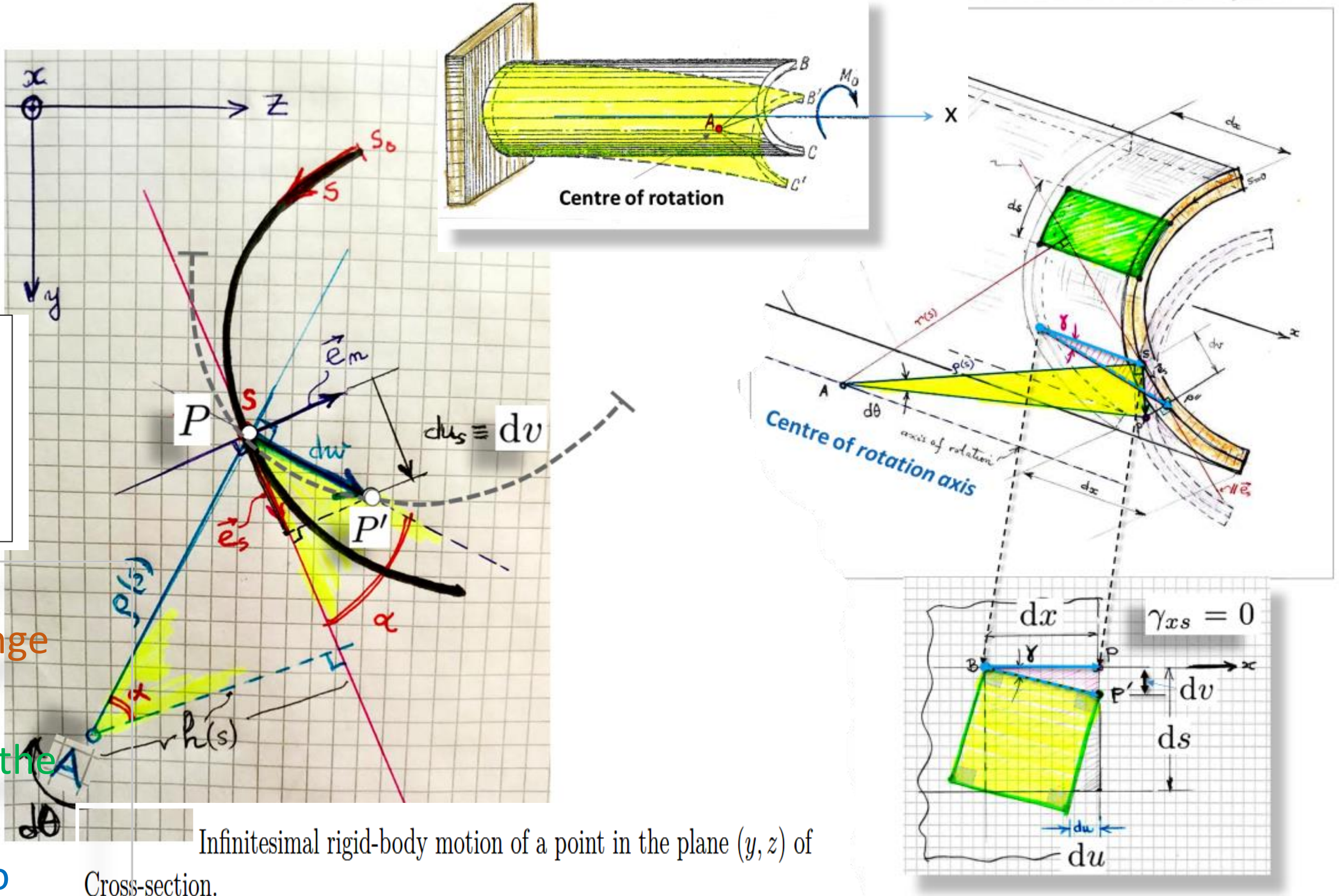
The main idea: Express the deplanation differential such that it can be integrated to obtain the axial displacement $u(x, s)$) at any point $P(x, s)$ of the section at on the mid-plane. In order to achieve this task, one has to find an expression for the deplanation differential $\mathrm{d}u(x, s)$), one should express $\mathrm{d}u$ in terms of $\mathrm{d}v$ which is at its turn expressed in terms of $\gamma(x) = \mathrm{d}v/\mathrm{d}x$, (Fig. 2.1). This is what we will do in the following.

Main geometric assumption:
The cross-section shape does not change (no distortion, ei vääristy)

So stiffeners should be added to keep the cross-section not distorted

Such assumption is quite impossible to achieve with very thin-walled cross sections.
This is one reason why, in practice computational tools are needed.

Kinematics of the displacement



Infinitesimal rigid-body motion of a point in the plane (y, z) of Cross-section.

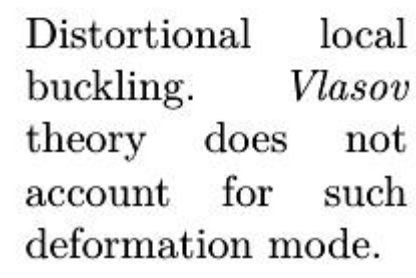
Validity of the Vlassov's theory (or model)

Note that local distortional buckling modes, for beams having very thin-walled cross-section (shell-beams), cannot be accounted with the Vlassov theory, since the cross-section geometry is distorted

Technically speaking, it is impossible anon-sense to try to put stiffeners to retain the cross-section shape as assumed in Vlassov's theory)

For a reliable analyses one should use computational technology and/or experimental approach

However, the computer compute and the engineer analyses. For that, the engineer needs courses of mechanics, in general, even with equations

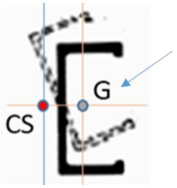


Distortional local buckling. *Vlasov* theory does not account for such deformation mode.

What is warping? What is the shear center?

The student should refer to the additional reading material and textbook for details on warping

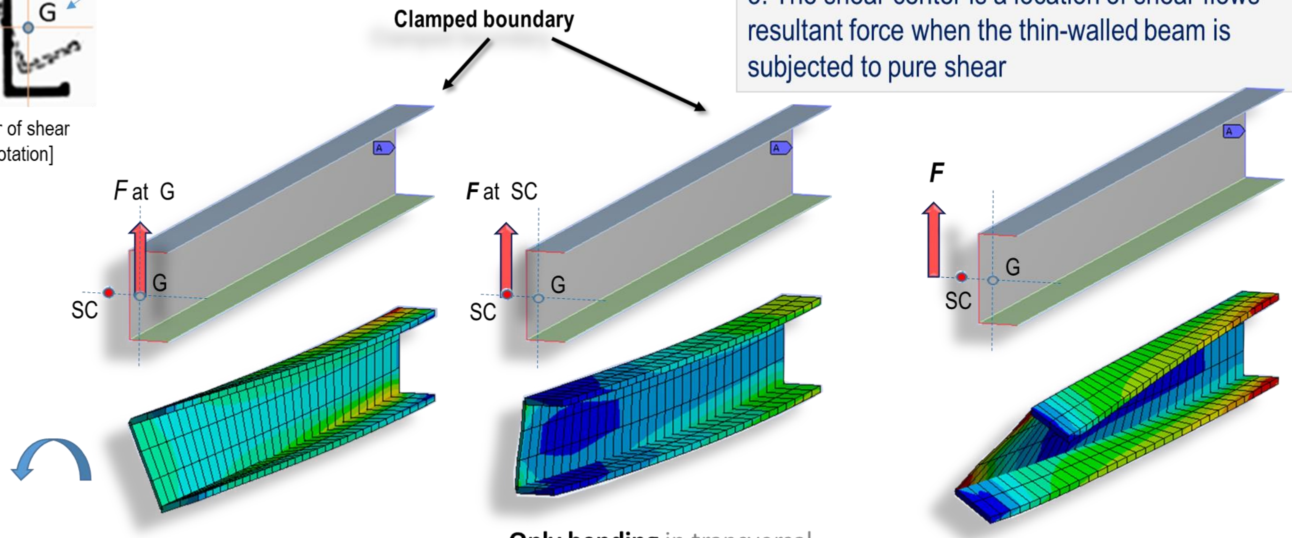
For the purpose of this course: use tables to find SC and Iw



(G) center of gravity [= area center (C)]

(SC) center of shear [center of rotation]

1. When transverse force applied at shear center it does not lead to torsion
2. The shear center (SC) is the center of rotation for a thin-walled section of beam subjected to pure torsion
3. The shear center is a location of shear flows resultant force when the thin-walled beam is subjected to pure shear



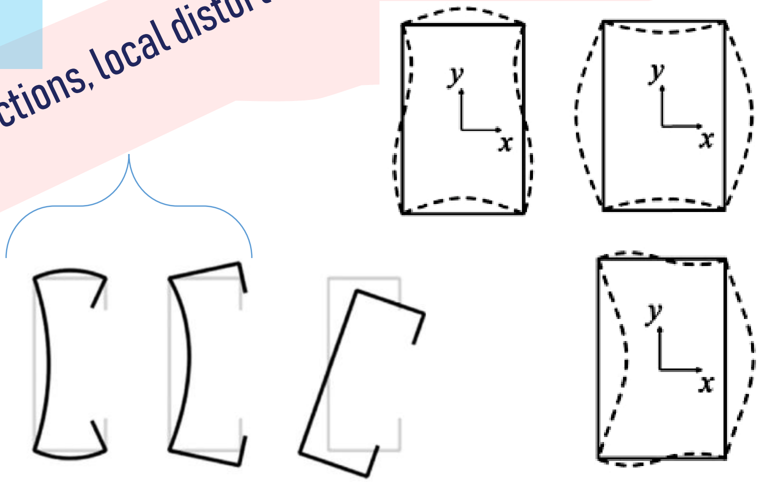
Combined and torsion

Only bending in transversal plane

Combined and torsion

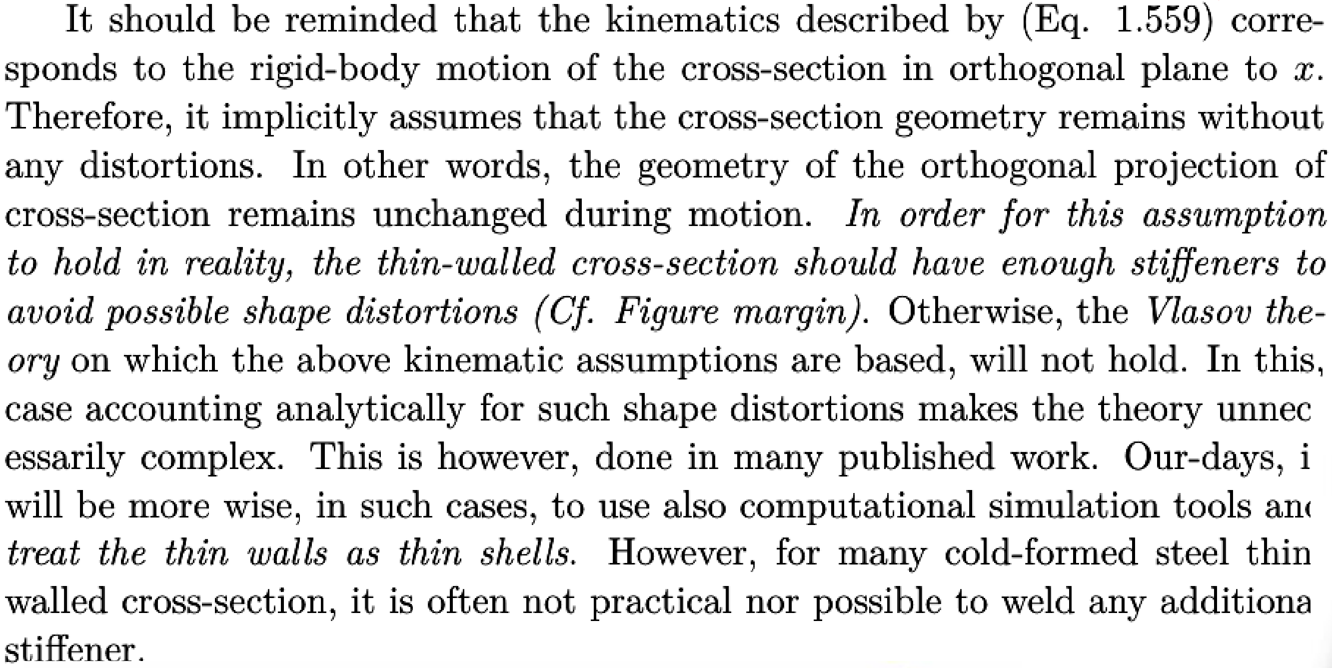
The cross-section shape is assumed not distorting in Vlassov's theory

For very thin-walled cross sections, local distortional deformation modes exits



Validity of the Vlassov's theory (or model)

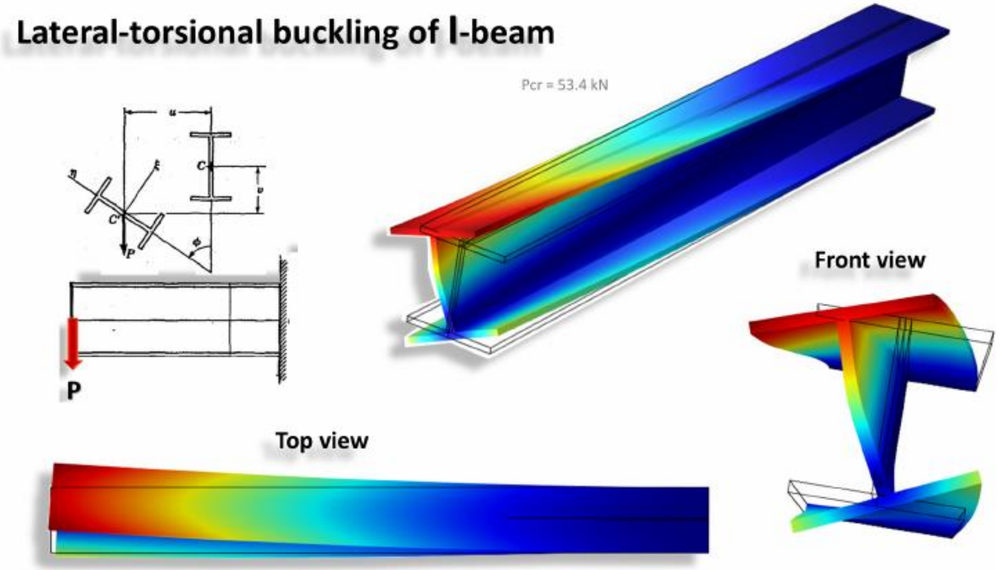
$$\vec{u} =(u - yv' - zw' - \omega\phi')\vec{i} + (v - (z - z_s)\phi)\vec{j} + (w + (y - y_s)\phi)\vec{k}.$$



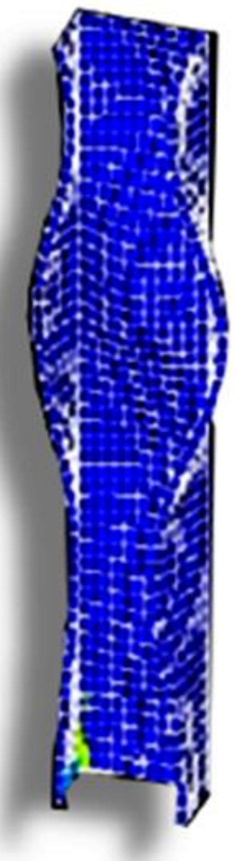
It should be reminded that the kinematics described by (Eq. 1.559) corresponds to the rigid-body motion of the cross-section in orthogonal plane to x. Therefore, it implicitly assumes that the cross-section geometry remains without any distortions. In other words, the geometry of the orthogonal projection of cross-section remains unchanged during motion. *In order for this assumption to hold in reality, the thin-walled cross-section should have enough stiffeners to avoid possible shape distortions (Cf. Figure margin).* Otherwise, the *Vlasov theory* on which the above kinematic assumptions are based, will not hold. In this, case accounting analytically for such shape distortions makes the theory unnecessarily complex. This is however, done in many published work. Our-days, i will be more wise, in such cases, to use also computational simulation tools anc *treat the thin walls as thin shells.* However, for many cold-formed steel thin walled cross-section, it is often not practical nor possible to weld any additiona stiffener.

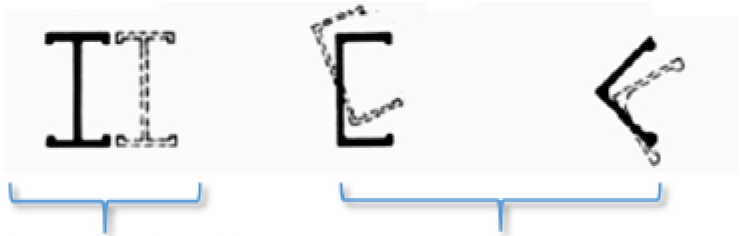
$$\begin{cases} u_Q(x) = u - yv' - zw' - \omega\phi', \\ v_Q(x) = v - (z - z_s)\phi, \\ w_Q(x) = w + (y - y_s)\phi, \end{cases}$$

Lateral-torsional buckling of I-beam

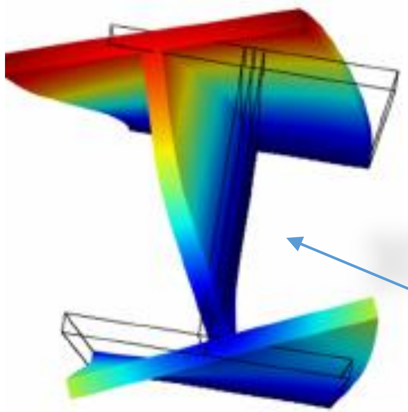


Local distortional modes Thin-shell





Front view

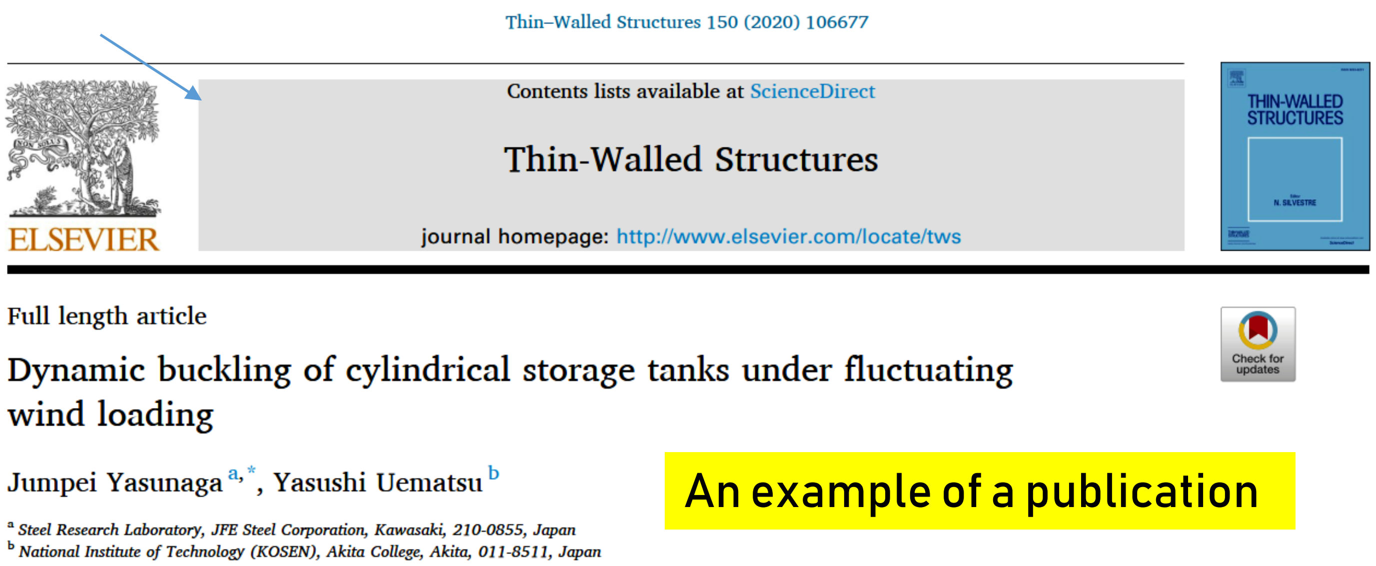


3D – FE-model

Distortion of the cross-section

Thin-walled structures are important for engineers

They deserve their own scientific journal



Thin–Walled Structures 150 (2020) 106677

Contents lists available at ScienceDirect

Thin-Walled Structures

journal homepage: http://www.elsevier.com/locate/tws

Full length article

Dynamic buckling of cylindrical storage tanks under fluctuating wind loading

Jumpei Yasunaga[a,*], Yasushi Uematsu[b]

[a] Steel Research Laboratory, JFE Steel Corporation, Kawasaki, 210-0855, Japan
[b] National Institute of Technology (KOSEN), Akita College, Akita, 011-8511, Japan

An example of a publication

ARTICLE INFO

Keywords:
Cylindrical storage tank
Wind tunnel experiment
Finite element method analysis
Buckling
Vibration
Time-history response analysis

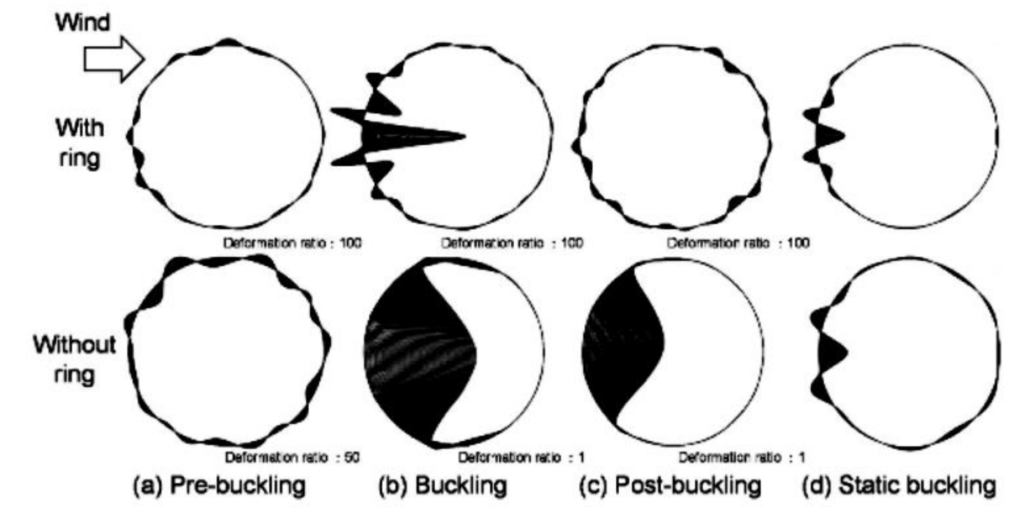
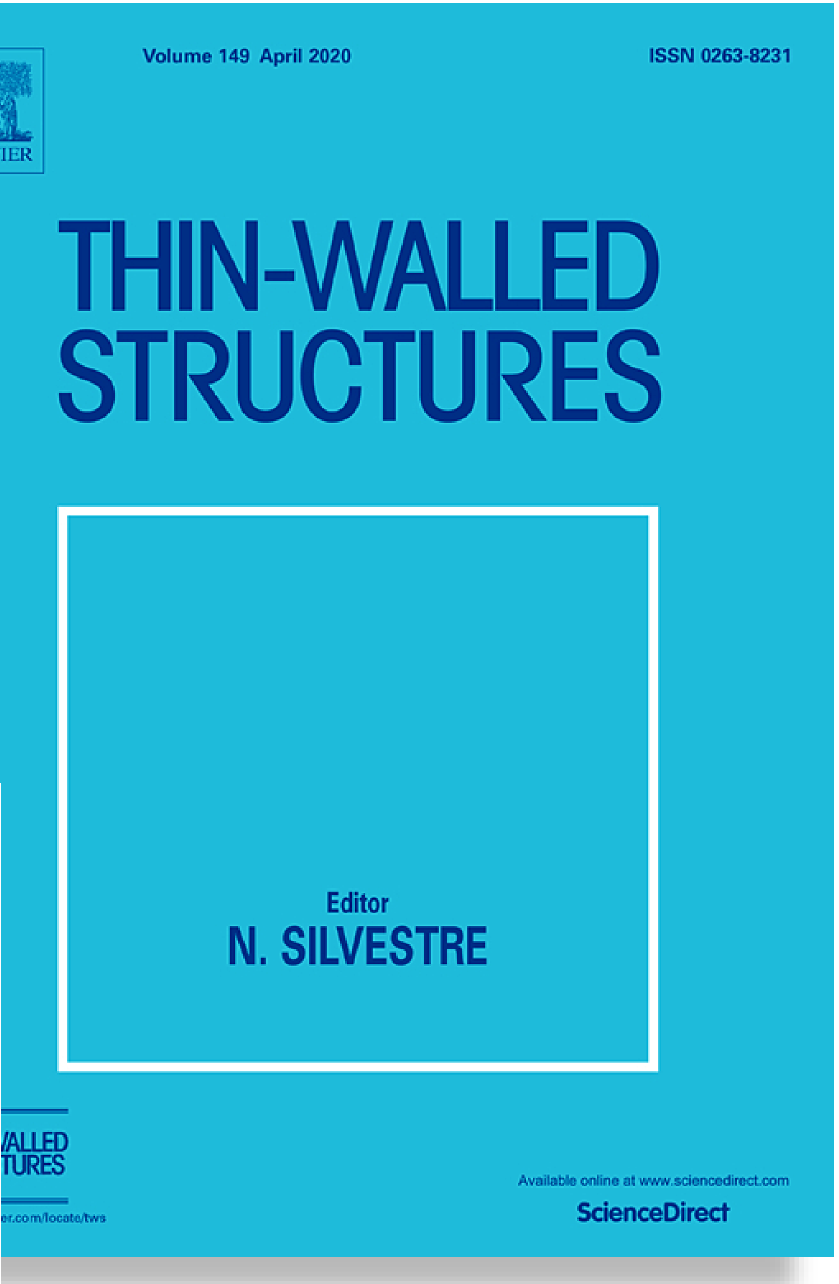
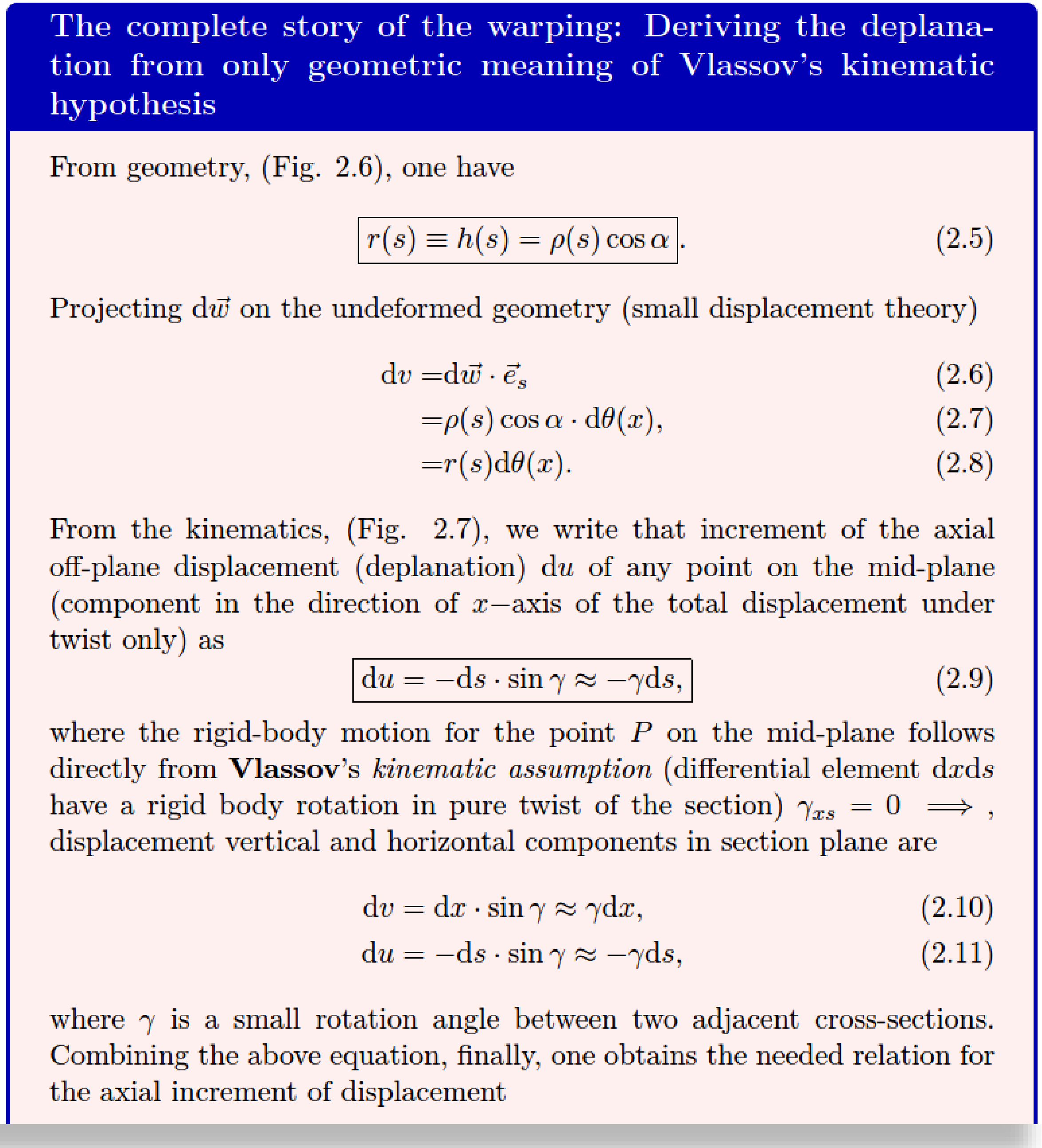


Fig. 11. Variation of deformation mode obtained from the time history response analysis and static buckling mode (top view of 3D model, steel tank, $H/D = 0.92$).



The sectorial coordinate $-\omega(s)$



The complete story of the warping: Deriving the deplanation from only geometric meaning of Vlassov's kinematic hypothesis

From geometry, (Fig. 2.6), one have

$$\boxed{r(s) \equiv h(s) = \rho(s)\cos\alpha}. \tag{2.5}$$

Projecting $\mathrm{d}\vec{w}$ on the undeformed geometry (small displacement theory)

$$\begin{aligned} \mathrm{d}v =& \mathrm{d}\vec{w}\cdot\vec{e}_s && (2.6)\\ =& \rho(s)\cos\alpha\cdot\mathrm{d}\theta(x), && (2.7)\\ =& r(s)\mathrm{d}\theta(x). && (2.8) \end{aligned}$$

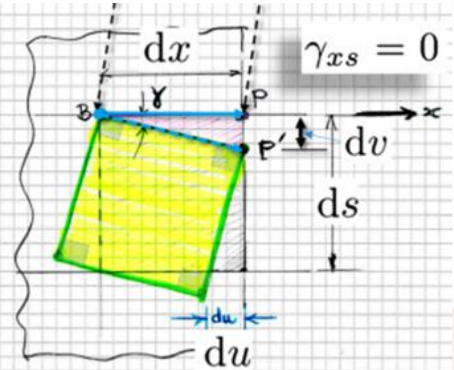
From the kinematics, (Fig. 2.7), we write that increment of the axial off-plane displacement (deplanation) $\mathrm{d}u$ of any point on the mid-plane (component in the direction of $x-$axis of the total displacement under twist only) as

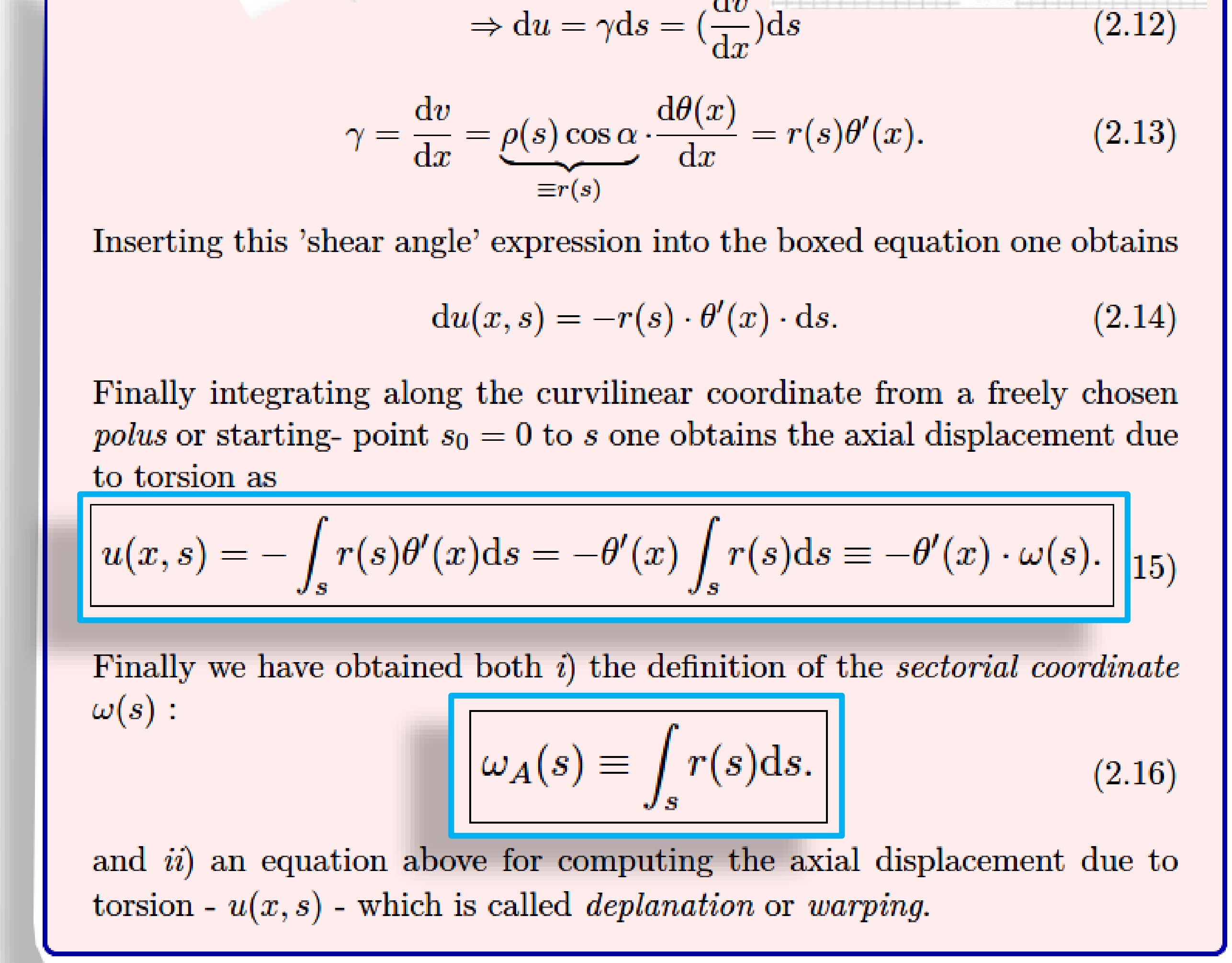
$$\boxed{\mathrm{d}u = -\mathrm{d}s\cdot\sin\gamma \approx -\gamma\mathrm{d}s,} \tag{2.9}$$

where the rigid-body motion for the point P on the mid-plane follows directly from **Vlassov**'s *kinematic assumption* (differential element $\mathrm{d}x\mathrm{d}s$ have a rigid body rotation in pure twist of the section) $\gamma_{xs} = 0 \implies$, displacement vertical and horizontal components in section plane are

$$\begin{aligned} \mathrm{d}v &= \mathrm{d}x\cdot\sin\gamma \approx \gamma\mathrm{d}x, && (2.10)\\ \mathrm{d}u &= -\mathrm{d}s\cdot\sin\gamma \approx -\gamma\mathrm{d}s, && (2.11) \end{aligned}$$

where γ is a small rotation angle between two adjacent cross-sections. Combining the above equation, finally, one obtains the needed relation for the axial increment of displacement





$$\Rightarrow \mathrm{d}u = \gamma\mathrm{d}s = (\frac{\mathrm{d}v}{\mathrm{d}x})\mathrm{d}s \tag{2.12}$$

$$\gamma = \frac{\mathrm{d}v}{\mathrm{d}x} = \underbrace{\rho(s)\cos\alpha}_{\equiv r(s)}\cdot\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} = r(s)\theta'(x). \tag{2.13}$$

Inserting this 'shear angle' expression into the boxed equation one obtains

$$\mathrm{d}u(x,s) = -r(s)\cdot\theta'(x)\cdot\mathrm{d}s. \tag{2.14}$$

Finally integrating along the curvilinear coordinate from a freely chosen *polus* or starting- point $s_0 = 0$ to s one obtains the axial displacement due to torsion as

$$\boxed{u(x,s) = -\int_s r(s)\theta'(x)\mathrm{d}s = -\theta'(x)\int_s r(s)\mathrm{d}s \equiv -\theta'(x)\cdot\omega(s).} \quad 15)$$

Finally we have obtained both *i)* the definition of the *sectorial coordinate* $\omega(s)$:

$$\boxed{\omega_A(s) \equiv \int_s r(s)\mathrm{d}s.} \tag{2.16}$$

and *ii)* an equation above for computing the axial displacement due to torsion - $u(x,s)$ - which is called *deplanation* or *warping*.

Normal Stress resultant from Vlassov twist

$$\boxed{u(x,s) = -\int_s r(s)\theta'(x)\mathrm{d}s = -\theta'(x)\int_s r(s)\mathrm{d}s \equiv -\theta'(x)\cdot\omega(s).}$$

$$\boxed{\omega_A(s) \equiv \int_s r(s)\mathrm{d}s.}$$

$$\boxed{\epsilon_{xx}(x,s) = \frac{\mathrm{d}}{\mathrm{d}x}u(x,s) = -\theta''(x)\cdot\omega(s)}$$

$$\boxed{\sigma_\omega(x,s) = E\epsilon_{xx}(x,s) = -E\omega(s)\theta''(x)}$$

warping normal stress or Vlassov's normal stress

$$\int_A \sigma_\omega(x,s)\mathrm{d}A = -\int_A E\omega(s)\theta''(x)\mathrm{d}A = -E\theta''(x)\int_A \omega(s)\mathrm{d}A = 0.$$

$$\int_A \omega(s)\mathrm{d}A \equiv S_\omega$$

sectorial static moment of the cross-section.

sectorial linear moments

$$\int_A \sigma_\omega y\mathrm{d}A = -E\theta''(x)\int_A \omega(s)y(s)\mathrm{d}A = 0,$$

$$\int \sigma_\omega z\mathrm{d}A = -E\theta''(x)\int \omega(s)z(s)\mathrm{d}A = 0.$$

$$S_{\omega y} = \int_A \omega(s)y(s)\mathrm{d}A,$$

$$S_{\omega z} = \int_A \omega(s)z(s)\mathrm{d}A.$$

Bi-moment

$$\boxed{B(x) = \int_A \sigma_{xx}\omega\mathrm{d}A = -E\theta''(x)\int_A \omega^2(s)\mathrm{d}A.}$$

$$I_\omega = \int_A \omega^2(s)\mathrm{d}A.$$

sectorial moment of inertia

Shear stresses

$$\tau = \tau_t + \tau_\omega,$$

$$\tau_t = \frac{M_t}{I_t}\cdot t(s)$$

$$\boxed{\sigma_\omega(x,s) = B(x)\cdot\frac{\omega(s)}{I_\omega}.}$$

$$\boxed{\sigma_{xx} = M_y\cdot\frac{z(s)}{I_y}.}$$

Shear stresses

$$\tau = \tau_t + \tau_\omega,$$

$$\tau_t = \frac{M_t}{I_t} \cdot t(s)$$

$$\sigma_\omega(x,s) = -E\omega(s)\theta''(x) = \frac{B(x)\omega(s)}{I_\omega},$$

$$\tau_\omega(x,s) = \frac{B'(x)S_\omega(s)}{t(s)I_\omega} = \frac{M_\omega(x)S_\omega(s)}{t(s)I_\omega},$$

Vlassov's theory,

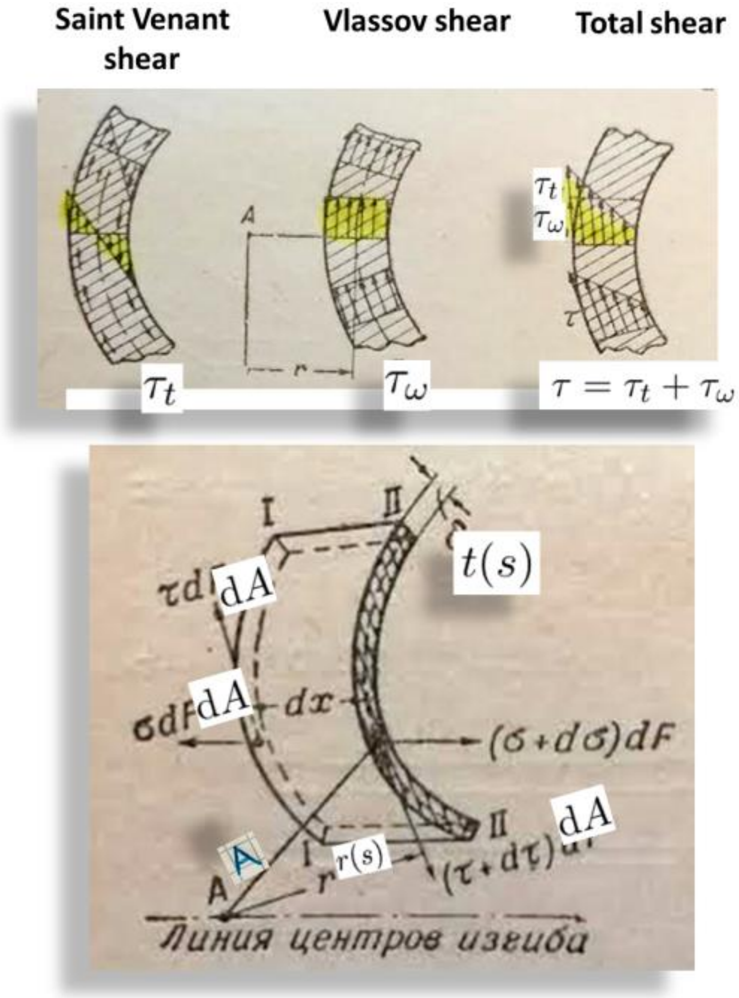
$$S_\omega = \int_A \omega(s)\mathrm{d}A = \int_s \omega(s)t(s)\mathrm{d}s,$$

$$I_\omega = \int_A \omega^2(s)\mathrm{d}A = \int_s \omega^2(s)t(s)\mathrm{d}s,$$

bi-moment and the warping moment (torsional)

$$\boxed{B(x) = -EI_\omega\theta''(x), \quad M_\omega = B' = -EI_\omega\theta''',}$$

$$\boxed{-EI_\omega\theta^{(IV)}(x) + GI_t\theta'' = m,}$$



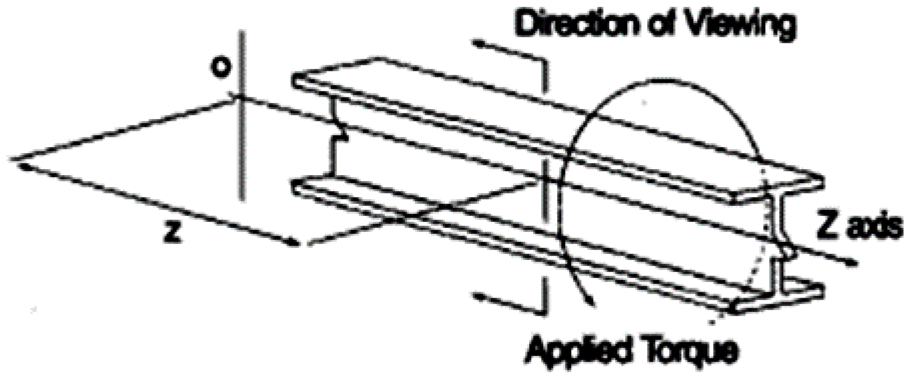
Shear stresses from free-torsion (Saint Venant) and non-uniform torsion (Vlassov). (these figures were adapted from **Belaiev** (1959).)

constitutive relations

$$M_t = GI_t\theta', \quad B = -EI_\omega\theta'', \quad M_\omega = B' = -EI_\omega\theta'''.$$

The phenomenon

Torsional stresses

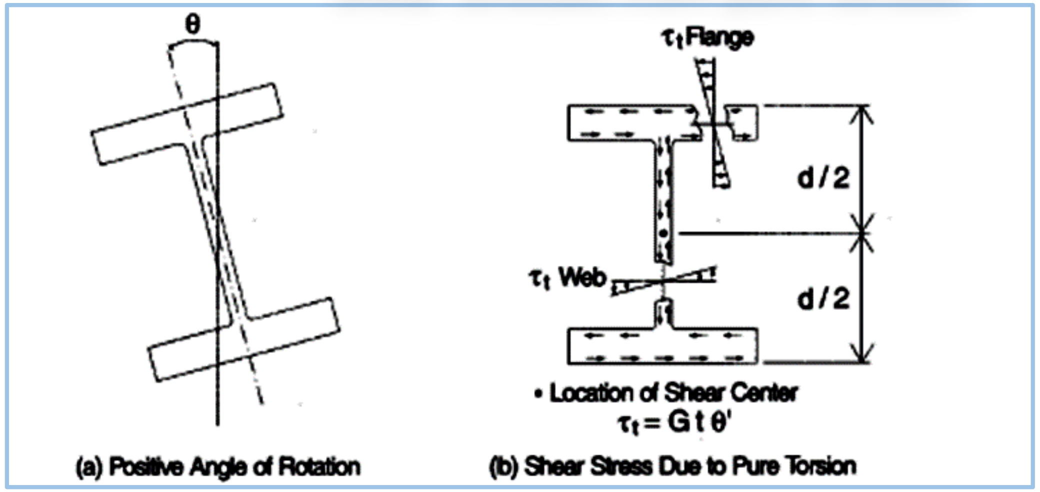


Pure torsion

Puhdas vääntö

Saint Venant's torsion

Shear stresses from pure torsion



(a) Positive Angle of Rotation

(b) Shear Stress Due to Pure Torsion

warping torsion

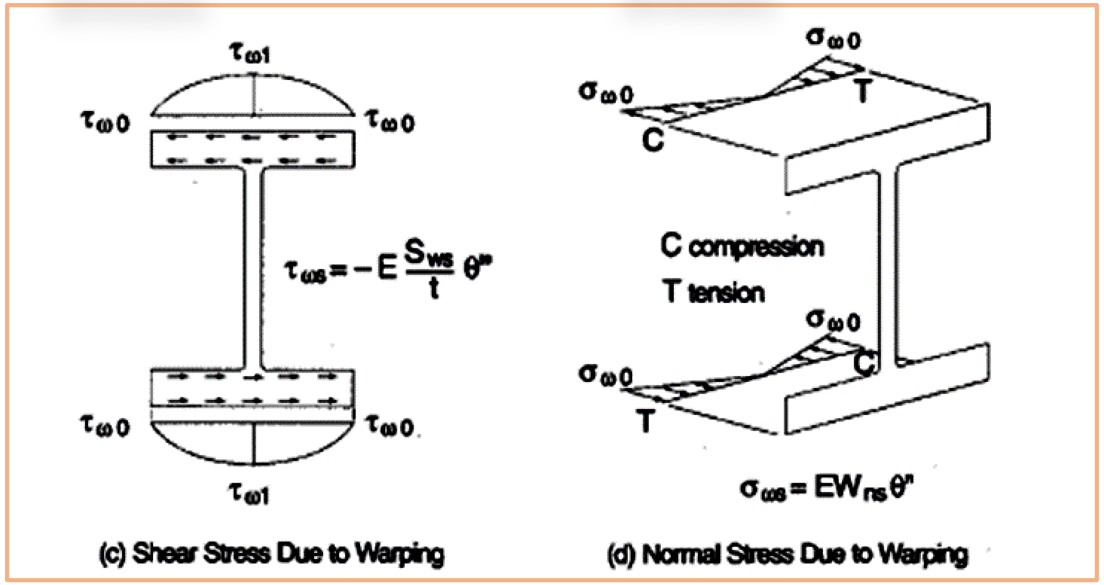
Non-uniform torsion

Estetty vääntö

Vlassov's torsion

Warping shear stresses

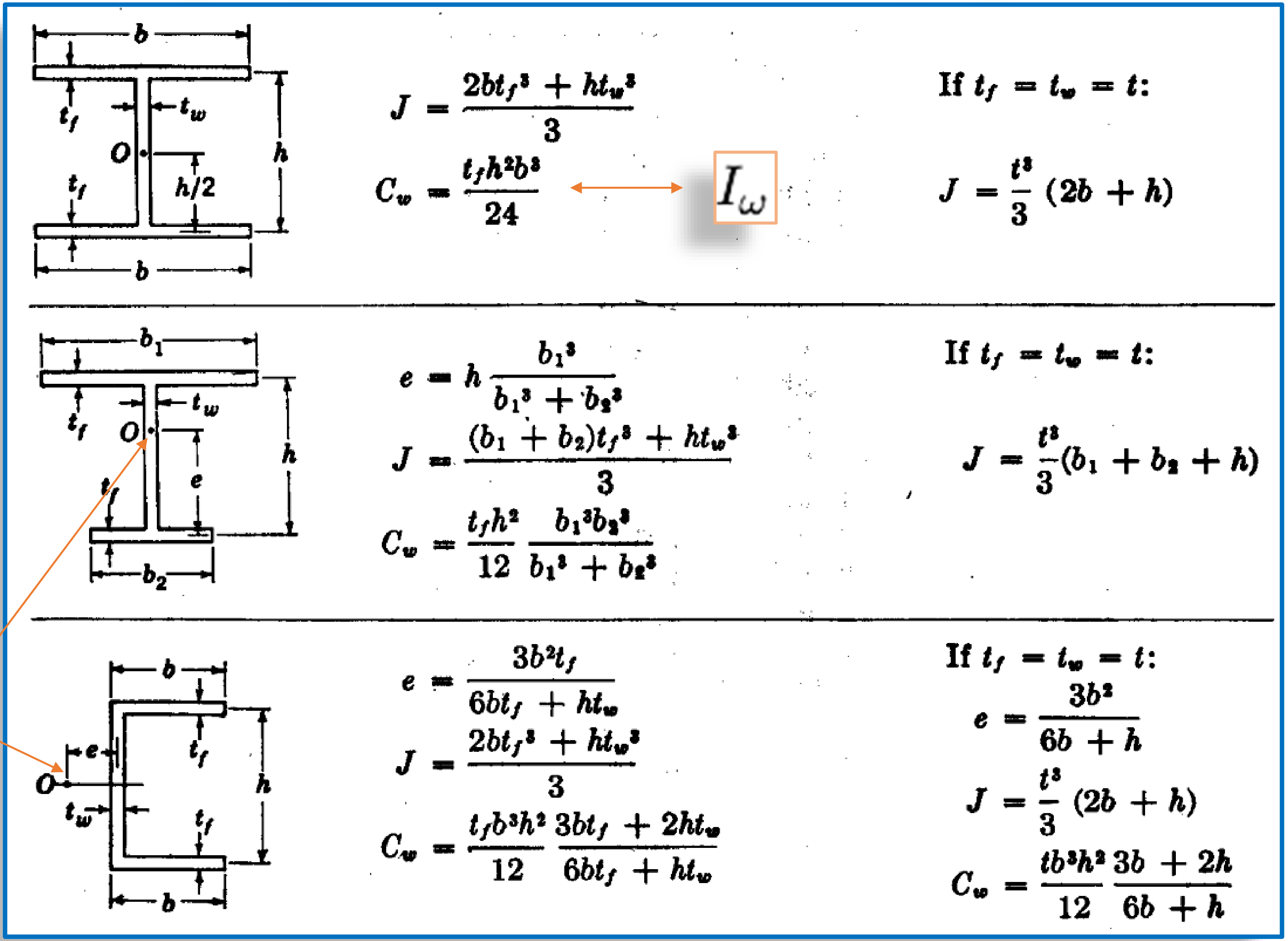
Warping normal stresses



(c) Shear Stress Due to Warping

(d) Normal Stress Due to Warping

Example of table giving shear center and the warping inertia moment I_ω



	$J = \dfrac{2bt_f^3 + ht_w^3}{3}$ $C_w = \dfrac{t_f h^2 b^3}{24} \longleftrightarrow I_\omega$	If $t_f = t_w = t$: $J = \dfrac{t^3}{3}(2b + h)$
	$e = h\dfrac{b_1^3}{b_1^3 + b_2^3}$ $J = \dfrac{(b_1 + b_2)t_f^3 + ht_w^3}{3}$ $C_w = \dfrac{t_f h^2}{12}\dfrac{b_1^3 b_2^3}{b_1^3 + b_2^3}$	If $t_f = t_w = t$: $J = \dfrac{t^3}{3}(b_1 + b_2 + h)$
	$e = \dfrac{3b^2 t_f}{6bt_f + ht_w}$ $J = \dfrac{2bt_f^3 + ht_w^3}{3}$ $C_w = \dfrac{t_f b^3 h^2}{12}\dfrac{3bt_f + 2ht_w}{6bt_f + ht_w}$	If $t_f = t_w = t$: $e = \dfrac{3b^2}{6b + h}$ $J = \dfrac{t^3}{3}(2b + h)$ $C_w = \dfrac{tb^3h^2}{12}\dfrac{3b + 2h}{6b + h}$

Shear Center

- Now to stay realistic (6 weeks stability course) we will use tables for theses cross-section constants
- **Torsion topic** is a wide subject. Torsion of beams with thin-walled open-cross sections deserves, at least, a full three-weeks course by itself

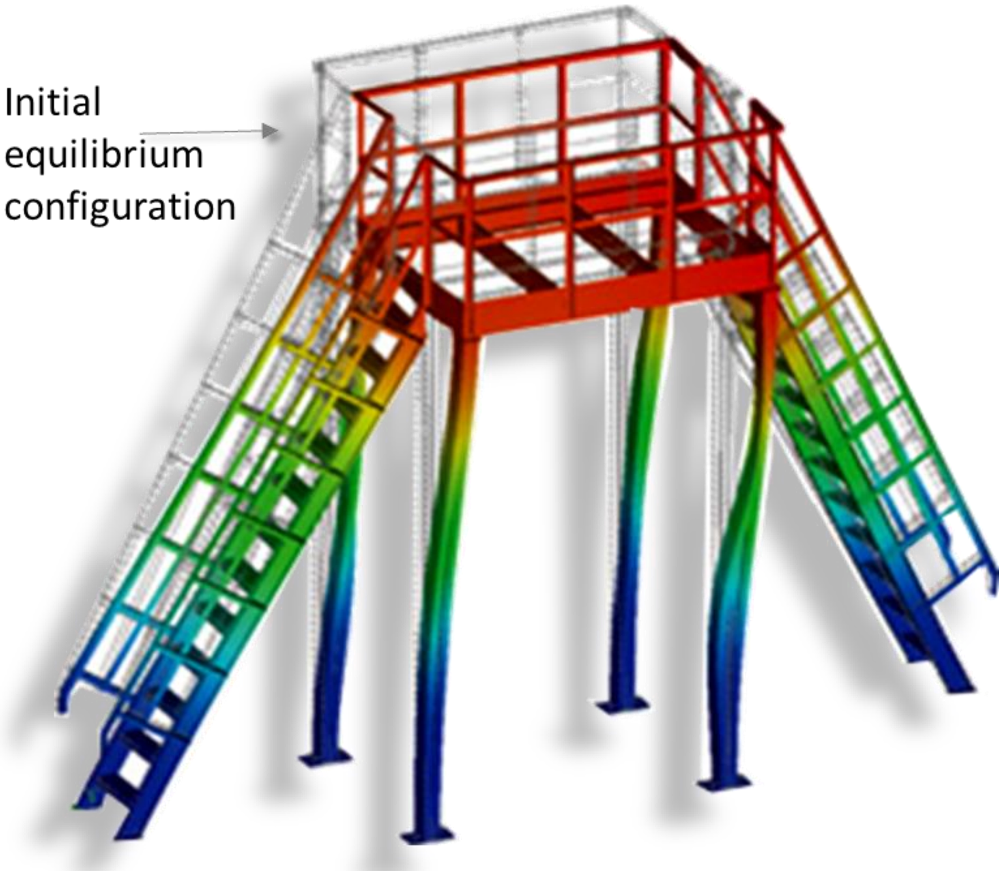
Main geometric assumption of the Vlassov's theory:

The cross-section shape does not change (no distortion, ei vääristy)

So stiffeners should be added to keep the cross-section not distorted

Such assumption is quite impossible to achieve with very thin-walled cross sections

This is one reason why, in practice computational tools are needed to perform reliable stability analysis and GNA for very thin-walled shell-beams …



Deriving the linear **equations** of loss **of stability – The IDEA**

The seed:

Initial stresses from pre-buckled configuration

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\text{linear part of strain increments}} + \underbrace{\underbrace{\int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part of strain increments}}}_{U(\sigma^0,\epsilon_2)} - \underbrace{\Delta W_{ext}(P,\epsilon_2)}_{\text{such that } \notin U(\sigma^0,\epsilon_2)}$$

Non-linear strains (quadratic part) from slightly buckled configuration

$$\epsilon_{ij} = \frac{1}{2}(\overbrace{u_{i,j}+u_{j,i}}+\overbrace{u_{k,i}u_{k,j}})$$ - using Einstein summation rule

$$\mathbf{E} = \frac{1}{2}\left((\nabla_X\mathbf{u})^{\mathrm{T}} + \nabla_X\mathbf{u} + (\nabla_X\mathbf{u})^{\mathrm{T}}\cdot\nabla_X\mathbf{u}\right),$$

$$\hat{\mathbf{u}} \equiv \delta\mathbf{u} \equiv \underbrace{\mathbf{u} = [u,v,w]^{\mathrm{T}}}_{\text{this stands for changes away from critical point}} \rightarrow \text{strain changes} \rightarrow \epsilon$$

$u^0 + \delta u \equiv u^0 + \hat{u}$, dead load (P^0) is kept constant during variation

The water & the sun:

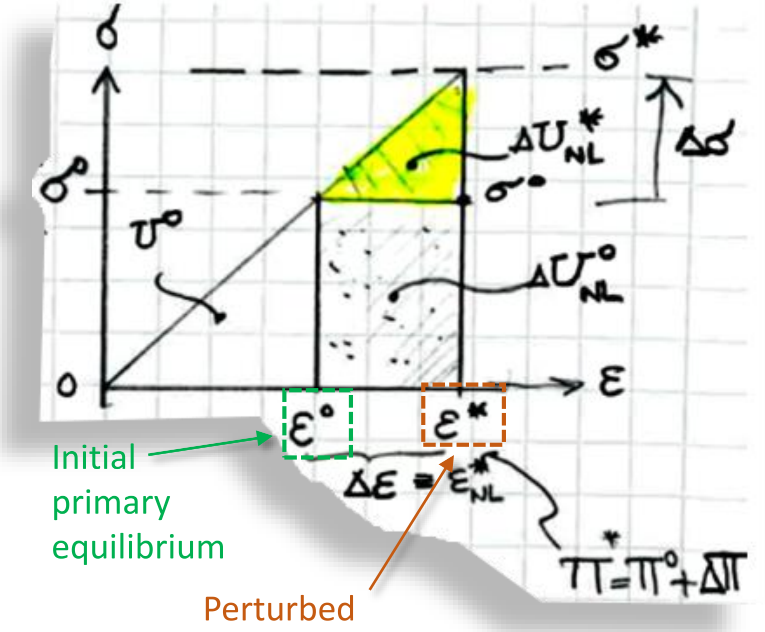
$$\boxed{\delta(\Delta\Pi) = \delta[\delta^2\Pi|_{\mathbf{u}^0}] = 0, \forall\delta\mathbf{u}, \quad \text{kin. admissible,}}$$

$$\Downarrow$$

The fruit:

The linear equations of loss of **stability**
Eigen-value problem (BVP)

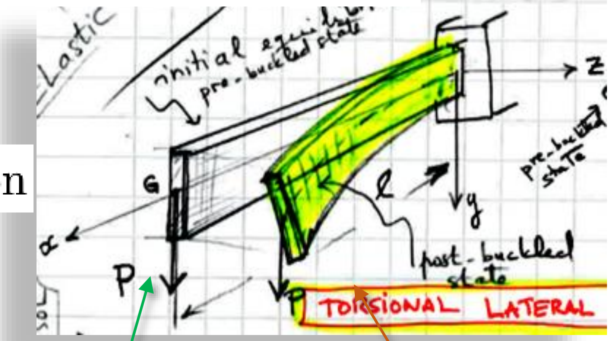
It is the **solution** of these equations which **provides** the **buckling load** and the corresponding mode



Initial primary equilibrium

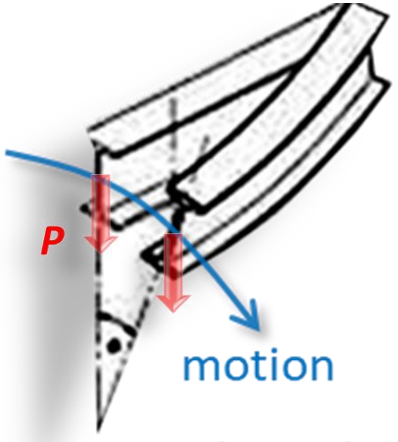
Perturbed equilibrium

no warping $I_\omega \approx 0$



Initial primary equilibrium

Perturbed (adjacent) equilibrium



Lateral torsional buckling

Equation example

$$EI_\omega\varphi^{(4)} - GI_v\varphi'' - \frac{(M_x^0)^2}{EI_y}\varphi = 0,$$

Total potential energy

Non-linear strains (quadratic part) from slightly buckled configuration

itial stresses from pre-buckled configuration

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\text{linear part of strain increments}} + \underbrace{\underbrace{\int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part of strain increments}}}_{U(\sigma^0,\epsilon_2)} - \underbrace{\Delta W_{ext}(P,\epsilon_2)}_{\text{such that } \notin U(\sigma^0,\epsilon_2)}$$

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2 \mathrm{d}x + \int_0^\ell \sigma_x^0 A\,[\underbrace{\frac{1}{2}(v')^2}_{\epsilon_2}]\,\mathrm{d}x, \quad -P = N^0(x) < 0$$

← Flexural buckling

Pure torsion

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t\phi'^2 \mathrm{d}x + \int_0^\ell\int_A \sigma_x^0\epsilon_2 \mathrm{d}A\mathrm{d}x,$$

Initial stresses: $\sigma_x^0 = \frac{M_z^0}{I_z}y$

no warping $I_\omega \approx 0$

New terms

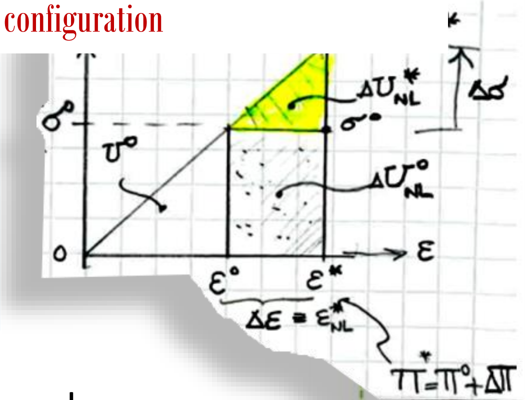
Warping

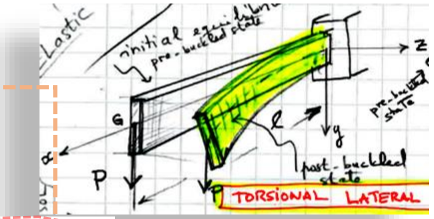
$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t\phi'^2 \mathrm{d}x + \boxed{\frac{1}{2}\int_0^\ell EI_\omega\phi''^2 \mathrm{d}x} +$$

$$+ \int_0^\ell\int_A \sigma_x^0 \underbrace{\frac{1}{2}[(w_Q')^2 + (v_Q')^2]}_{\epsilon_2} \mathrm{d}A\mathrm{d}x$$

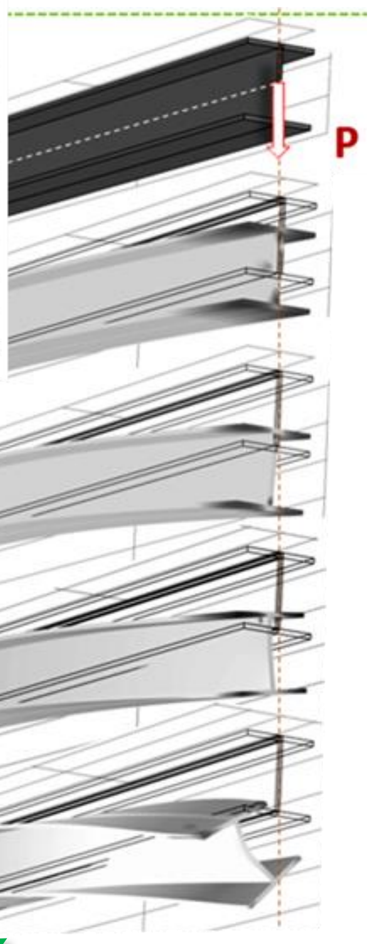
Flexion in both directions

Initial stresses: $\tau_{xy}^0(x,y) = \frac{Q_y^0(x)}{bI_z}S_z^+(y)$





Load increases



Stability loss:

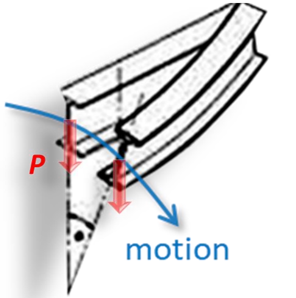
$\Pi = \Pi^0 + \Delta\Pi$

$\delta(\Delta\Pi) = 0$, $P = P_{cr,min}$

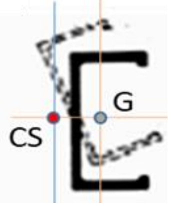
⟹ **buckling load** and mode

Beams having **thin-walled open cross-sections** can have **torsional** modes of **stability loss** due to their relatively low torsional rigidity .

… and for narrow cross-sections, too



Lateral torsional buckling



In short …

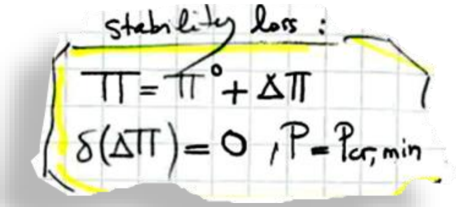
bending **Pure torsion** **Warping**

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y {w''}^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t {\phi'}^2 \mathrm{d}x + \boxed{\frac{1}{2}\int_0^\ell EI_\omega {\phi''}^2 \mathrm{d}x} +$$

$$+ \int_0^\ell \int_A \sigma_x^0 \underbrace{\frac{1}{2}[(w'_Q)^2 + (v'_Q)^2]} \mathrm{d}A \mathrm{d}x$$

Initial stresses from pre-buckled configuration

Non-linear strains (quadratic part) from slightly buckled configuration



stability loss :

$\Pi = \Pi^0 + \Delta\Pi$

$\delta(\Delta\Pi) = 0$, $P = P_{cr,min}$

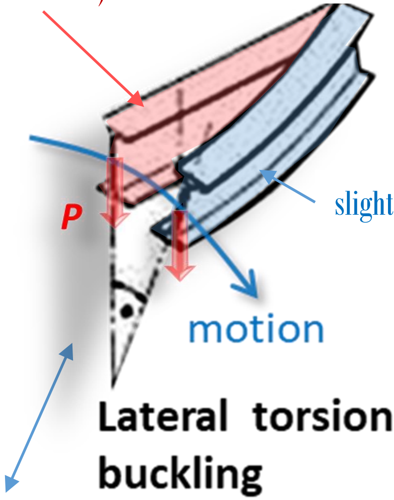
$\Longrightarrow$ **buckling load** and mode

$$\boxed{\delta(\Delta\Pi) = \delta[\delta^2\Pi|_{\mathbf{u}^0}] = 0, \forall\delta\mathbf{u}, \quad \text{kin. admissible,}}$$

$\Downarrow$

The **linear equations** of loss of **stability**
Eigen-value problem (BVP)

We will derive BVP for lateral torsional and combined flexural-torsional buckling



Lateral torsional buckling

It is the **solution** of these equations which **provides** the **buckling load** and the corresponding mode

$$EI_\omega \varphi^{(4)} - GI_t \varphi'' - \frac{(M_x^0)^2}{EI_y}\varphi = 0,$$

Lets start the story from the beginning …

Lateral-torsional buckling of beams

kiepahdus

Rectangular **narrow** cross-section $\Longrightarrow$ no warping $I_\omega \approx 0$

What to do?

- derive the stability loss equations for lateral torsional buckling when the warping is negligible

Assumptions

- negligible additional vertical deflection v at buckling
- accounts for the effect of shear stress
- negligible or no warping at all

START:

Narrow cross-section

$I_y << I_z,$

no warping $I_\omega \approx 0$

Pure torsion

Initial stress

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y {w''}^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t {\phi'}^2 \mathrm{d}x + \int_0^\ell \int_A \sigma_x^0 \epsilon_2 \mathrm{d}A\mathrm{d}x,$$

$$\Downarrow$$

$$\delta(\Delta\Pi[w,\phi]) = 0, \quad \forall \delta w, \delta\phi \quad \text{kin. admissible} \Longrightarrow$$

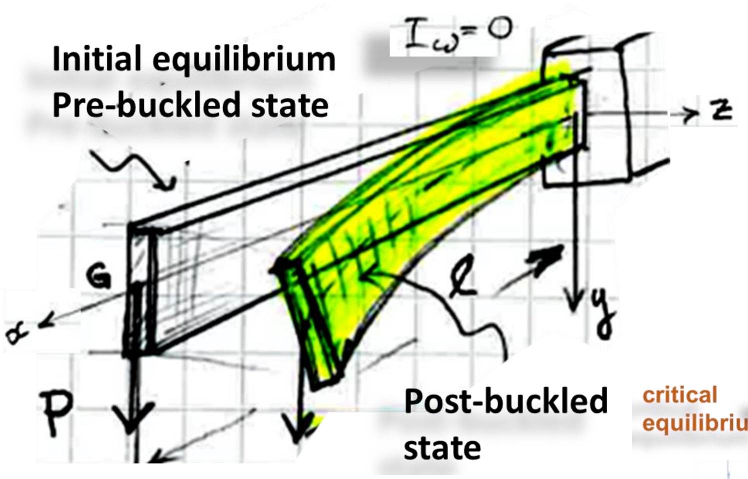
$$\Downarrow$$

GOAL:

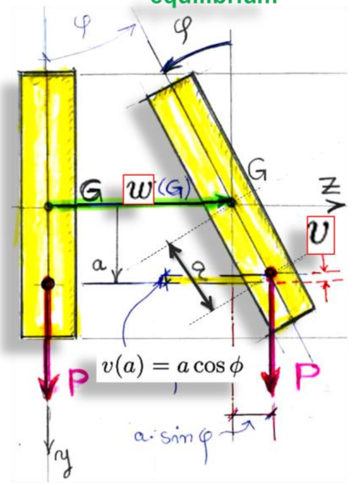
Stability equation

$$\begin{cases} EI_y\,(w'')'' - (M_z^0\phi)'' &= 0 \\ (GI_t\phi')' + M_z^0 w'' &= 0. \end{cases}$$

complete model $\Longrightarrow$



critical equilibrium ⟹ infinitesimal perturbation toward adjacent equilibrium



$$\hat{\mathbf{u}} \equiv \delta\mathbf{u} \equiv \underbrace{\hat{\mathbf{u}} = (0, v, w)}\equiv \mathbf{u}$$

infinitesimal displacement increment between critical point and disturbed state

The following slides show the details of how we obtain the stability loss equations: … $\Longrightarrow$

Lateral-torsional buckling of beams

kiepahdus

- In flexural buckling of columns the thrust (puristus) was axial and normal to the cross-section of the beam-column
- Now we address stability of beam having a thin-walled open cross-section
- The loading is transversal to the axis of the beam

$I_y << I_z,$ **Narrow** cross-section

no warping $I_\omega \approx 0$

Pure torsion

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y {w''}^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t {\phi'}^2 \mathrm{d}x + \int_0^\ell \int_A \sigma_x^0 \epsilon_2 \mathrm{d}A\mathrm{d}x,$$

Now, this case of load

Initial stresses

kinematics of the displacement increments of the mid-plane $z = 0$:

$$\begin{cases} w(x,y) &= w(G) + y\sin\phi \approx w(x) + y\phi(x) \\ u(x,y) &= 0, \\ v(x,y) &= 0 \end{cases}$$

Work conjugates:

$$\sigma_x^0 = \frac{M_z^0}{I_z} y$$

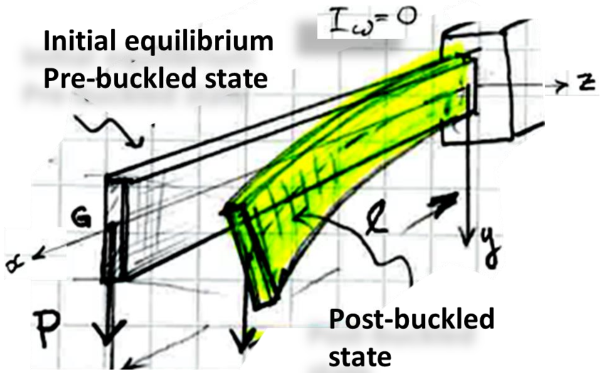
$$\epsilon_2 = \frac{1}{2}(w_{,x})^2$$

Hypothesis (which holds)

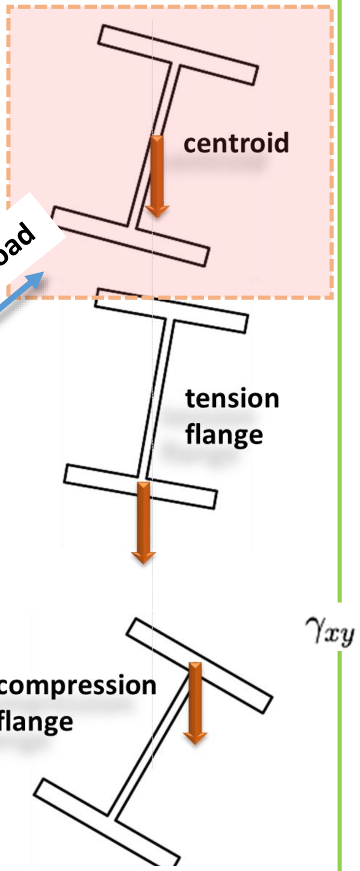
- negligible additional vertical deflection *v* at buckling
- accounts for the effect of shear stress
- negligible or no warping at all

Rectangular **narrow** cross-section

$\Longrightarrow$ no warping $I_\omega \approx 0$



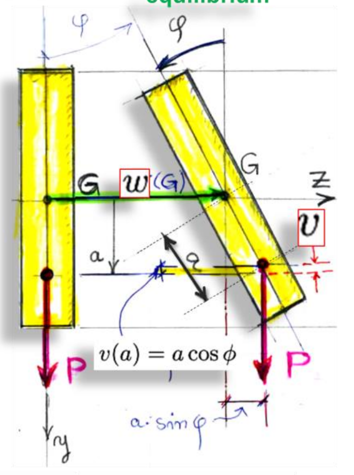
Effect of location of the load



critical equilibrium ⇒ infinitesimal perturbation toward adjacent equilibrium

$$\begin{aligned} \epsilon_2 &= \frac{1}{2}(\omega_z^2 + \omega_y^2) \\ &= \frac{1}{2}(\underbrace{u_{,z}}_{=0} - w_{,x})^2 + \frac{1}{2}(\underbrace{v_{,x}}_{=0} - \underbrace{u_{,y}}_{=0})^2 \\ &= \frac{1}{2}(w_{,x})^2. \end{aligned}$$

$$\gamma_{xy} = \gamma_{xy}^L + \gamma_{xy}^{NL} \equiv \gamma_{xy}^L + \gamma_{xy}^*$$



$v(a) = a\cos\phi$

$$\hat{\mathbf{u}} \equiv \delta\mathbf{u} \equiv \underbrace{\hat{\mathbf{u}} = (0, v, w)}\equiv \mathbf{u}$$

infinitesimal displacement
crement between critical
oint and disturbed state

Initial stresses

Work conjugates:

$$\tau_{xy}^0(x,y) = \frac{Q_y^0(x)}{bI_z} S_z^+(y) = \frac{(M_z^0(x))'}{bI_z} S_z^+(y)$$

$$\gamma_{xy}^* = -w_x w_y$$

$$\gamma_{xy} = \gamma_{xy}^L + \gamma_{xy}^{NL} \equiv \gamma_{xy}^L + \gamma_{xy}^*$$

Lateral-torsional buckling of beams

Pure torsion

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t\phi'^2 \mathrm{d}x + \int_0^\ell\int_A \sigma_x^0 \epsilon_2 \mathrm{d}A\mathrm{d}x,$$

Initial stresses

Bending

$$\boxed{\sigma_x^0 = \frac{M_z^0}{I_z}y}$$

Shear stresses from transversal load

$$\boxed{\tau_{xy}^0(x,y) = \frac{Q_y^0(x)}{bI_z}S_z^+(y) = \frac{(M_z^0(x))'}{bI_z}S_z^+(y)}$$

$$\Downarrow$$

This is now a complete model

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t\phi'^2 \mathrm{d}x + \underbrace{\int_0^\ell (M_z^0\phi)'w'\mathrm{d}x}_{\text{bending \& shear}}}$$

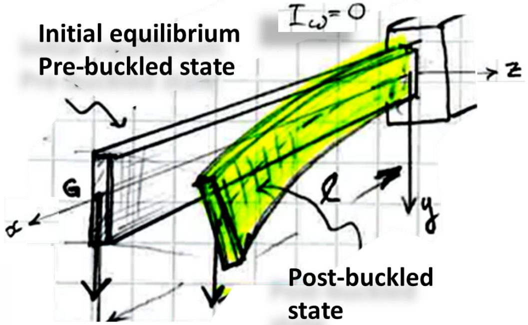
NB: For pedagogical reasons and to lower the complexity of the procedure, I decided that in the following derivation of equations of stability, we first start by omitting the effect of shear stresses resulting from transversal load and account only for bending initial stresses. This way should be easier fro the student to follow. Then we add the contribution of initial shear stresses (of the transversal load) when shear effects to complete the total potential energy increment and find its effect to the stability loss equations.

Rectangular **narrow** cross-section $\Longrightarrow$ $\boxed{\text{no warping } I_\omega \approx 0}$

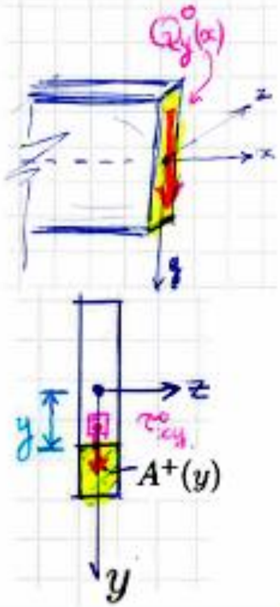
$$\Downarrow$$

$$I_y << I_z,$$

$$\text{no warping } I_\omega \approx 0$$



Sign convention

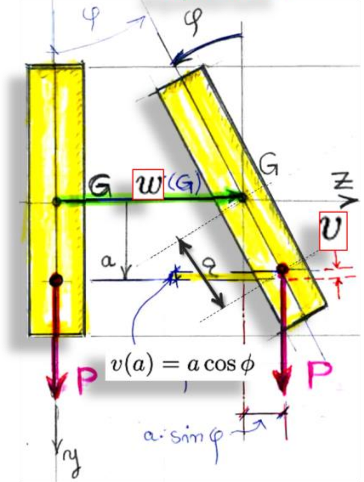


$$\boxed{\tau_{xy}^0(x,y) = \frac{Q_y^0(x)}{bI_z}S_z^+(y)}$$

$$b\int_y^{y=h/2} y\mathrm{d}y \equiv S_z^+(y)$$

Example: Transverse load *P* or distributed load *q* applied at the centroid

critical equilibrium ⇒ infinitesimal perturbation toward adjacent equilibrium



$$\hat{\mathbf{u}} \equiv \delta\mathbf{u} \equiv \underbrace{\hat{\mathbf{u}} = (0,v,w)}_{} \equiv \mathbf{u}$$

infinitesimal displacement increment between critical point and disturbed state

Lateral-torsional buckling of beams

Pure torsion

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t\phi'^2 \mathrm{d}x + \int_0^\ell \int_A \sigma_x^0 \boxed{\epsilon_2} \mathrm{d}A\mathrm{d}x,$$

Initial stress:

$$\boxed{\sigma_x^0 = \frac{M_z^0}{I_z} y}$$

kinematics of the displacement increments of the mid-plane $z = 0$:

$$\begin{cases} w(x,y) &= w(G) + y\sin\phi \approx w(x) + y\phi(x) \\ u(x,y) &= 0, \\ v(x,y) &= 0 \end{cases}$$

$$\Longrightarrow$$

$$\begin{aligned} \epsilon_2 &= \frac{1}{2}(\omega_z^2 + \omega_y^2) \\ &= \frac{1}{2}(\underbrace{u_{,z}}_{=0} - w_{,x})^2 + \frac{1}{2}(\underbrace{v_{,x}}_{=0} - \underbrace{u_{,y}}_{=0})^2 \\ &= \frac{1}{2}(w_{,x})^2. \end{aligned}$$

$$\Downarrow$$

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t\phi'^2 \mathrm{d}x + \int_0^\ell \int_A M_z^0 w'\phi' \mathrm{d}x}$$

when shear effects omitted

(this model is not complete. Shear force effects should be accounted for in lateral-torsional buckling, *ref.* lecturer pdf-material)

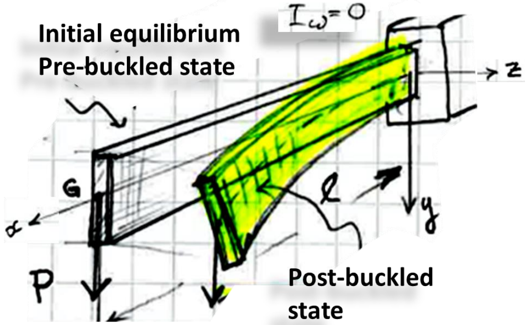
How we obtain this term?

Rectangular **narrow** cross-section

$$\Downarrow$$

$$I_y << I_z,$$
$$\text{no warping } I_\omega \approx 0$$

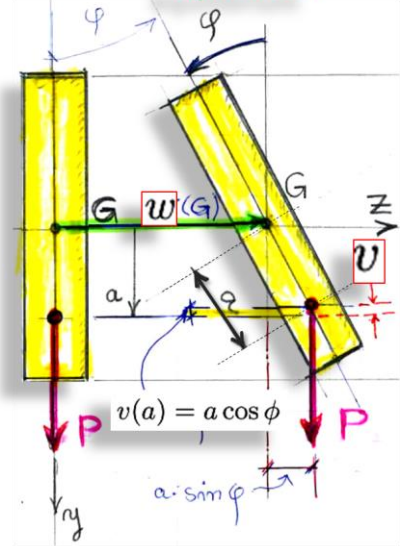
$$\Longrightarrow \boxed{\text{no warping } I_\omega \approx 0}$$



critical equilibrium ⇒ infinitesimal perturbation toward adjacent equilibrium

Example: Load *P* or distributed load *q* applied at the centroid

centroid
G or C



$$\hat{\mathbf{u}} \equiv \delta\mathbf{u} \equiv \underbrace{\hat{\mathbf{u}} = (0, v, w)} \equiv \mathbf{u}$$

infinitesimal displacement increment between critical point and disturbed state

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t \phi'^2 \mathrm{d}x + \boxed{\int_0^\ell \int_A M_z^0 w'\phi' \mathrm{d}x}}$$

?

when shear effects omitted
(this model is not complete)

$$\begin{cases} w(x,y) & = w(G) + y\sin\phi \approx w(x) + y\phi(x) \\ u(x,y) & = 0, \\ v(x,y) & = 0 \end{cases}$$

$$\boxed{\int_0^\ell \int_A \sigma_x^0 \epsilon_2 \mathrm{d}A\mathrm{d}x} = \int_0^\ell \frac{M_z^0(x)}{I_z}\int_A y\cdot[\frac{1}{2}w'(x)^2 + \frac{1}{2}(y\phi'(x))^2 + y\phi'(x)w'(x)]\mathrm{d}A\mathrm{d}x$$

$$\begin{aligned} \epsilon_2 &= \frac{1}{2}(\omega_z^2 + \omega_y^2) \\ &= \frac{1}{2}(\underbrace{u_{,z}}_{=0} - w_{,x})^2 + \frac{1}{2}(\underbrace{v_{,x}}_{=0} - \underbrace{u_{,y}}_{=0})^2 \\ &= \frac{1}{2}(w_{,x})^2. \end{aligned}$$

$$= \frac{1}{2}\int_0^\ell \frac{M_z^0}{I_z} w'^2 \mathrm{d}x \underbrace{\int_A y\mathrm{d}A}_{S_z=0} + \frac{1}{2}\int_0^\ell \frac{M_z^0}{I_z}\phi'^2 \mathrm{d}x \underbrace{\int_A y^3 \mathrm{d}A}_{=0} +$$

$$+ \int_0^\ell \frac{M_z^0}{I_z} w'\phi' \mathrm{d}x \underbrace{\int_A y^2 \mathrm{d}A}_{=I_z}$$

$$= \boxed{\int_0^\ell M_z^0 w'\phi' \mathrm{d}x}$$

$$\boxed{\delta(\Delta\Pi[w,\phi]) = 0, \quad \forall \delta w, \delta\phi \quad \text{kin. admissible} \implies} \implies$$

Euler-Lagrange equations (Field equations):

- Stability equation
- Boundary conditions

Lateral-torsional buckling of beams

Pure torsion

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t\phi'^2 \mathrm{d}x + \int_0^\ell \int_A \sigma_x^0 \boxed{\epsilon_2} \mathrm{d}A\mathrm{d}x,$$

Initial stress:

$$\sigma_x^0 = \frac{M_z^0}{I_z} y$$

kinematics of the displacement increments of the mid-plane $z = 0$:

$$\begin{cases} w(x,y) &= w(G) + y\sin\phi \approx w(x) + y\phi(x) \\ u(x,y) &= 0, \\ v(x,y) &= 0 \end{cases}$$

$$\boxed{\epsilon_2} = \frac{1}{2}(\omega_z^2 + \omega_y^2)$$

$$= \frac{1}{2}(\underbrace{u_{,z}}_{=0} - w_{,x})^2 + \frac{1}{2}(\underbrace{v_{,x}}_{=0} - \underbrace{u_{,y}}_{=0})^2$$

$$= \frac{1}{2}(w_{,x})^2.$$

$$\tau_{xy}^0(x,y) = \frac{Q_y^0(x)}{bI_z} S_z^+(y) = \frac{(M_z^0(x))'}{bI_z} S_z^+(y)$$

$$\Downarrow$$

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t\phi'^2 \mathrm{d}x + \underbrace{\int_0^\ell (M_z^0\phi)'w'\mathrm{d}x}_{\text{bending \& shear}}}$$

How we got this term?

This is now a complete model

Example: Transverse load *P* or distributed load *q* applied at the centroid

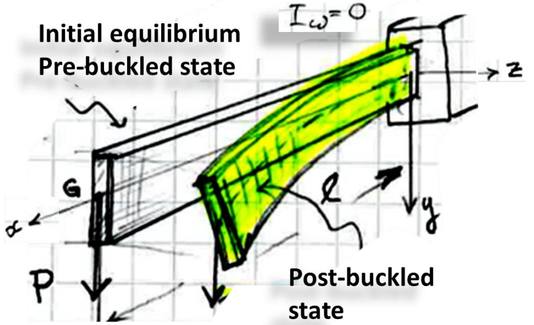
Rectangular **narrow** cross-section

$$\Downarrow$$

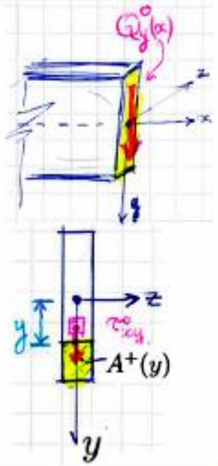
$I_y << I_z,$

no warping $I_\omega \approx 0$

$\Longrightarrow$ no warping $I_\omega \approx 0$



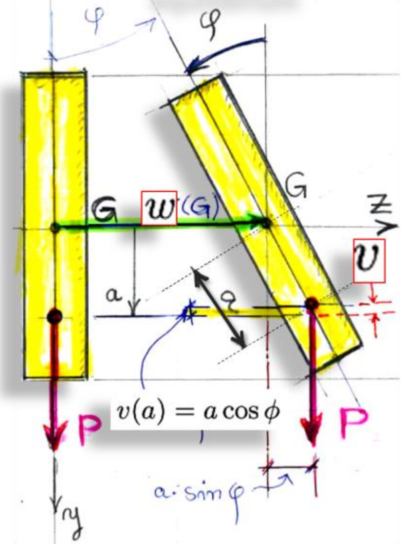
Sign convention



$$\tau_{xy}^0(x,y) = \frac{Q_y^0(x)}{bI_z} S_z^+(y)$$

$$b\int_y^{y=h/2} y\mathrm{d}y \equiv S_z^+(y)$$

critical equilibrium ⇒ infinitesimal perturbation toward adjacent equilibrium



$v(a) = a\cos\phi$

$$\hat{\mathbf{u}} \equiv \delta\mathbf{u} \equiv \underbrace{\hat{\mathbf{u}} = (0, v, w)}_{} \equiv \mathbf{u}$$

infinitesimal displacement increment between critical point and disturbed state

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t \phi'^2 \mathrm{d}x + \underbrace{\int_0^\ell (M_z^0\phi)' w' \mathrm{d}x}_{\text{bending \& shear}}}$$

?

How we got this term?

This is now a complete model

$$\begin{cases} w(x,y) & = w(G) + y\sin\phi \approx w(x) + y\phi(x) \\ u(x,y) & = 0, \\ v(x,y) & = 0 \end{cases}$$

$$\begin{aligned} \epsilon_2 &= \frac{1}{2}(\omega_z^2 + \omega_y^2) \\ &= \frac{1}{2}(\underbrace{u_{,z}}_{=0} - w_{,x})^2 + \frac{1}{2}(\underbrace{v_{,x}}_{=0} - \underbrace{u_{,y}}_{=0})^2 \\ &= \frac{1}{2}(w_{,x})^2. \end{aligned}$$

Work of shear initial stresses is now added

$$\begin{aligned} \Delta U^*_{NL}(\tau^0_{xy}) &= \int_0^\ell \int_A \tau^0_{xy}\gamma^*_{xy} \mathrm{d}A\mathrm{d}x \\ &= \int_0^\ell \frac{Q_y^0}{bI_z}\phi w' \mathrm{d}x \underbrace{\int_A S_z(y)\mathrm{d}A}_{=bI_z} + \\ &- \int_0^\ell \frac{Q_y^0}{bI_z}\phi^2 \mathrm{d}x \underbrace{\int_A yS_z(y)\mathrm{d}A}_{=0} \\ &= \boxed{+\int_0^\ell Q_y^0 \phi w' \mathrm{d}x = +\int_0^\ell (M_z^0)'\phi w' \mathrm{d}x.} \end{aligned}$$

$$\gamma_{xy} = 2e_{xy} - \omega_y\omega_x = \gamma^L_{xy} + \gamma^{NL}_{xy} \equiv \gamma^L_{xy} + \gamma^*_{xy}$$

$$\gamma^*_{xy} = -\omega_x\omega_y = -w_x w_y = -\phi w' + y\phi^2,$$

Euler-Lagrange equations (Field equations):

$$\boxed{\delta(\Delta\Pi[w,\phi]) = 0, \quad \forall \delta w, \delta\phi \quad \text{kin. admissible} \implies} \Longrightarrow$$

- Stability equation
- Boundary conditions

Deriving the stability loss equations …

when initial shear stress effects omitted, for pedagogical simplicity. They will be added at the end of the derivation (this model is not complete)

$$\delta(\Delta\Pi[w,\phi]) = 0, \quad \forall \delta w, \delta\phi \quad \text{kin. admissible} \implies$$

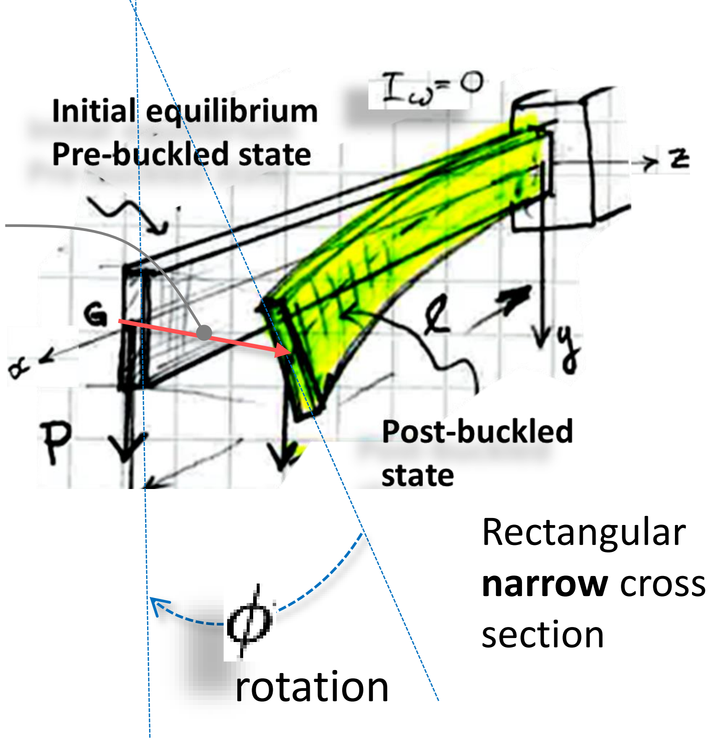
$$\delta(\Delta\Pi[w,\phi]) = \int_0^\ell EI_y w''\delta w''\mathrm{d}x + \int_0^\ell GI_t\phi'\delta\phi'\mathrm{d}x +$$
$$+ \int_0^\ell \int_A M_z^0 w'\delta\phi'\mathrm{d}x + \int_0^\ell \int_A M_z^0 \delta w'\phi'\mathrm{d}x = 0,$$
$$\forall \delta w, \delta\phi \quad \text{kin. admissible.}$$

$$\Downarrow$$

$$\delta(\Delta\Pi[w,\phi]) = \int_0^\ell \left[EI_y\left(w''\right)'' - \left(M_z^0\phi'\right)'\right]\delta w\mathrm{d}x +$$
$$- \int_0^\ell \left[(GI_t\phi')' + \left(M_z^0 w'\right)'\right]\delta\phi\mathrm{d}x +$$
$$- \left[\left(GI_t\phi' + M_z^0 w'\right)\delta\phi\right]_0^\ell +$$
$$+ \left[\left(-\left(EI_y w''\right)' + M_z^0\phi'\right)\delta w\right]_0^\ell +$$
$$+ \left[EI_y w''\delta w'\right]_0^\ell = 0,$$
$$\forall \delta w, \delta\phi \quad \text{kin. admissible.}$$

Lateral w deflection

no warping $I_\omega \approx 0$



Initial equilibrium Pre-buckled state

Post-buckled state

Rectangular **narrow** cross-section

rotation

Should account for this part: (shear initial stresses) to have a complete model

$$\delta(\Delta U_{NL}^*) = +\int_0^\ell (M_z^0)' w'\delta\phi\mathrm{d}x + \int_0^\ell (M_z^0)'\phi\delta w'\mathrm{d}x.$$

The variational machinery (thanks to Lagrange and some others …)

Deriving the stability loss equations …

when shear effects accounted
(this model is **complete**)

The variational machinery (thanks to Lagrange and some others …)

$$\delta(\Delta\Pi[w,\phi]) = 0, \quad \forall\delta w, \delta\phi \quad \text{kin. admissible} \implies$$

Only bending initial stresses:

$$\Downarrow$$

$$\delta(\Delta\Pi[w,\phi]) = \int_0^\ell EI_y w''\delta w''\mathrm{d}x + \int_0^\ell GI_t\phi'\delta\phi'\mathrm{d}x + \\ + \int_0^\ell \int_A M_z^0 w'\delta\phi'\mathrm{d}x + \int_0^\ell \int_A M_z^0\delta w'\phi'\mathrm{d}x = 0, \\ \forall\delta w, \delta\phi \quad \text{kin. admissible.}$$

Should account for this part: (shear initial stresses)

$$\delta(\Delta U_{NL}^*) = + \int_0^\ell (M_z^0)'w'\delta\phi\mathrm{d}x + \int_0^\ell (M_z^0)'\phi\delta w'\mathrm{d}x.$$

$$\Downarrow$$

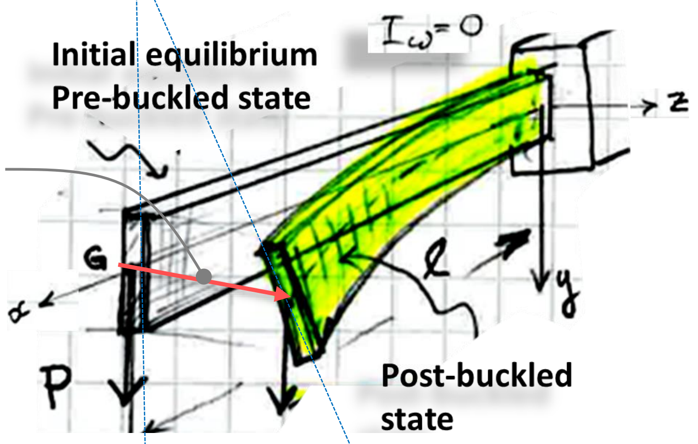
$$\underline{\delta w}: \qquad EI_y\,(w'')'' - \left(M_z^0\phi'\right)' - \underbrace{\left((M_z^0)'\phi\right)'}_{\text{from } \tau^0_{\mathrm{xy}}\cdot\gamma^*_{\mathrm{xy}}} = 0$$

$$\underline{\delta\phi}: \qquad (GI_t\phi')' + \left(M_z^0 w'\right)' - \underbrace{(M_z^0)'w'}_{\text{from } \tau^0\cdot\gamma^*} = 0.$$

$$\implies$$

Lateral w deflection

no warping $I_\omega \approx 0$



Rectangular **narrow** cross-section

ϕ

rotation

Stability equation:

$$\boxed{\begin{cases} EI_y\,(w'')'' - (M_z^0\phi)'' & = 0 \\ (GI_t\phi')' + M_z^0 w'' & = 0. \end{cases}}$$

complete model

Boundary conditions:

$$\begin{aligned} &\text{at} \quad x = \ell, & & & &\text{at} \quad x = 0 \\ &GI_t\phi' + M_z^0 w' = 0, & & & &w' = 0 \\ &-(EI_y w'')' + M_z^0\phi' = 0, & &\text{or} & &w = 0 \\ &EI_y w'' = 0, & & & &\phi = 0. \end{aligned}$$

From where comes the Standard EN lateral torsional buckling stress formula?

This is in the Eng. PRACTICE $\Longleftarrow$ $\Longleftarrow$ $\Longleftarrow$ **This is given by the THEORY** no warping $I_\omega \approx 0$

Critical lateral buckling stress in EN 1955-1-1 (section 6.3.3) for wooden beams

$$\sigma_{\mathrm{m,crit}} = \frac{M_{\mathrm{y,crit}}}{W_{\mathrm{y}}} = \frac{\pi\sqrt{E_{0,05} I_{\mathrm{z}} G_{0,05} I_{\mathrm{tor}}}}{\ell_{\mathrm{ef}} W_{\mathrm{y}}}$$

Critical stress in torsional buckling for a wooden beam in uniform bending as given in the standard (check 1955!).

$$\sigma_{cr} = \frac{M_{0,cr}}{W_y^{(e)}} = \frac{\pi}{W_y^{(e)}\ell}\sqrt{EI_y GI_t}$$

$W_y^{(e)}$ is the elastic bending resistance.

$\Uparrow$

It is the **solution** of **the differential equation** of **Stability loss** under uniform bending

$$\begin{cases} EI_y\,(w'')'' - (M_z^0\phi')' & = 0, \\ (GI_t\phi')' + (M_z^0 w')' & = 0 \end{cases}$$

Let’s derive this formula

Pure bending

Puhdas taivutus

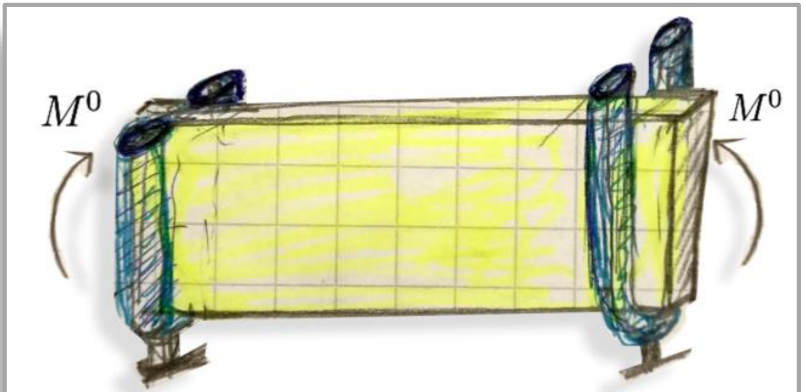
Note that now the shear force is identically zero since the bending moment is constant so shear contribution can be simply ignored.

$$\begin{cases} EI_y\,(w'')'' - (M_z^0\phi')' &= 0, \\ (GI_t\phi')' + (M_z^0 w')' &= 0 \end{cases}$$

incomplete model, no shear

$$\begin{cases} EI_y\,(w'')'' - (M_z^0\phi)'' &= 0 \\ (GI_t\phi')' + M_z^0 w'' &= 0. \end{cases}$$

complete model



$$\begin{cases} \phi(0) = 0, \\ w(0) = 0, \\ w''(0) = 0 \end{cases} \qquad \begin{cases} \phi(\ell) = 0, \\ w(\ell) = 0, \\ w''(\ell) = 0 \end{cases}$$

no warping $I_\omega \approx 0$

A constant external moment $M_z^0 = M_0$ at both ends

$$\Downarrow$$

$$\begin{cases} EI_y w^{(4)} - M_0\phi'' &= 0, \\ GI_t\phi'' + M_0 w'' &= 0. \end{cases}$$

$$\Downarrow$$

$$\begin{cases} w^{(4)} + k_t^2 w'' = 0, \\ \phi'' = -\frac{M_0}{GI_t} w''. \end{cases}$$

$$k_t^2 = M_0^2/(GI_tEI_y)$$

Solution of **the differential** under **uniform bending:**

Notice the **analogy** with Euler buckling of a simply supported column

$$P_{cr} = EI_y \cdot \pi^2/\ell^2$$

$$w_n(x) = A_n \sin\left(\frac{n\pi x}{\ell}\right),$$

$$\left(\frac{n\pi}{\ell}\right)^2 \underbrace{\left[\left(\frac{n\pi}{\ell}\right)^2 - k_t^2\right]}_{=0} A_n \sin\left(\frac{n\pi x}{\ell}\right) = 0, \quad n = 1, 2, \ldots$$

$$\Rightarrow M_n = \left(\frac{n\pi}{\ell}\right)\sqrt{EI_yGI_t} \quad \text{Eigen-values.}$$

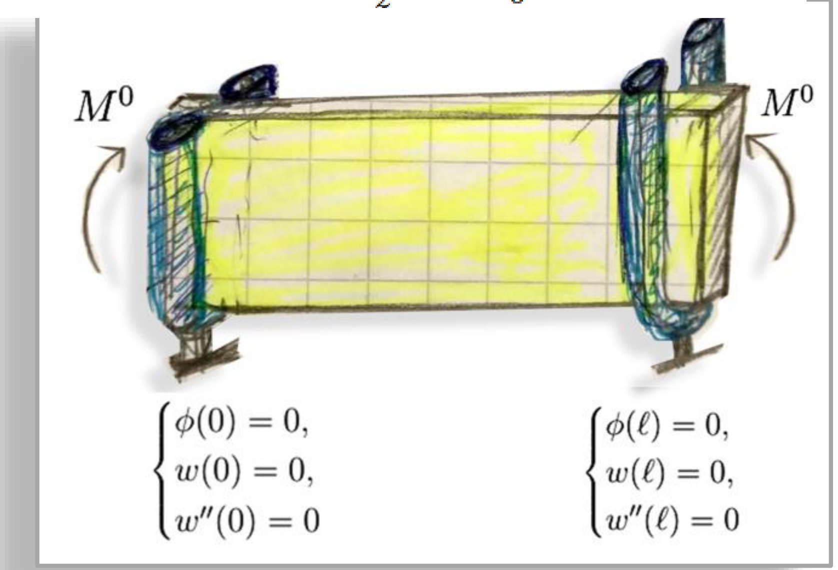
The buckling (critical) end-moment $\boxed{M_{0,cr} = \frac{\pi}{\ell}\sqrt{EI_yGI_t}}$

The critical stress $\boxed{\sigma_{cr} = \frac{M_{0,cr}}{W_y^{(e)}} = \frac{\pi}{W_y^{(e)}\ell}\sqrt{EI_yGI_t}}$

Pure bending

Puhdas taivutus

A constant external moment $M_z^0 = M_0$ at both ends



$$\begin{cases} \phi(0) = 0, \\ w(0) = 0, \\ w''(0) = 0 \end{cases} \qquad \begin{cases} \phi(\ell) = 0, \\ w(\ell) = 0, \\ w''(\ell) = 0 \end{cases}$$

no warping $I_\omega \approx 0$

$$\begin{cases} EI_y w^{(4)} - M_0 \phi'' &= 0, \\ GI_t \phi'' + M_0 w'' &= 0. \end{cases}$$

$$\begin{cases} w^{(4)} + k_t^2 w'' = 0, \\ \phi'' = -\dfrac{M_0}{GI_t} w''. \end{cases}$$

$$k_t^2 = M_0^2 / (GI_t EI_y)$$

$$\phi'' = -\frac{M_0}{GI_t} w'' = -\frac{M_0}{GI_t} A_n \left(\frac{n\pi}{\ell}\right)^2 \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Downarrow$$

$$\boxed{M_{0,cr} = \frac{\pi}{\ell}\sqrt{EI_y GI_t}}$$

The buckling (critical) end-moment

$$\boxed{\sigma_{cr} = \frac{M_{0,cr}}{W_y^{(e)}} = \frac{\pi}{W_y^{(e)} \ell}\sqrt{EI_y GI_t}}$$

The critical stress

The buckling modes

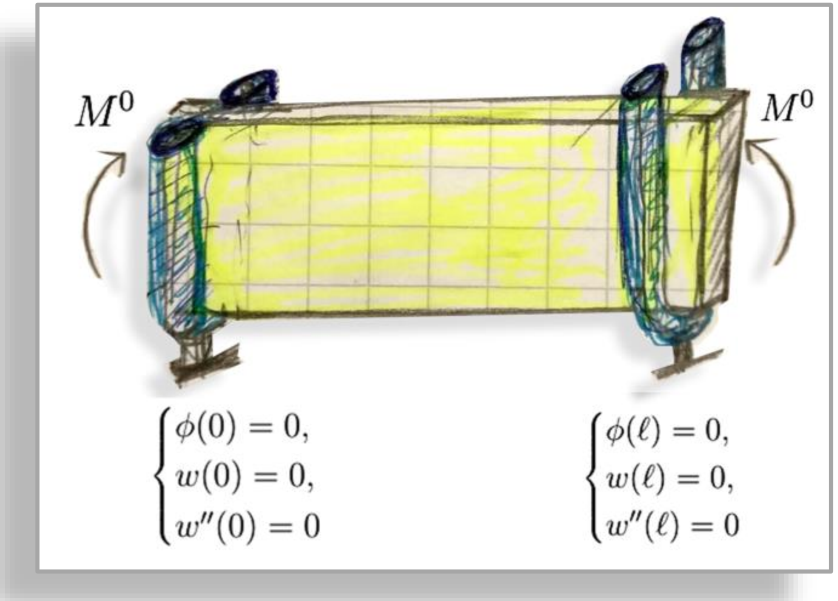
$$\begin{cases} v_{cr}(x) &= A \sin\left(\frac{\pi x}{\ell}\right) \\ \phi_{cr}(x) &= -B\sqrt{\dfrac{EI_y}{GI_t}} \sin\left(\frac{\pi x}{\ell}\right) \end{cases}$$

Pure bending

Puhdas taivutus

A constant external moment $M_z^0 = M_0$ at both ends

no warping $I_\omega \approx 0$



$$\begin{cases} \phi(0) = 0, \\ w(0) = 0, \\ w''(0) = 0 \end{cases} \qquad \begin{cases} \phi(\ell) = 0, \\ w(\ell) = 0, \\ w''(\ell) = 0 \end{cases}$$

Résumé:

no warping $I_\omega \approx 0$

$$\boxed{M_{0,cr} = \frac{\pi}{\ell}\sqrt{EI_yGI_t}}$$

The buckling (critical) end-moment

$$\boxed{\sigma_{cr} = \frac{M_{0,cr}}{W_y^{(e)}} = \frac{\pi}{W_y^{(e)}\ell}\sqrt{EI_yGI_t}}$$

The critical stress

EN Standard Formula:

$$\sigma_{m,crit} = \frac{M_{y,crit}}{W_y} = \frac{\pi\sqrt{E_{0,05}I_zG_{0,05}I_{tor}}}{\ell_{ef}W_y}$$

For cross-section with non-negligible warping, the critical moment is [we will derive this formula later]

$$M_{0,cr} = \frac{\pi}{\ell}\sqrt{EI_yGI_t}\sqrt{1 + \frac{\pi^2}{\ell^2}\frac{EI_\omega}{GI_t}}$$

cross-section warping

(Timoshenko)

Rayleigh-Ritz energy method

Rayleigh-quotient

$P_{cr} \leq P_{cr,\text{approx.}}$

The energy criterion in the form $\delta(\Delta\Pi) = 0$ means that solutions of the stability problem make the change in the total potential energy (1.407) stationary. This fact can be used to find approximations for the critical buckling load. The method is called **Rayleigh-Ritz**. The idea, is to postulate cinematically admissible displacement fields, now for instance, $w(x)$ and $\phi(x)$, and to solve the buckling load from the stationarity condition

$$\boxed{\delta(\Delta\Pi(a_i; P)) = 0, \quad \forall \delta a_i \implies \frac{\partial}{\partial a_j}\Delta\Pi(a_1, a_2, \ldots, a_n; P) = 0,} \quad (1.440)$$

where a_i are the parameters in the displacements approximation. The above stationarity condition leads to the homogeneous system of equations (Eq. 1.441) below:

$$\boxed{\mathbf{K} - P\mathbf{S} = 0,} \quad (1.441)$$

Discrete Eigen-value problem

where, the effect of pre-stresses

$$M_z^0(x; P) = \det[\mathbf{K} - P\mathbf{S}] = 0, \quad (1.442)$$

from the reference equilibrium state are, naturally, solved in the primary equilibrium configuration in the framework of linear elasticity an

geometric-matrix terms S_{ij}

theory. The bending moment distribution $\bar{M}_z^0(x)$ is the one one obtains by setting $P = 1$.

Stiffness-matrix terms K_{ij}

Rayleigh-Ritz energy method

Approximation of buckling load using *Rayleigh-quotient*

High Cantilever beam

Here we start with no warping case

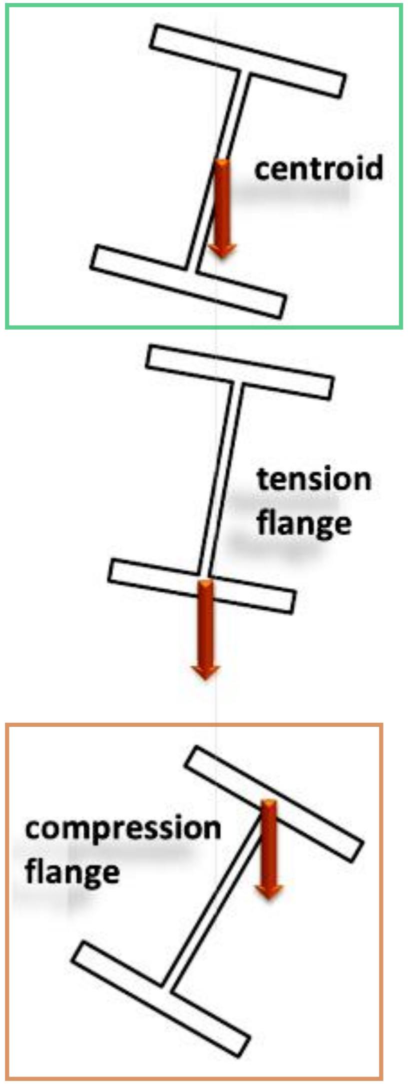
Two illustrative examples to study the effect of the location of the load

using the

Rayleigh-quotient

Also called **Rayleigh–Ritz ratio**

Effect of location of the load



Rayleigh-Ritz energy method

Approximation of buckling load using *Rayleigh-quotient*

High Cantilever beam

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2\mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t\phi'^2\mathrm{d}x + \underbrace{\int_0^\ell (M_z^0\phi)'w'\mathrm{d}x}_{\text{bending \& shear}}$$

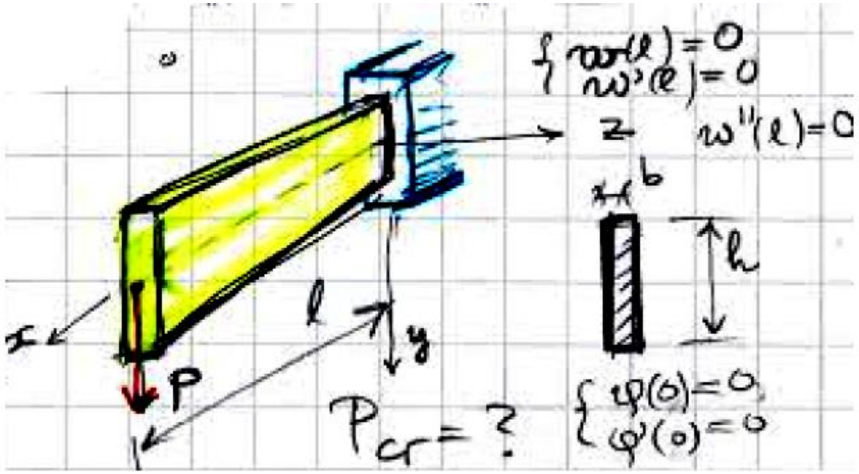
$\Downarrow$

Stability criteria $\delta(\Delta\Pi) = 0$

$$M_z^0(x;P) \equiv P\cdot\bar{M}_z^0(x)$$

initial bending moment

High Cantilever beam



Here no warping case

- 1) by approximating separately $\bar{w}(x)\approx w(x)$ and $\bar{\phi}(x)\approx\phi(x)$ in the energy functional (1.444) and using the criticality condition (stationarity). This is a more general approach.

$\delta(\Delta\Pi) = 0$ A more general method

- 2) approximating only $w(x)$ in the Rayleigh-quotient (1.450) after eliminating the second unknown function $\phi(x)$ using the second equilibrium equation. (not a general method. In general, it may become impossible to proceed explicitly with the elimination for other types of problem.)

Equilibrium criteria

$\Delta\Pi = 0$

Rayleigh-quotient

$$\boxed{P_{cr}^2 = \frac{\int_0^\ell EI_y w''^2\,\mathrm{d}x}{\int_0^\ell (\bar{M}_z^0)^2 w'^2/GI_t\,\mathrm{d}x}}$$

Approximation of buckling load using *Rayleigh-quotient*

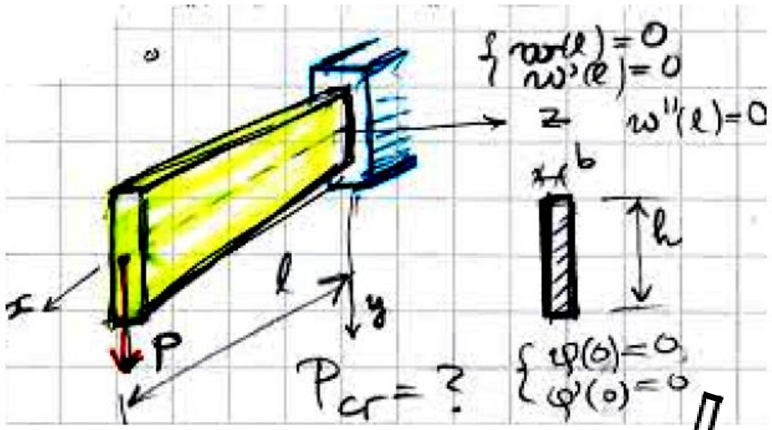
High Cantilever beam

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t \phi'^2 \mathrm{d}x + \int_0^\ell M_z^0 w' \phi' \mathrm{d}x}$$

$$M_z^0(x;P) \equiv P \cdot \bar{M}_z^0(x)$$

initial bending moment

High Cantilever beam



eliminate $\phi(x)$ from the energy-functional integrating

$$GI_t\phi'' + (M_0 w')' = 0 \implies GI_t\phi' + (M_0 w') = C$$

$$w'(\ell) = 0, \phi(\ell) = 0 \implies C = 0,$$

$$\boxed{\implies \phi' = -\frac{M_z^0}{GI_t} w'}$$

Load P at the torsion centre G: centroid

The simplest polynomial

$$\boxed{\bar{w}''(x) = A(\ell - x) \implies \bar{w}'(x) = Ax(\ell - x/2)}$$

Fulfills kinematic boundary conditions: $\begin{cases} w(0) = 0 \\ w'(0) = 0 \\ \phi(0) = 0 \end{cases}$

$$\boxed{P_{cr}^2 = \frac{\int_0^\ell EI_y w''^2 \,\mathrm{d}x}{\int_0^\ell (\bar{M}_z^0)^2 w'^2 / GI_t \,\mathrm{d}x}} \implies \bar{P}_{cr} = \sqrt{(35 GI_t EI_y)/(2\ell^4)} = \frac{4.18}{\ell^2}\sqrt{GI_t EI_y}.$$

Approximation from *R*-quotient

Rayleigh-quotient

$$P_{cr} \le P_{cr,\text{approx.}}$$

Exact ≤ Approximation

$$P_{cr} = \frac{4.013}{\ell^2}\sqrt{GI_t EI_y}$$

Exact analytical solution

NB. I computed this example with ignoring shear effect. Student! Redo the exercise and account for shear.

Approximation of buckling load using RR-quotient

High Cantilever beam

Exam example - 2018

Let $L = 2\ell$, thus the bending moment due to the own weight is $M_z^{\mathrm{o}} = \frac{q\ell^2}{2}(1-(\frac{x}{\ell})^2)$, when the origin is located at the mid span. The energy integral is

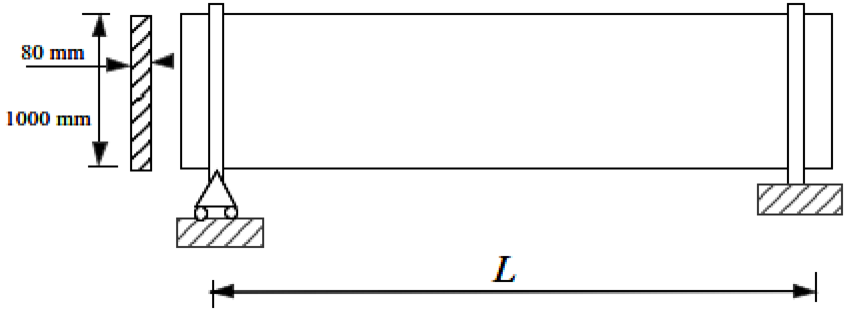
$$\Pi = \int_0^{\ell}\left[EI_y(w'')^2 + GI_t(\phi')^2 + 2(M_z^{\mathrm{o}}\phi)'w'\right]\mathrm{d}x$$

The beam is simply supported at each end when the approximations for the deflection and rotation can be of polynomial form, satisfying the boundary conditions $w'(0) = w(\pm\ell) = \phi'(0) = \phi(\pm\ell) = 0$ and are $w = w_{\mathrm{o}}(1-(\frac{x}{\ell})^2)$ and $\phi = \phi_{\mathrm{o}}(1-(\frac{x}{\ell})^2)$. Trigonometric functions $w = w_{\mathrm{o}}\cos(\frac{\pi x}{\ell})$ and $\phi = \phi_{\mathrm{o}}\cos(\frac{\pi x}{\ell})$ give better approximation.

$$\Pi = \int_0^{\ell}\left[EI_y(\frac{-2w_{\mathrm{o}}}{\ell^2})^2 + GI_t(\frac{-2x\phi_{\mathrm{o}}}{\ell^2})^2 + 2\left(\frac{q\ell^2}{2}\phi_{\mathrm{o}}(1-(\frac{x}{\ell})^2)^2\right)'(\frac{-2xw_{\mathrm{o}}}{\ell^2})\right]\mathrm{d}x =$$

$$= \frac{4EI_y}{\ell^3}w_{\mathrm{o}}^2 + \frac{4GI_t}{3\ell}\phi_{\mathrm{o}}^2 + \frac{16q\ell}{15}w_{\mathrm{o}}\phi_{\mathrm{o}} \Rightarrow \begin{cases} \frac{\partial\Pi}{\partial w_{\mathrm{o}}} = \frac{8EI_y}{\ell^3}w_{\mathrm{o}} + \frac{16q\ell}{15}\phi_{\mathrm{o}} \\ \frac{\partial\Pi}{\partial\phi_{\mathrm{o}}} = \frac{8GI_t}{3\ell}\phi_{\mathrm{o}} + \frac{16q\ell}{15}w_{\mathrm{o}} \end{cases} \Rightarrow$$

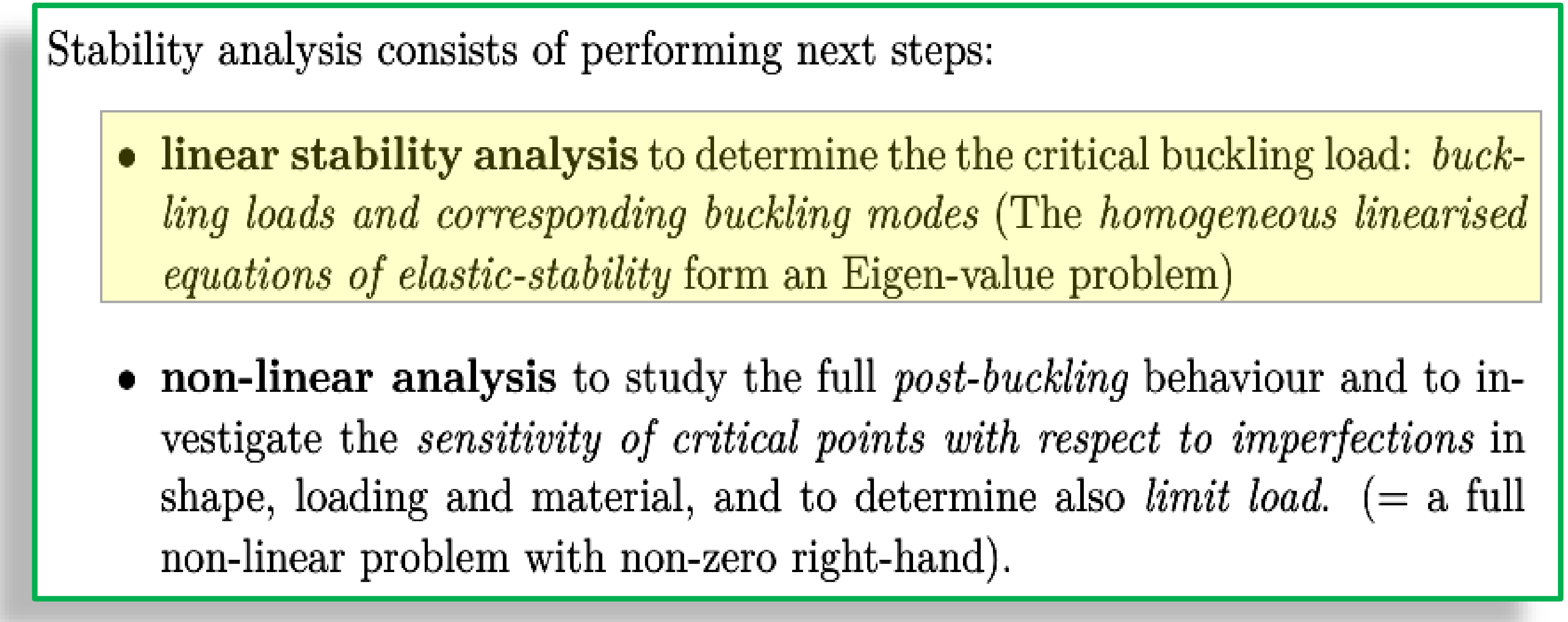
$$\begin{bmatrix} \frac{8EI_y}{\ell^3} & \frac{16q\ell}{15} \\ \frac{16q\ell}{15} & \frac{8GI_t}{3\ell} \end{bmatrix}\begin{Bmatrix} w_{\mathrm{o}} \\ \phi_{\mathrm{o}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \ell^6 = \frac{75}{4}\frac{EI_yGI_t}{q^2} \Rightarrow L = 2\ell = 33.1\,\mathrm{m}$$



What is the critical length of a simply supported beam with respect to lateral buckling, when its cross-section is a narrow rectangle ($80\,\mathrm{mm}\times 1000\,\mathrm{mm}$)? The Young's modulus and the shear modulus are $E = 36\,\mathrm{kN/mm^2}$ and $G = 15{,}4\,\mathrm{kN/mm^2}$ respectively. The loading due to the own weight is $g = 24\,\mathrm{kN/mm^3}$.

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^{\ell} EI_y w''^2\mathrm{d}x + \frac{1}{2}\int_0^{\ell} GI_t\phi'^2\mathrm{d}x + \underbrace{\int_0^{\ell}(M_z^0\phi)'w'\mathrm{d}x}_{\text{bending \& shear}}}$$

Computational stability analysis



Stability analysis consists of performing next steps:

- **linear stability analysis** to determine the the critical buckling load: *buckling loads and corresponding buckling modes* (The *homogeneous linearised equations of elastic-stability* form an Eigen-value problem)
- **non-linear analysis** to study the full *post-buckling* behaviour and to investigate the *sensitivity of critical points with respect to imperfections* in shape, loading and material, and to determine also *limit load*. (= a full non-linear problem with non-zero right-hand).

Linear buckling Analysis

(you will have a computer exercise on this)

Post-buckling Analysis

also known as

Non-linear buckling analysis

also **GNA**

(you will have a computer exercise on this)

Two steps:

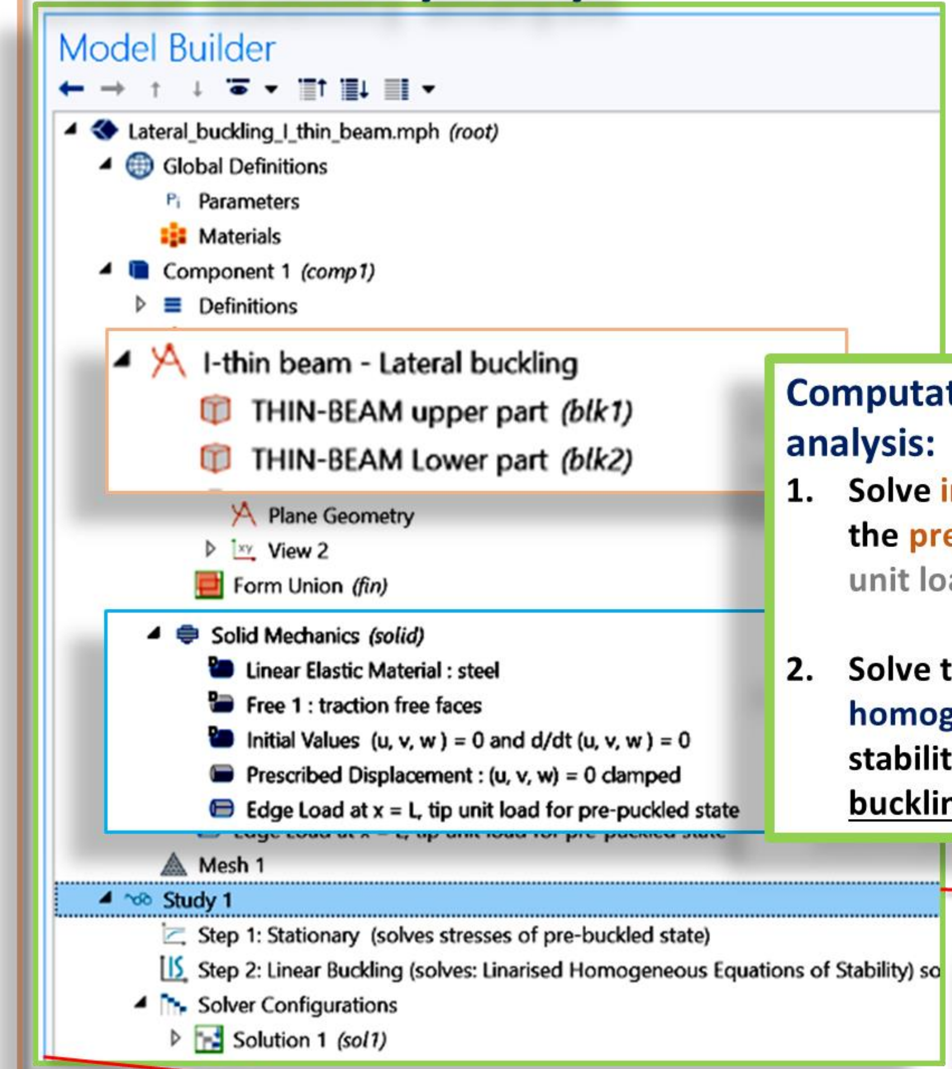
1. Solve **initial stress state** in the pre-buckled state for unit loading
2. Solve the **linearized homogeneous equations of stability** to obtain the critical load and buckling mode

Linear buckling Analysis

(you will have a computer exercise on this)

FE-buckling analysis

Linear stability analysis



Model Builder

- Lateral_buckling_I_thin_beam.mph *(root)*
  - Global Definitions
    - Parameters
    - Materials
  - Component 1 *(comp1)*
    - Definitions
    - I-thin beam - Lateral buckling
      - THIN-BEAM upper part *(blk1)*
      - THIN-BEAM Lower part *(blk2)*
      - Plane Geometry
      - View 2
      - Form Union *(fin)*
    - Solid Mechanics *(solid)*
      - Linear Elastic Material : steel
      - Free 1 : traction free faces
      - Initial Values (u, v, w) = 0 and d/dt (u, v, w) = 0
      - Prescribed Displacement : (u, v, w) = 0 clamped
      - Edge Load at x = L, tip unit load for pre-puckled state
    - Mesh 1
  - Study 1
    - Step 1: Stationary (solves stresses of pre-buckled state)
    - Step 2: Linear Buckling (solves: Linarised Homogeneous Equations of Stability) so
    - Solver Configurations
      - Solution 1 *(sol1)*

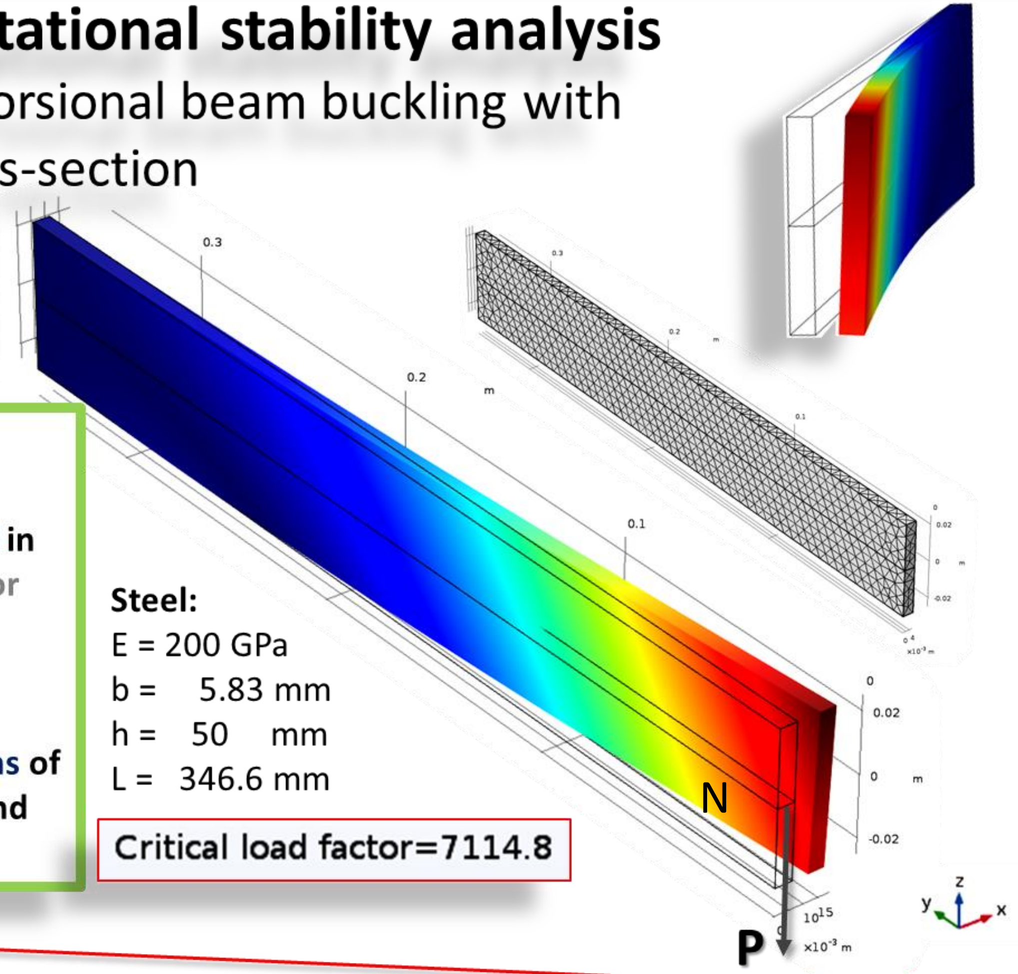
Example using COMSOL

Computational stability analysis:

1. Solve **initial stress state** in the **pre-buckled** state for unit loading
2. Solve the **linearized homogeneous equations** of stability: **Critical load** and **buckling mode**

Computational stability analysis

Lateral torsional beam buckling with thin cross-section



Steel:

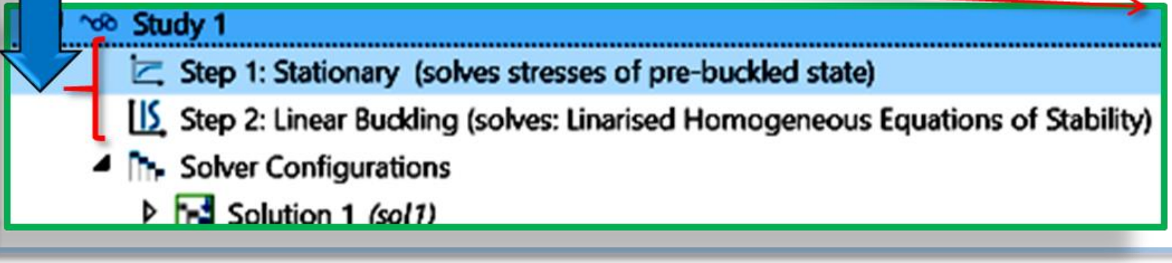
E = 200 GPa

b = 5.83 mm

h = 50 mm

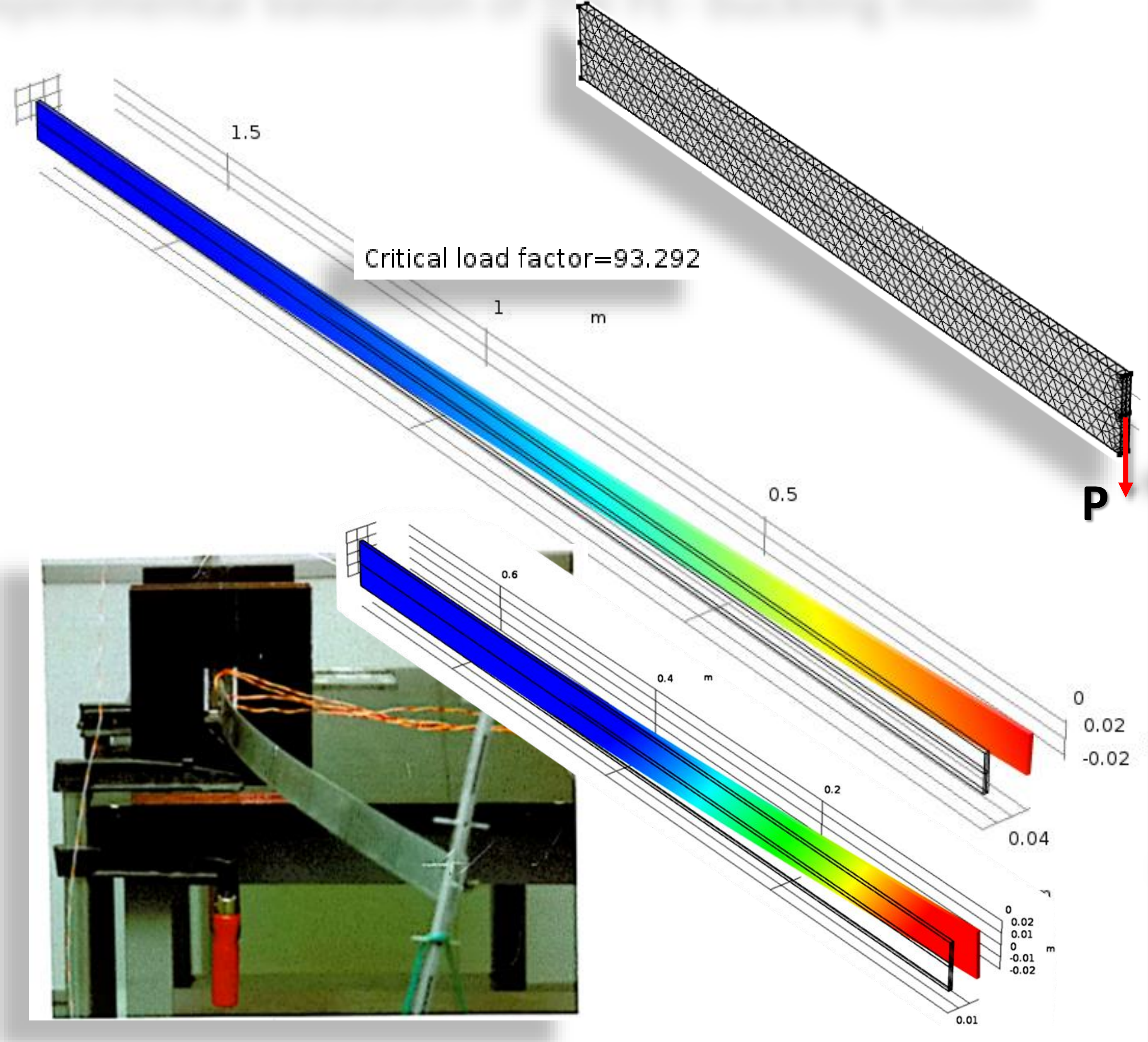
L = 346.6 mm

Critical load factor=7114.8



- Study 1
  - Step 1: Stationary (solves stresses of pre-buckled state)
  - Step 2: Linear Buckling (solves: Linarised Homogeneous Equations of Stability)
  - Solver Configurations
    - Solution 1 *(sol1)*

Experimental Validation of the FE- buckling model



Critical load factor=93.292

Critical load factor=-64.651

Ref: Experiments by R. Kouhia & P. Hassinen (TKK)

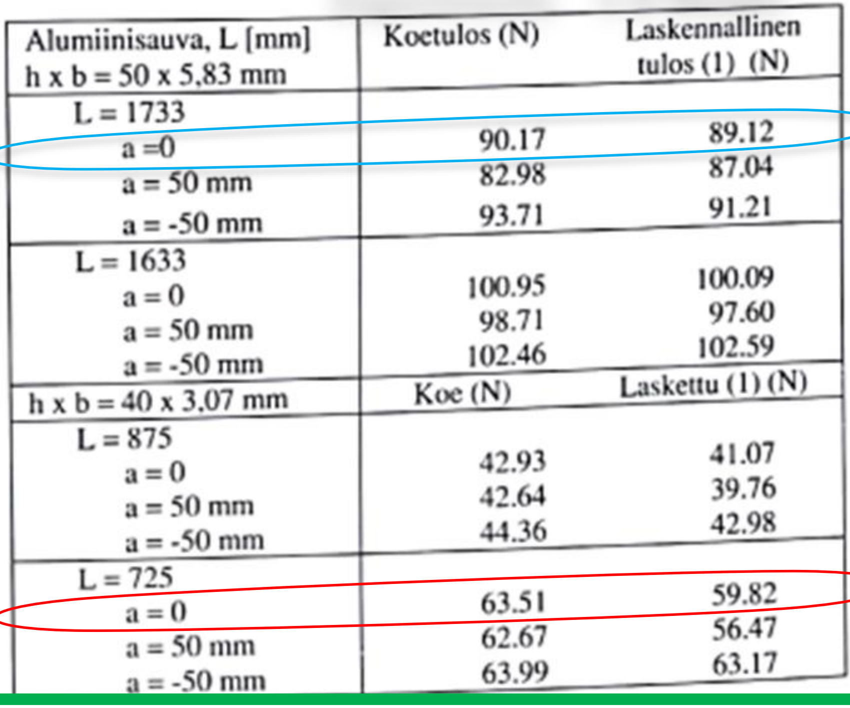
Material Aluminum: E = 70 GPa, ν = 0.33

- **Experiment:** (Southwell-plot) **63.5 N and 90.2 N**
- **FE-model (3-D): 64.6 N and 93.3 N**
- **Analytical (beam model): 59.8 N and 89.1 N**

Experiments **1-D Model**

a = 0

a = 0



Alumiinisauva, L [mm] h x b = 50 x 5,83 mm	Koetulos (N)	Laskennallinen tulos (1) (N)
L = 1733		
a =0	90.17	89.12
a = 50 mm	82.98	87.04
a = -50 mm	93.71	91.21
L = 1633		
a = 0	100.95	100.09
a = 50 mm	98.71	97.60
a = -50 mm	102.46	102.59
h x b = 40 x 3,07 mm	Koe (N)	Laskettu (1) (N)
L = 875		
a = 0	42.93	41.07
a = 50 mm	42.64	39.76
a = -50 mm	44.36	42.98
L = 725		
a = 0	63.51	59.82
a = 50 mm	62.67	56.47
a = -50 mm	63.99	63.17

$$P_{cr} = 4.013/\ell^2 \cdot \sqrt{EI_y GI_t}\left[\pm 1 - \frac{a}{\ell}\sqrt{\frac{EI_y}{GI_t}}\right]$$

(**Prandtl** 1889 & **Timoshenko** 1910)

FE-post-buckling analysis

Post-buckling analysis

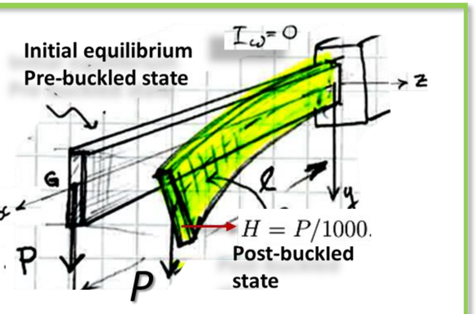
$E = 70$ GPa, $\nu = 0.33$

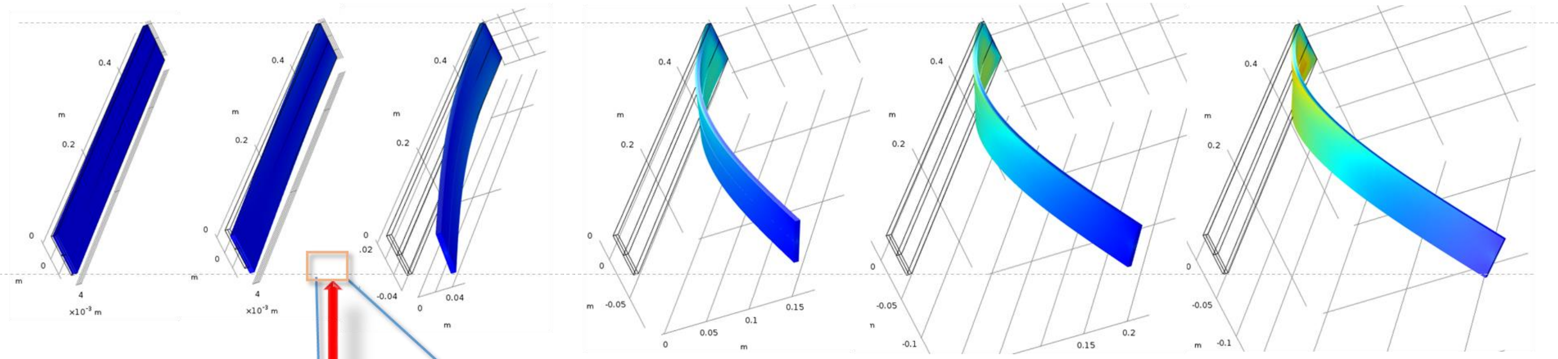
$\ell = 0.5$ m, $b = 5.83$

$h = 50$ mm $\quad H = P/1000.$

$P_{cr} = 1.176$ kN.

perturbation load was at $y = h/4$ away from the center of mass





0P

0.4P

$P_{cr} = 1.176$ kN.

Buckling occurs here

0.6P

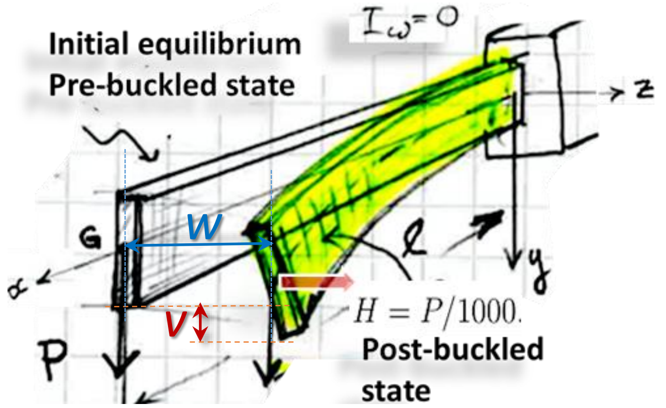
0.7P

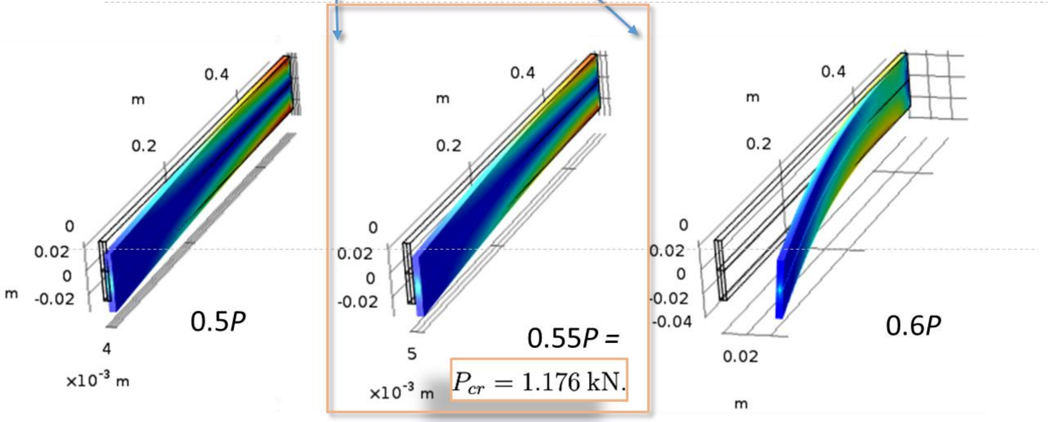
0.8P

N.B. displacement scale is 1:1

1P

P = 2 kN





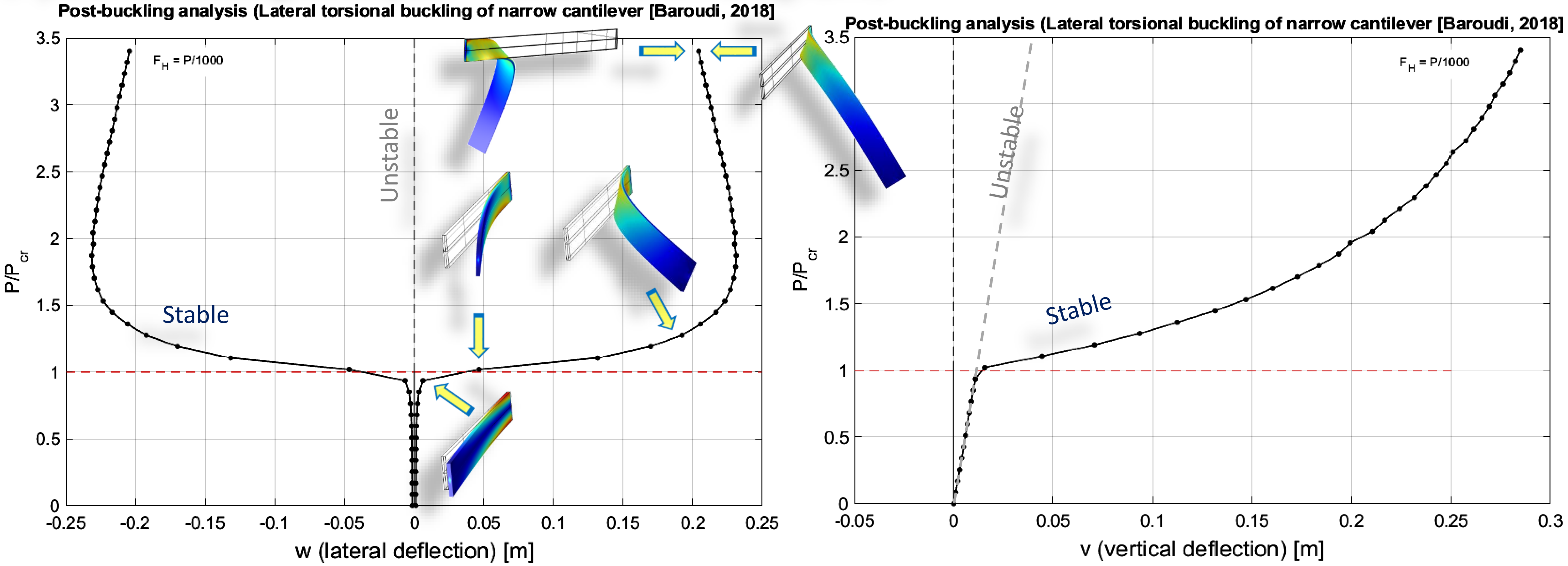
0.5P

0.55P = $P_{cr} = 1.176$ kN.

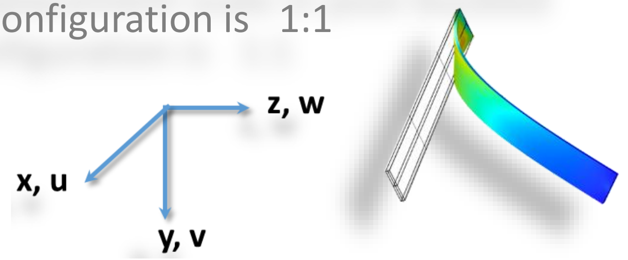
0.6P

[Post-buckling analysis] A thin aluminium cantilever with a vertical tip load $P = 2$ kN and a horizontal perturbation force $H = P/1000$. The critical load being $P_{cr} = 1.176$ kN. Simulation data: $\ell = 0.5$ m, $b = 5.83$ mm, $h = 50$ mm. $E = 70$ GPa, $\nu = 0.33$. Location of the horizontal perturbation load was at $y = h/4$ away from the center of mass of the cross-section.

FE-post-buckling analysis – bifurcation diagrams



Displacement scale in post-buckled configuration is 1:1



Post-buckled configuration

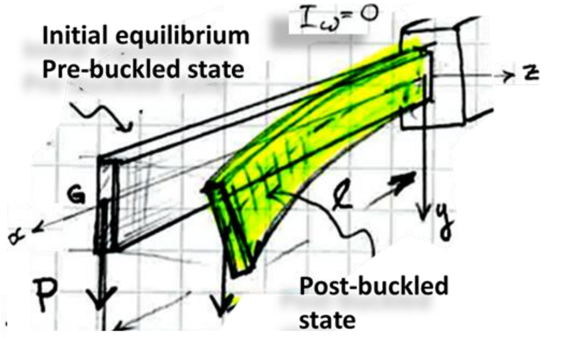
$E = 70$ GPa, $\nu = 0.33$

$\ell = 0.5$ m, $b = 5.83$

$h = 50$ mm $\quad H = P/1000.$

$\boxed{P_{cr} = 1.176\,\text{kN}.}$

perturbation load was at $y = h/4$ away from the center of mass



Lateral-torsional buckling for beams with warping

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t \phi'^2 \mathrm{d}x + \underbrace{\frac{1}{2}\int_0^\ell EI_\omega \phi''^2 \mathrm{d}x}_{\text{new contribution to } \Delta U} +$$

$$+ \underbrace{\int_0^\ell (M_z^0\phi)' w' \mathrm{d}x}_{\text{both bending \& shear initial stresses}} + \underbrace{\int_0^\ell M_z^0 \beta_y (\phi')^2 \mathrm{d}x}_{\text{new contribution to } \Delta W(\tau_{xs}^0)} +$$

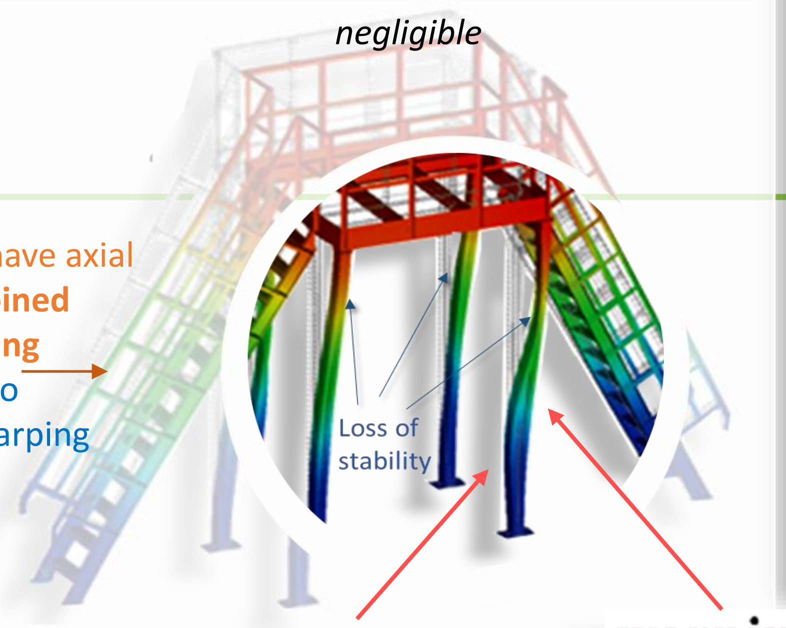
$$\underbrace{+\frac{a_y}{2}\int_0^\ell q_y \phi^2 \mathrm{d}x}_{\text{new contribution to } W_{\text{ext}}} .$$

Strain energy from warping

Thin-walled shells shear is negligible

Note that in this case we have axial compression, so it **is combined torsion and flexural buckling**

I use this illustration just to demonstrate restrained warping

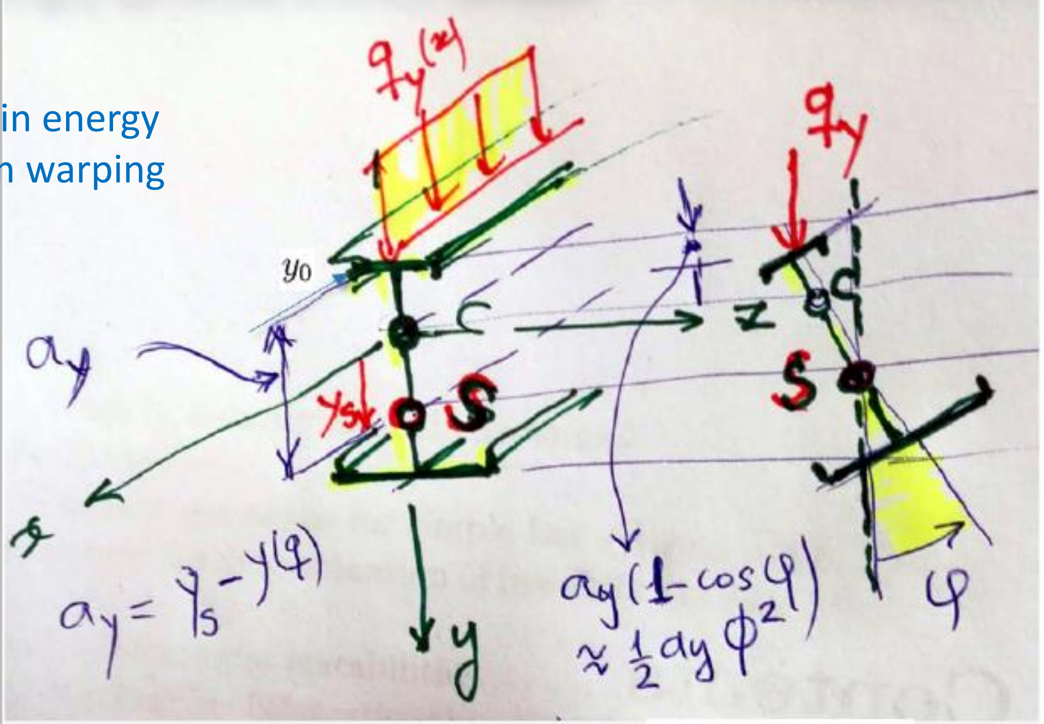


Restrained warping

warping

Lateral-Torsional buckling $I_\omega \neq 0.$

Singly symmetric cross section



S- shear center (rotation center): $(y_s, z_s = 0)$

C- center of mass

Moment arm of external force: $a_y \equiv (y_s - y_0)$

Narrow rectangular cross-section with **no warping**:

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t \phi'^2 \mathrm{d}x + \int_0^\ell (M_z^0\phi)' w' \mathrm{d}x + 1/2 Pa\phi(\ell)^2$$

Narrow rectangular cross-section with no warping and end-point load at *a*

Stability Equations

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t \phi'^2 \mathrm{d}x + \underbrace{\frac{1}{2}\int_0^\ell EI_\omega \phi''^2 \mathrm{d}x}_{\text{new contribution to } \Delta U} +$$

$$+ \underbrace{\int_0^\ell (M_z^0\phi)' w' \mathrm{d}x}_{\text{both bending \& shear initial stresses}} + \underbrace{\int_0^\ell M_z^0 \beta_y (\phi')^2 \mathrm{d}x}_{\text{new contribution to } \Delta W(\tau_{xs}^0)} +$$

$$\underbrace{+\frac{a_y}{2}\int_0^\ell q_y \phi^2 \mathrm{d}x}_{\text{new contribution to } W_{\text{ext}}} .$$

Showing variation of the new contributions only:

$$\delta(\frac{1}{2}\int_0^\ell EI_\omega \phi''^2 \mathrm{d}x) = \int_0^\ell EI_\omega \phi'' \delta\phi'' \mathrm{d}x,$$

$$\delta(-\frac{a_y}{2}\int_0^\ell q_y \phi^2 \mathrm{d}x) = -\frac{a_y}{2}\int_0^\ell q_y \phi \delta\phi \mathrm{d}x,$$

$$\delta(\int_0^\ell M_z^0 \beta_y (\phi')^2 \mathrm{d}x) = \int_0^\ell M_z^0 \beta_y \phi' \delta\phi' \mathrm{d}x.$$

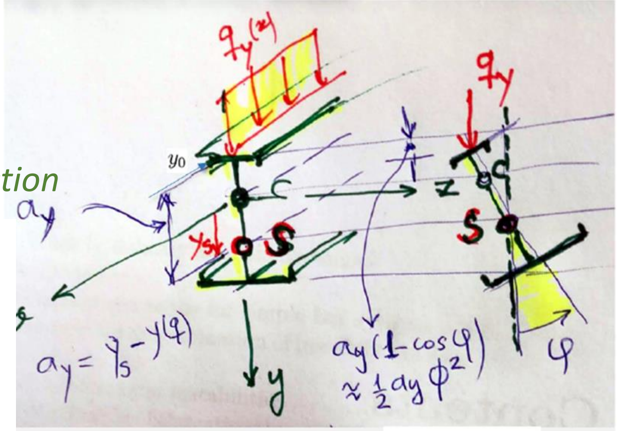
Thin-walled open-cross section (shells) shear is negligible

This term goes to zero when we ignore the effect of initial shear stresses (often we can do so)

Stability loss **criteria:** $\delta(\Delta\Pi) = 0 \Downarrow$

$$\begin{cases} (EI_y w'')'' - (M_z^0\phi)'' = 0, \\ (EI_\omega \phi'')'' - (GI_t\phi')' - M_z^0 w'' - 2\beta_y \cancel{(M_z^0\phi')'} + e_y q_y^0 \phi = 0 \end{cases}$$

Singly symmetric cross section



S- shear center (rotation center): $(y_s, z_s = 0)$

C- center of mass

Moment arm of external force: $a_y \equiv (y_s - y_0)$

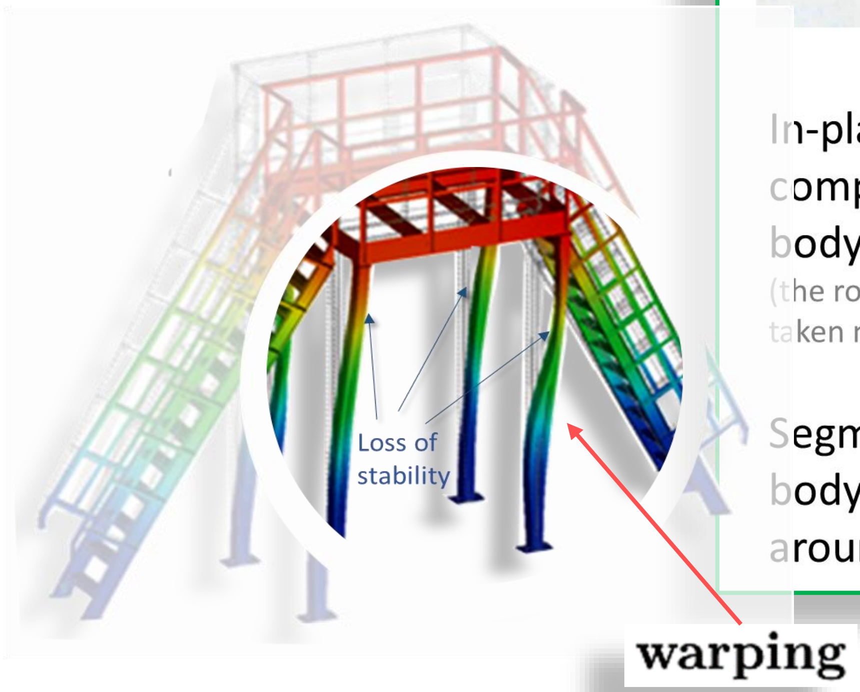
$$a_y = e_y = y_q - y_S$$

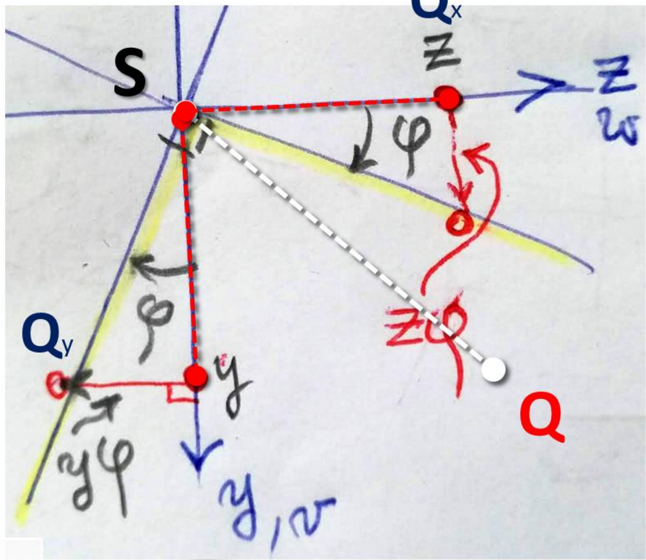
For details in deriving (1.485), refer to the lecture-notes by Prof. J. Paavola

Kinematics of the cross-section for in-plane motion

Kinematics = geometry of the motion

Out-of-plane motion
(= deplanation = warping)
should be added into the axial components of the motion

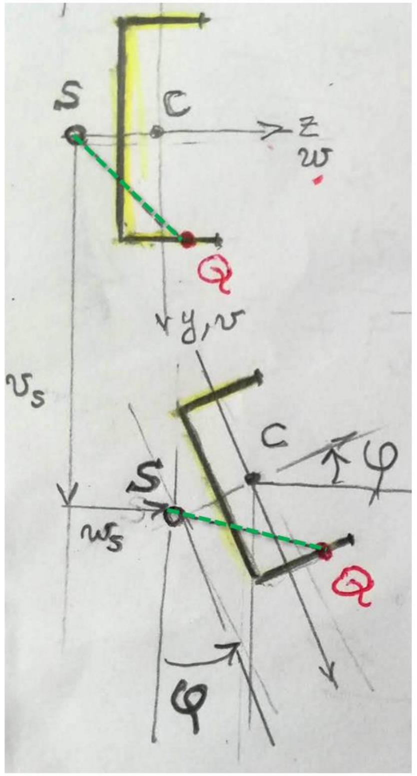


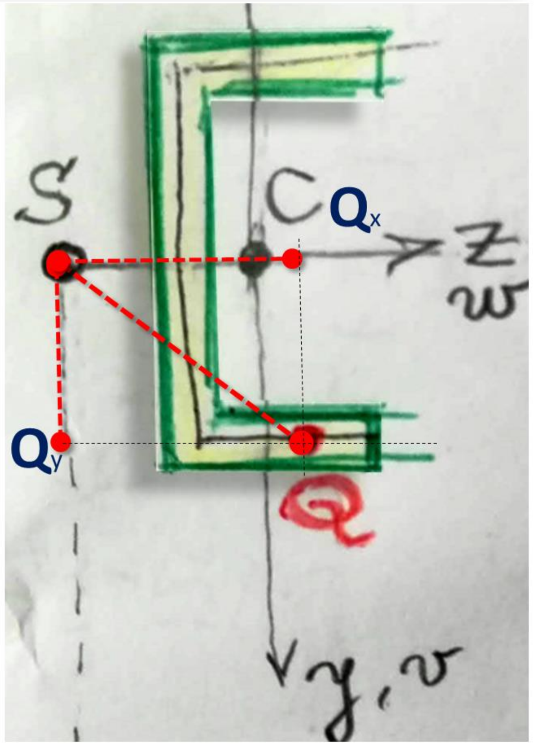


In-plane small displacement components in a small rigid-body rotation

(the rotation direction in this subfigure is taken negative)

Segment SQ has only rigid-body translation and rotation around he shear center S





Segment SQ moves as a rigid-body in the cross-section plan.

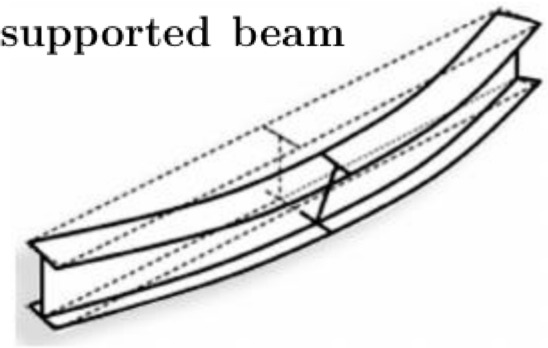
$$\begin{cases} w_{Q'} = w_c + (y_Q - y_S)\overset{\approx \varphi}{\sin\varphi} \\ v_Q = v_c - \underbrace{(z_Q - z_S)}_{\overline{SQ}}\sin\varphi \end{cases}$$

Rotation in this subfigure is correctly positive

Example: edges subjected to constant moment only

Consider such *simply supported beam* with singly-symmetric constant cross-section which is loaded at both ends by a constant moment $M_z^0 = \bar{M}_z^0$

simply supported beam



Lateral buckling of I-beam subject to end moments.

$$\begin{cases} (EI_y w'')'' - (M_z^0 \phi)'' = 0, \\ (EI_\omega \phi'')'' - (GI_t \phi')' - M_z^0 w'' \text{ + shear neglected } + e_y q_y^0 \phi = 0 \end{cases}$$

$e_y = 0$

$$\Downarrow$$

$$EI_y w^{(4)} - \bar{M}_z^0 \phi'' = 0,$$
$$EI_\omega \phi^{(4)} - GI_t \phi'' - \bar{M}_z^0 w'' = 0.$$

Trials should fulfill the PDE and the BCs

Boundary conditions:

$$w(0) = w(\ell) = 0,\ w''(0) = w''(\ell) = 0$$
$$\phi(0) = \phi(\ell) = 0,\ \phi''(0) = \phi''(\ell) = 0$$

These differential equations can be solved in many ways :

1. One way is to eliminate the **rotation** $\phi(x)$ from the first equation and **insert** it in the **second equation**. Then, one solves the last PDE in terms of the rotation $\phi(x)$ only.
2. However, the system of **PDE** with constant coefficients is quit straight-forward to solve by taking *trial solutions*

$$w(x) = A \sin(\pi x/\ell),$$
$$\phi(x) = B \sin(\pi x/\ell).$$

$$\det \begin{bmatrix} [\pi/\ell]^2 EI_y & \bar{M}_z^0 \\ -\bar{M}_z^0 & [\pi/\ell]^2 EI_\omega + GI_t \end{bmatrix} = 0.$$

The buckling moment

$$M_{cr} = \frac{\pi}{\ell} \sqrt{EI_y[EI_\omega(\pi/\ell)^2 + GI_t]}$$

Example: Simply supported beam subjected to transversal constant load

Simply supported beam

$e_y = 0$ ← For cases where the load is along the center-line

$$\begin{cases} (EI_y w'')'' - (M_z^0 \phi)'' = 0, \\ (EI_\omega \phi'')'' - (GI_t \phi')' - M_z^0 w'' \text{[redacted]} + e_y q_y^0 \phi = 0 \end{cases}$$

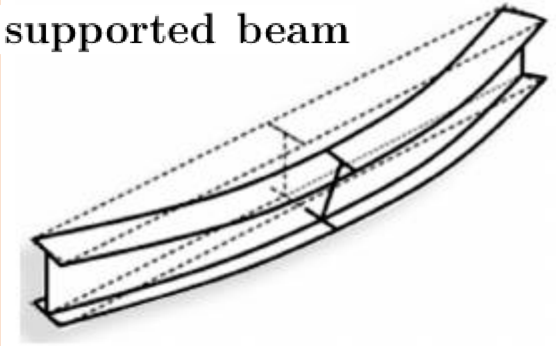
$\Downarrow$

$$M_z^0 = \frac{q_y}{2} x(\ell - x)$$

Insert the pre-stress bending moment

Constant transverse distributed load q_y

simply supported beam



Lateral buckling of I-beam subject to end moments.

For distributed load acting along the center-line, we obtain:

$$EI_y w^{(4)} - \frac{q_y}{2}[x(\ell - x)\phi]'' = 0,$$

$$EI_\omega \phi^{(4)} - GI_t \phi'' - \frac{q_y}{2} x(\ell - x) w'' = 0.$$

a system of coupled equations

This PDE is not easy to solve. **Timoshenko** solved it using *infinite series*.

Boundary conditions:

$$w(0) = w(\ell) = 0,\ w''(0) = w''(\ell) = 0$$

$$\phi(0) = \phi(\ell) = 0,\ \phi''(0) = \phi''(\ell) = 0$$

The solution was given by Timoshenko

$$(q_y \ell)_{cr} = \gamma \sqrt{EI_y GI_t}/\ell^2$$

$$\gamma = f(\frac{GI_t \ell^2}{EI_\omega})$$

← Stability coefficient

Example: Simply supported beam subjected to transversal constant load

Simply supported beam : $M_z^0 = \frac{q_y}{2} x(\ell - x)$

For distributed constant load acting along the center-line

$$EI_y w^{(4)} - \frac{q_y}{2}[x(\ell - x)\phi]'' = 0,$$

$$EI_\omega \phi^{(4)} - GI_t \phi'' - \frac{q_y}{2} x(\ell - x) w'' = 0.$$

$$\Downarrow$$

The solution was given by Timoshenko

$$(q_y \ell)_{cr} = \gamma \sqrt{EI_y GI_t / \ell^2}$$

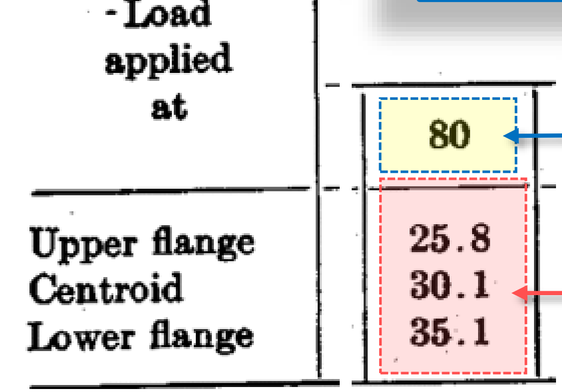
$$\gamma = f\left(\frac{GI_t \ell^2}{EI_\omega}\right)$$

Effect of load locations:

upper flange

centroid

lower flange



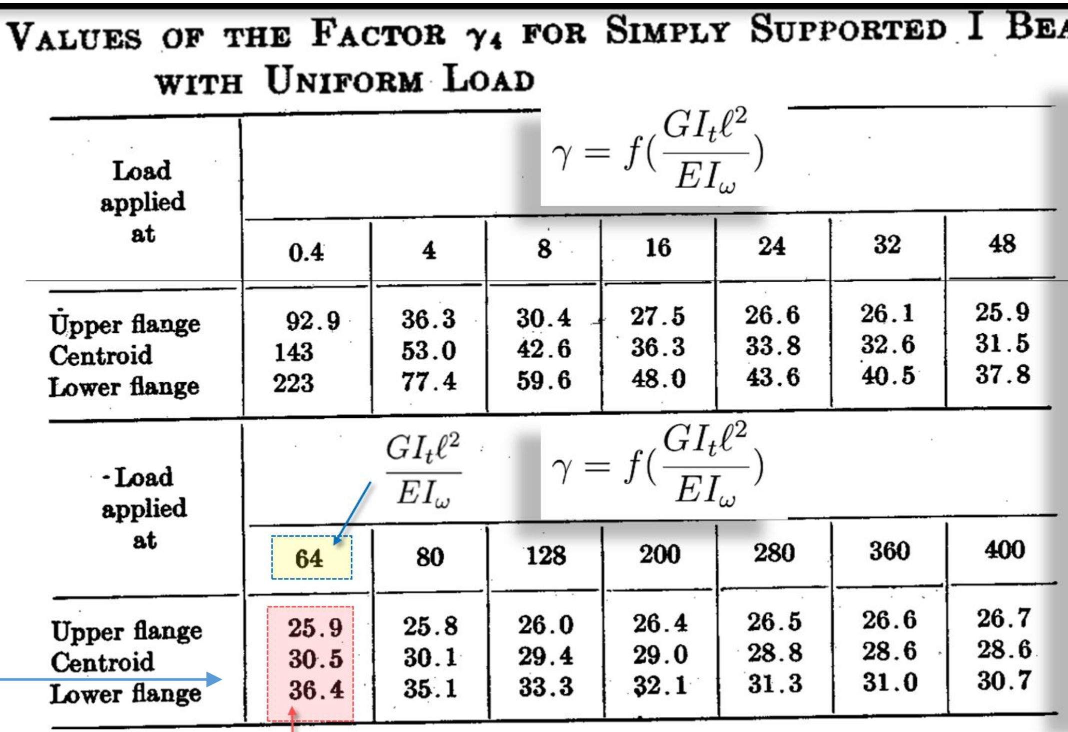
Load applied at	80
Upper flange	25.8
Centroid	30.1
Lower flange	35.1

$\frac{GI_t \ell^2}{EI_\omega}$

γ

Lateral torsional buckling.

Some values for a doubly symmetric I-beam cross-section for various locations (upper flange, centroid and lower flange) of the loading



VALUES OF THE FACTOR γ_4 FOR SIMPLY SUPPORTED I BEAMS WITH UNIFORM LOAD

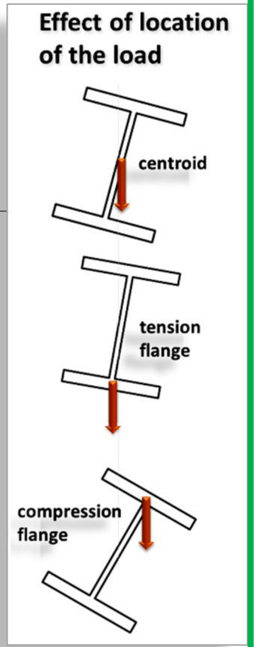
$\gamma = f(\frac{GI_t \ell^2}{EI_\omega})$

Load applied at	0.4	4	8	16	24	32	48
Upper flange	92.9	36.3	30.4	27.5	26.6	26.1	25.9
Centroid	143	53.0	42.6	36.3	33.8	32.6	31.5
Lower flange	223	77.4	59.6	48.0	43.6	40.5	37.8
Load applied at	64	80	128	200	280	360	400
Upper flange	25.9	25.8	26.0	26.4	26.5	26.6	26.7
Centroid	30.5	30.1	29.4	29.0	28.8	28.6	28.6
Lower flange	36.4	35.1	33.3	32.1	31.3	31.0	30.7

$\frac{GI_t \ell^2}{EI_\omega}$

$\gamma = f(\frac{GI_t \ell^2}{EI_\omega})$

γ



$$(q_y \ell)_{cr} = \gamma \sqrt{EI_y GI_t / \ell^2}$$

Timoshenko *Elastic Stability of structures.*

values for γ for a doubly symmetric I-beam

I-beam Cantilever Analytical solution

$$EI_y w^{(4)} - [P(\ell - x)\phi]'' = 0,$$
$$EI_\omega \phi^{(4)} - GI_t \phi'' - P(\ell - x) w'' = 0$$

$$P_{cr} = \gamma_2 \sqrt{EI_y GI_t}/\ell^2$$

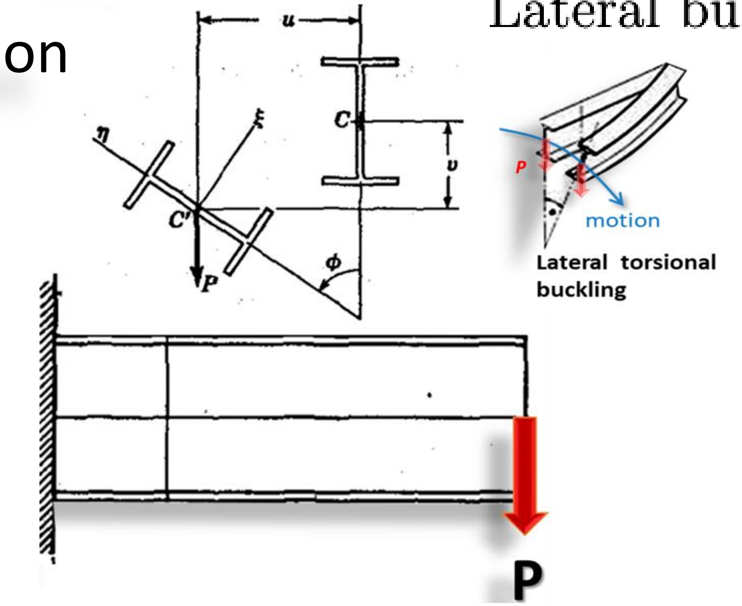
Timoshenko in 1910.

Stability coefficient

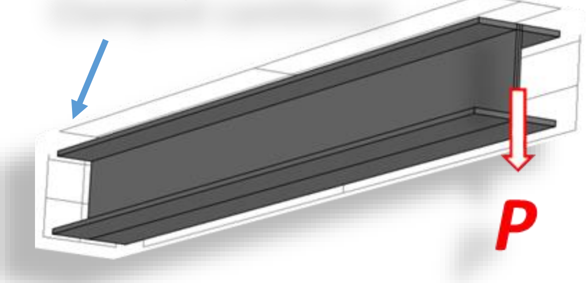
$$\gamma_2 = 4.013/[1 - \sqrt{EI_\omega/GI_t\ell^2}]^2$$

Lateral buckling of I-beam cantilever

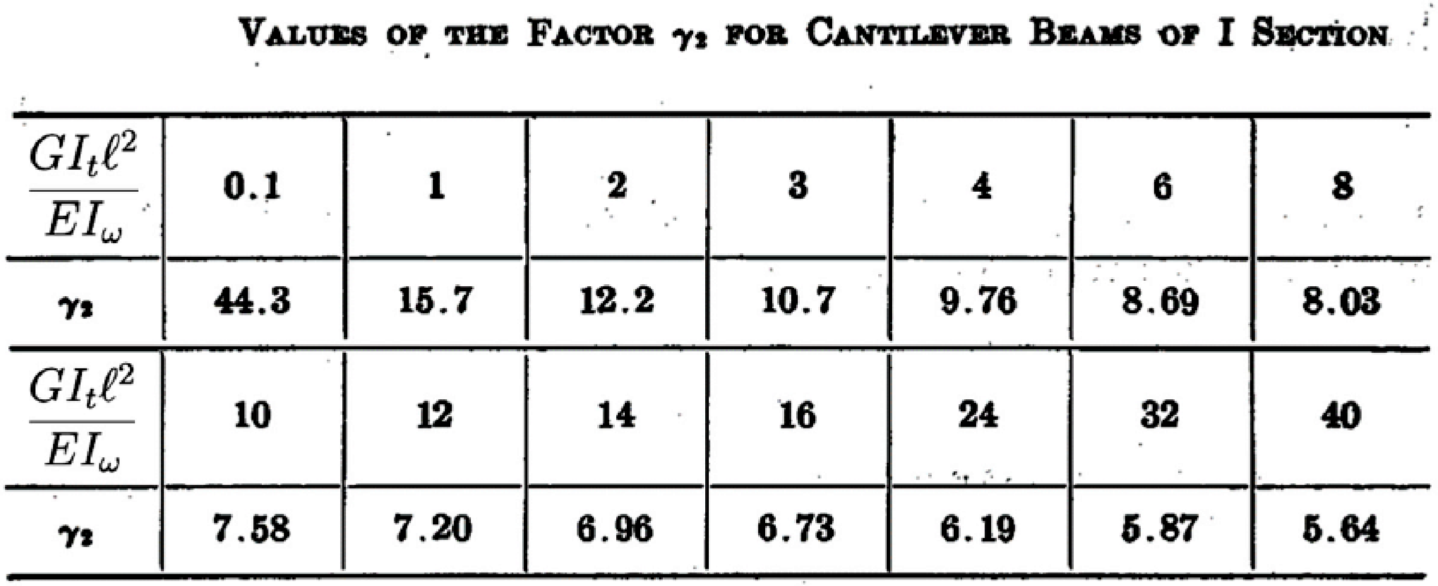
Lateral buckling of I-beam cantilever



Clamped cantilever



$$P_{cr} = \gamma_2 \sqrt{EI_y GI_t}/\ell^2 \qquad \gamma_2 = 4.013/[1 - \sqrt{EI_\omega/GI_t\ell^2}]^2$$



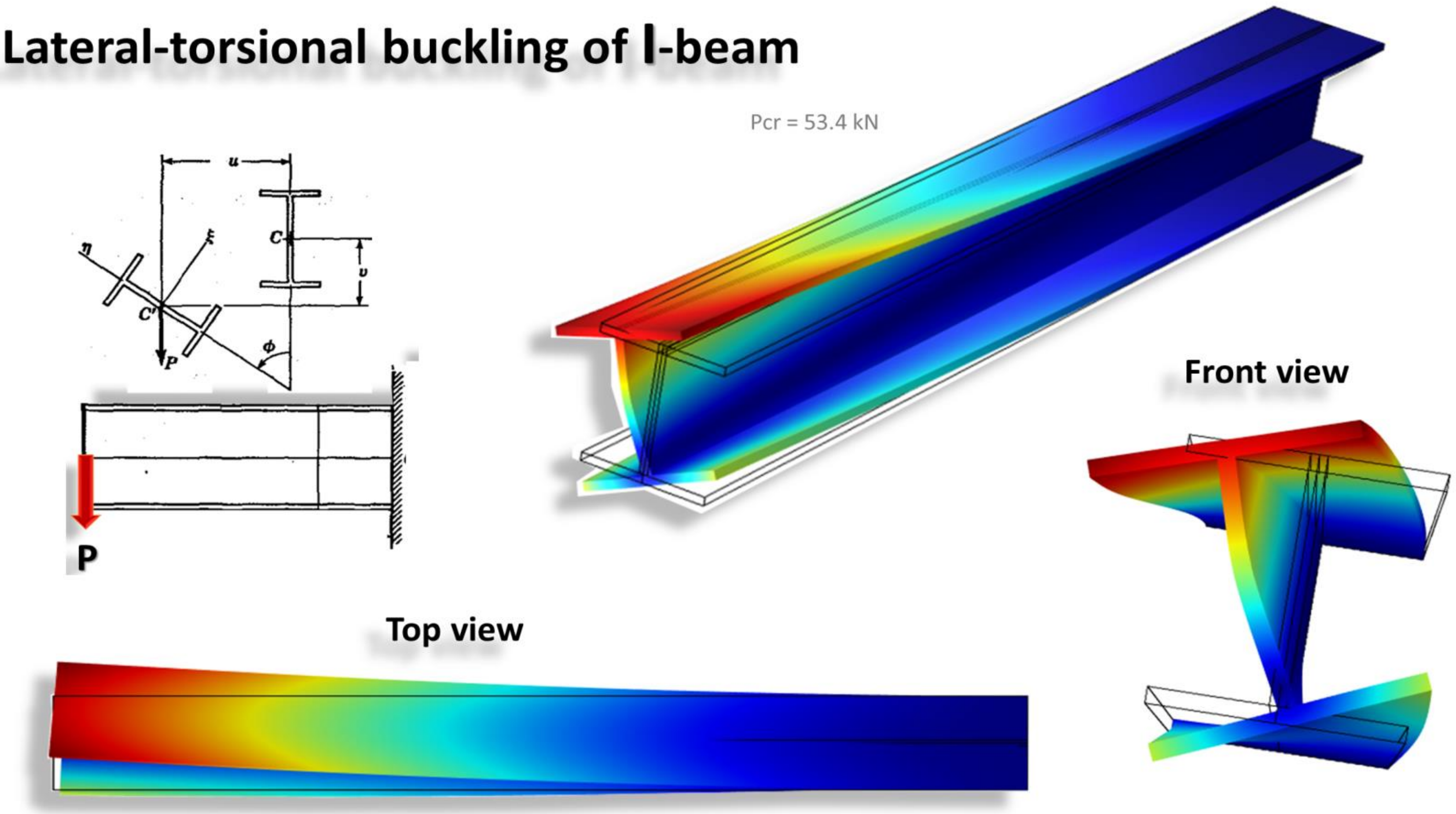
VALUES OF THE FACTOR γ_2 FOR CANTILEVER BEAMS OF I SECTION

$\frac{GI_t\ell^2}{EI_\omega}$	0.1	1	2	3	4	6	8
γ_2	44.3	15.7	12.2	10.7	9.76	8.69	8.03
$\frac{GI_t\ell^2}{EI_\omega}$	10	12	14	16	24	32	40
γ_2	7.58	7.20	6.96	6.73	6.19	5.87	5.64

FE-computational example

FE-Buckling analysis:

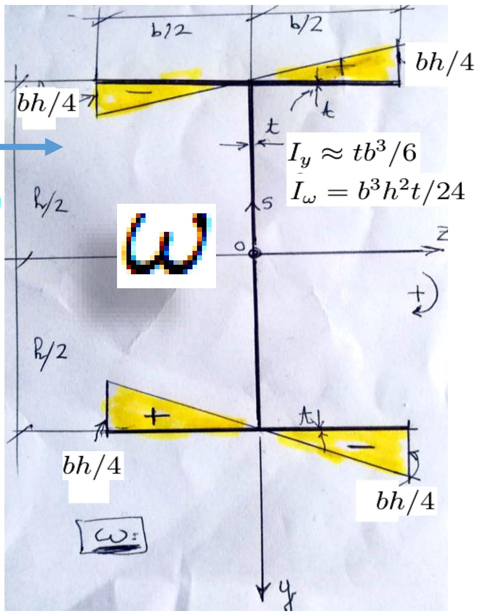
Lateral-torsional buckling of I-beam



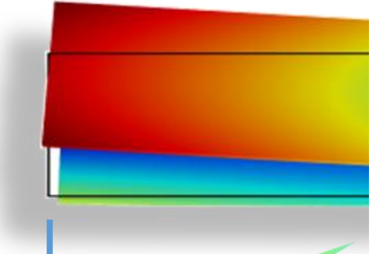
aluminium with $E = 70$ GPa, and $\nu = 0.33$. The thickness is constant 1 cm and the web has $a = 10$ cm hight and the flanges of $a = 10$ cm width

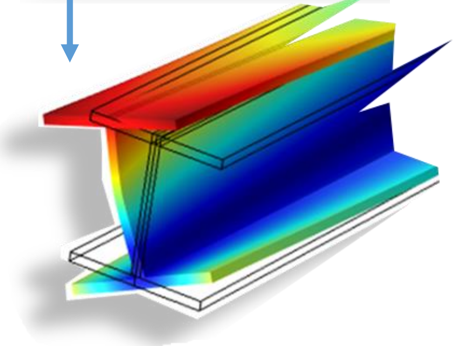
Lateral torsional buckling of doubly symmetric I-beam. The transversal load is at the cross-section centroid.$P_{cr} = 53.4$ kN. Note the small amount of distortion of the web (flexural mode of the web)

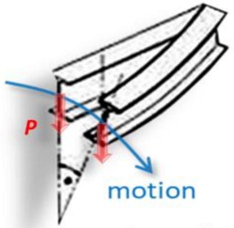
The **out-of-plane axial displacement** is proportional to the sectorial coordinate ω



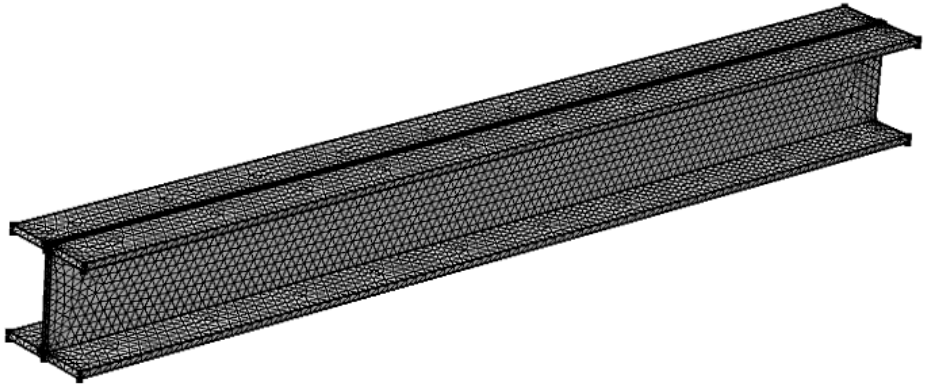
View from top







Lateral torsional buckling



Analytical *versus* FE-solution:

Lateral buckling of I-beam cantilever

Energetic solution

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell GI_t \phi'^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell EI_\omega \phi''^2 \mathrm{d}x + \int_0^\ell (M_z^0\phi)' w' \mathrm{d}x + 1/2 Pa\phi(\ell)^2$$

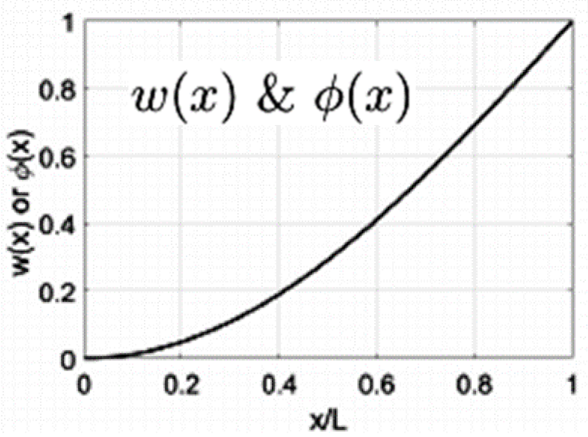
Now *a* = 0

The kinematic boundary conditions

$$\begin{cases} w(0) & = w'(0) = 0, \\ \phi(0) & = \phi'(0) = 0. \end{cases}$$

example of simple candidate fulfils the kinematic constraints

$$\begin{cases} w(x) & = w_0(1 - \cos\frac{\pi x}{2\ell}) \\ \phi(x) & = \phi_0(1 - \cos\frac{\pi x}{2\ell}) \end{cases}$$



$$\begin{cases} w'(x) & = w_0\pi/2\ell \sin\frac{\pi x}{2\ell} \\ \phi'(x) & = \phi_0\pi/2\ell \sin\frac{\pi x}{2\ell} \end{cases}$$

$$\Downarrow$$

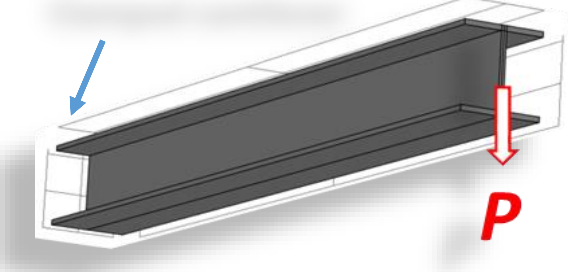
$$P_{cr}^{(FE)} = 53.4 \text{ kN}$$

Analytical solution:

$$P_{cr} = \gamma_2\sqrt{EI_yGI_t}/\ell^2, \quad \text{where}$$

$$\gamma_2 = 4.013/[1 - \sqrt{EI_\omega/GI_t\ell^2}]^2$$

Clamped cantilever

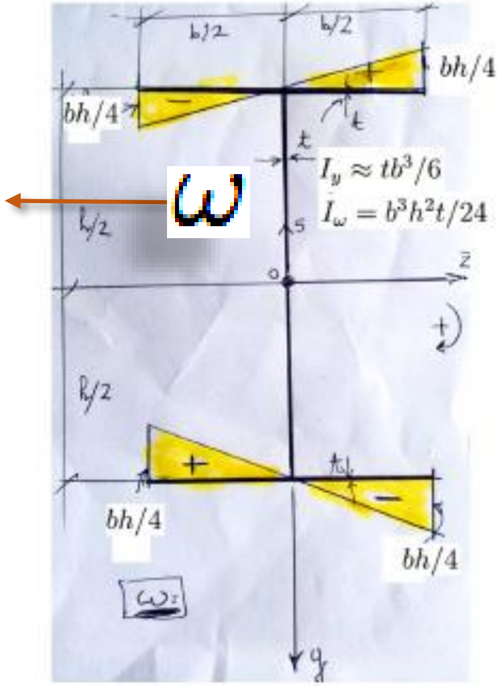


$$A = \int_s t(s)\mathrm{d}s = (2b+h)t,$$

$$I_y = \int_A z^2\mathrm{d}A = \int_s z^2(s)t(s)\mathrm{d}s \approx tb^3/6,$$

$$I_\omega = \int_A \omega^2(s)\mathrm{d}A = \int_s \omega^2(s)t(s)\mathrm{d}s = b^3h^2t/24,$$

$$I_t \approx \frac{1}{3}\sum_i \ell_i t_i^3 = \frac{1}{3}(h+2b)t^3.$$



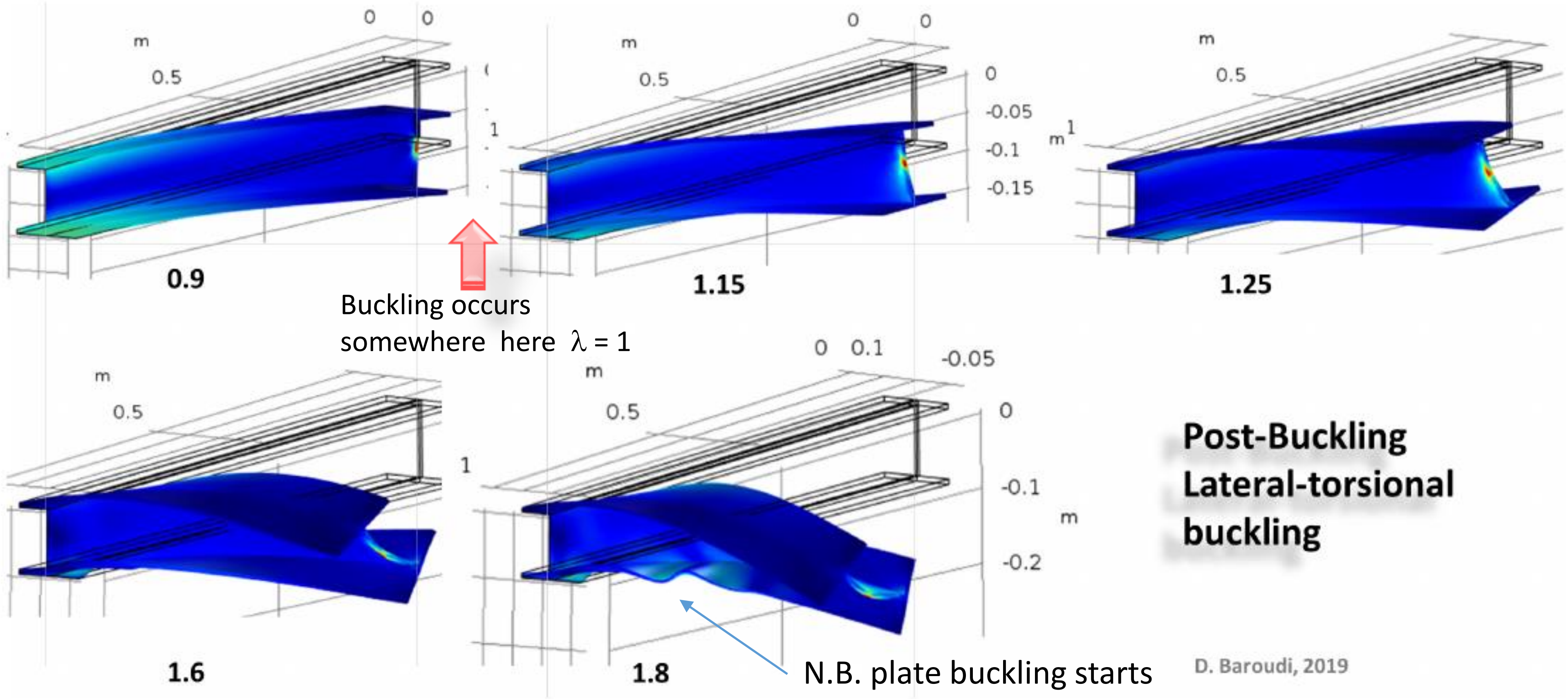
Sectrorial coordinate ω

HW: Find the approximation of the buckling load using Rayleigh-Ritz and compare it to analytical

The shear center and the centroid coincide

doubly symmetric open thin walled cross-section,

Post-buckling analysis



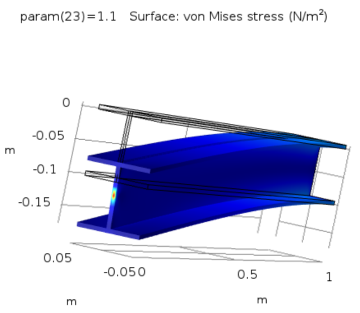
Buckling occurs somewhere here λ = 1

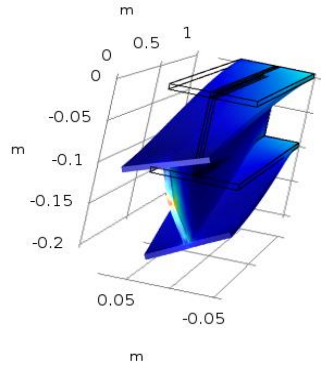
Post-Buckling Lateral-torsional buckling

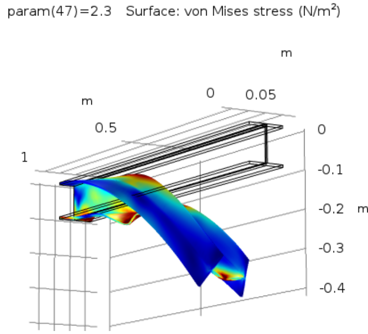
N.B. plate buckling starts (ala-laipan lommahdus)

D. Baroudi, 2019

FE-post-buckling analysis of an aluminium I-beam cantilever. The transversal tip-load is at the centroid. The scalar numbers $\lambda = P/P_{cr}$ in the sub-figures correspond to the the scaled transversal load. Note that for $\lambda \geq 1.8$ local (plate-)buckling (lommahdus) of the lower flange occurs.





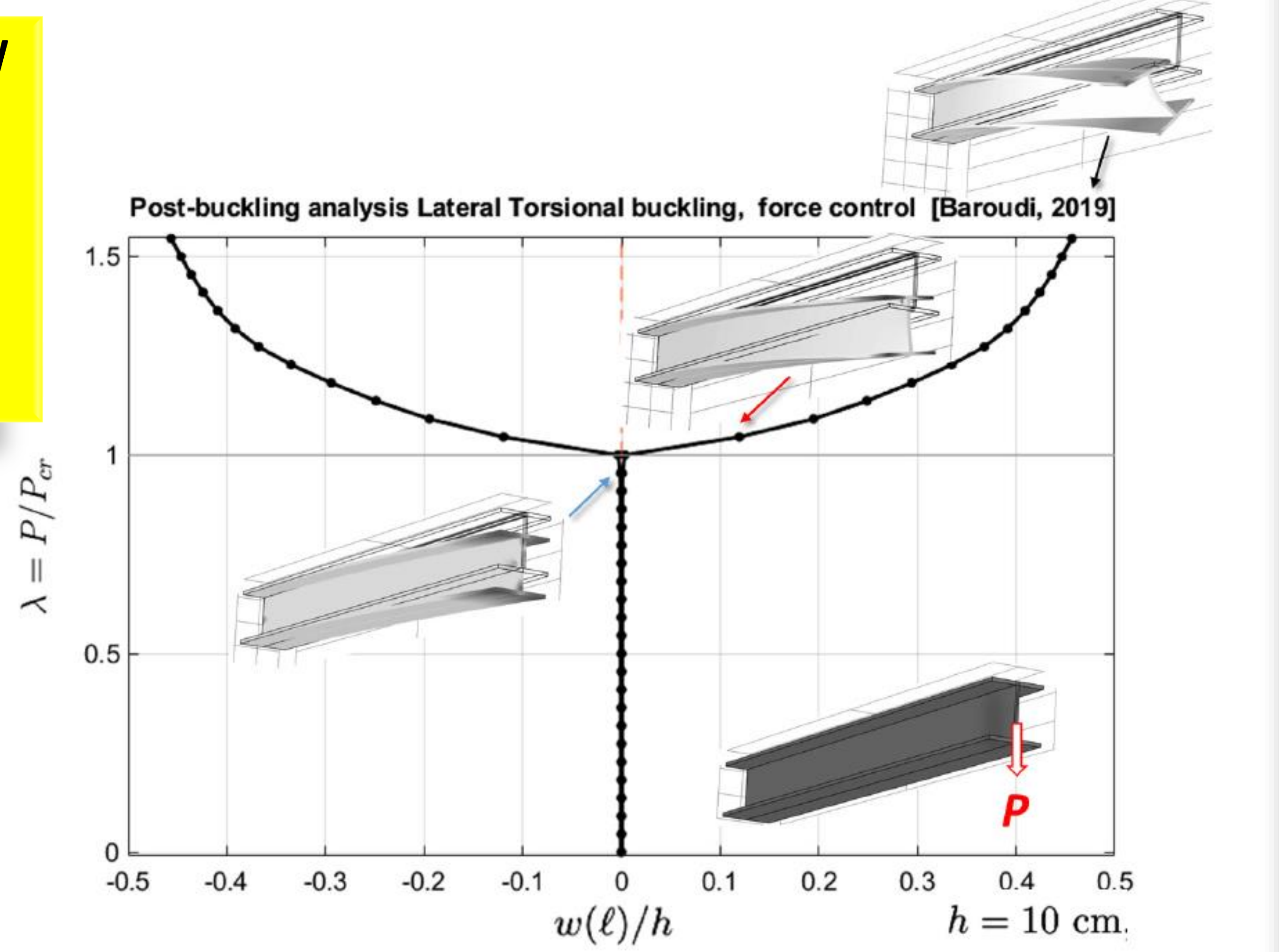


FE- based Post –buckling analysis

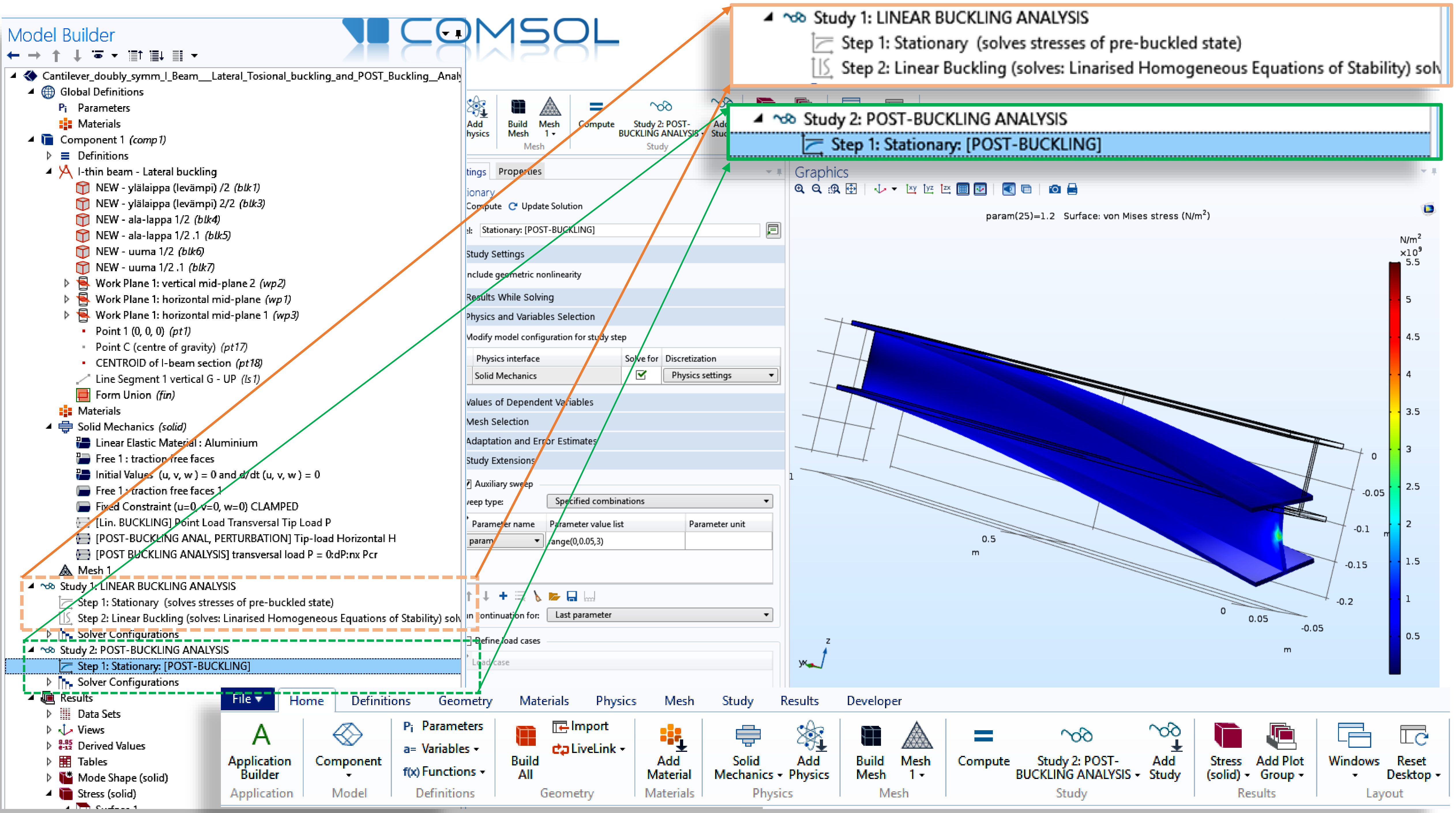
Computer class HW for next week #4

DO FE-based

- **Buckling analysis**
- **Post-buckling analysis**



[redacted] Equilibrium paths. FE-post-buckling analysis of an aluminium I-beam cantilever. The transversal tip-load is at the centroid. [redacted]



Study 1: LINEAR BUCKLING ANALYSIS
Step 1: Stationary (solves stresses of pre-buckled state)
Step 2: Linear Buckling (solves: Linarised Homogeneous Equations of Stability) solv

Study 2: POST-BUCKLING ANALYSIS
Step 1: Stationary: [POST-BUCKLING]

Model Builder

- Cantilever_doubly_symm_I_Beam__Lateral_Tosional_buckling_and_POST_Buckling__Analy
  - Global Definitions
    - Parameters
    - Materials
  - Component 1 *(comp1)*
    - Definitions
    - I-thin beam - Lateral buckling
      - NEW - ylälaippa (levämpi) /2 *(blk1)*
      - NEW - ylälaippa (levämpi) 2/2 *(blk3)*
      - NEW - ala-lappa 1/2 *(blk4)*
      - NEW - ala-lappa 1/2 .1 *(blk5)*
      - NEW - uuma 1/2 *(blk6)*
      - NEW - uuma 1/2 .1 *(blk7)*
      - Work Plane 1: vertical mid-plane 2 *(wp2)*
      - Work Plane 1: horizontal mid-plane *(wp1)*
      - Work Plane 1: horizontal mid-plane 1 *(wp3)*
      - Point 1 (0, 0, 0) *(pt1)*
      - Point C (centre of gravity) *(pt17)*
      - CENTROID of I-beam section *(pt18)*
      - Line Segment 1 vertical G - UP *(ls1)*
      - Form Union *(fin)*
    - Materials
    - Solid Mechanics *(solid)*
      - Linear Elastic Material : Aluminium
      - Free 1 : traction free faces
      - Initial Values (u, v, w) = 0 and d/dt (u, v, w) = 0
      - Free 1 : traction free faces 1
      - Fixed Constraint (u=0, v=0, w=0) CLAMPED
      - [Lin. BUCKLING] Point Load Transversal Tip Load P
      - [POST-BUCKLING ANAL, PERTURBATION] Tip-load Horizontal H
      - [POST BUCKLING ANALYSIS] transversal load P = 0:dP:nx Pcr
    - Mesh 1
  - Study 1: LINEAR BUCKLING ANALYSIS
    - Step 1: Stationary (solves stresses of pre-buckled state)
    - Step 2: Linear Buckling (solves: Linarised Homogeneous Equations of Stability) solv
    - Solver Configurations
  - Study 2: POST-BUCKLING ANALYSIS
    - Step 1: Stationary: [POST-BUCKLING]
    - Solver Configurations
  - Results
    - Data Sets
    - Views
    - Derived Values
    - Tables
    - Mode Shape (solid)
    - Stress (solid)
      - Surface 1

ionary

Compute   Update Solution

el: Stationary: [POST-BUCKLING]

Study Settings

nclude geometric nonlinearity

Results While Solving

Physics and Variables Selection

Modify model configuration for study step

	Physics interface	Solve for	Discretization
	Solid Mechanics	☑	Physics settings

Values of Dependent Variables

Mesh Selection

Adaptation and Error Estimates

Study Extensions

☑ Auxiliary sweep

veep type: Specified combinations

Parameter name	Parameter value list	Parameter unit
param	range(0,0.05,3)	

un continuation for: Last parameter

☐ Define load cases

Load case

Graphics

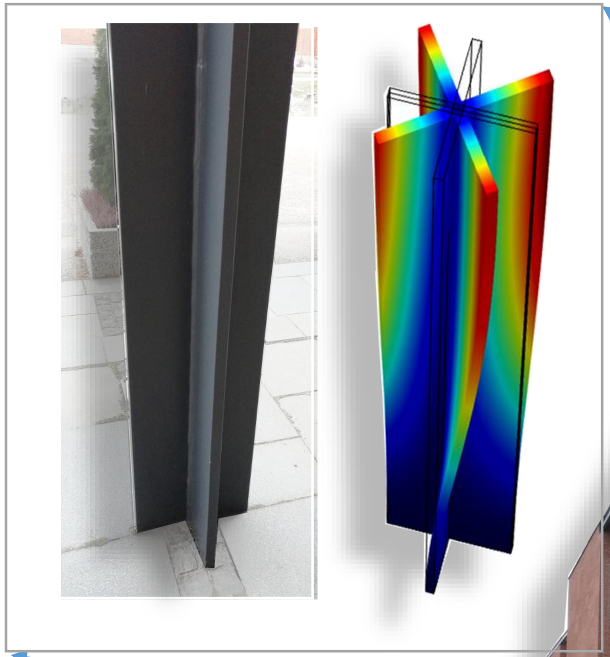
param(25)=1.2 Surface: von Mises stress (N/m²)

Torsional buckling

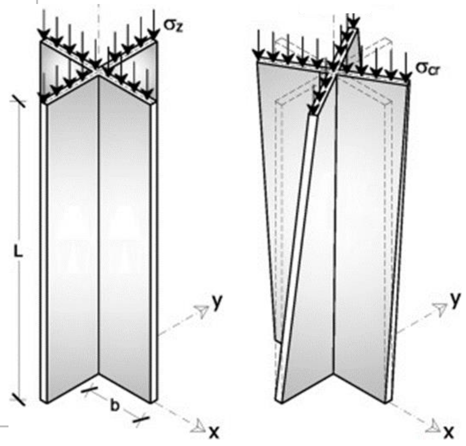
- **Lateral-torsional buckling:** beams loaded transversally with respect to center-line axis [previous topic]
- **Torsional buckling:** axial thrust (compression) normal to the cross-section [this topic]

For columns with **thin-walled open cross-sections**, the torsional rigidity is dramatically smaller as compared to the same but closed section.

When torsional rigidity is much small as compared to flexural rigidity in the principal directions **loss of stability** through **torsional mode** may **occur**.

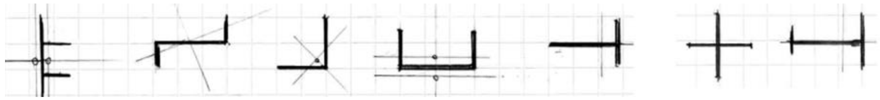


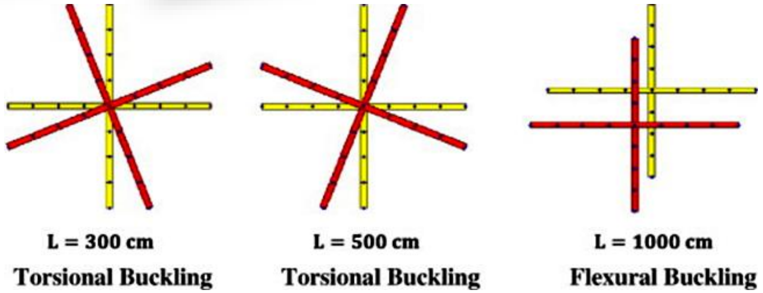
HW: determine the buckling load for this specific x-shaped column



Pure torsional buckling mode

Thin-walled open cross-sections



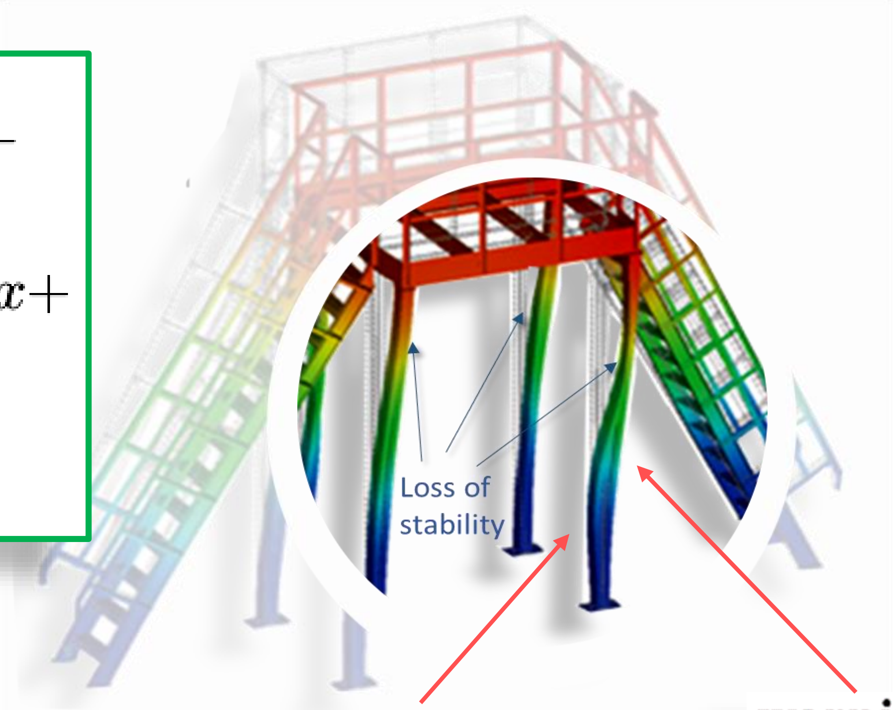


Combined torsional and flexural buckling

- This expression is general & accounts for **combined torsional** and **flexural buckling**
- the loading is **axial centric thrust**

Total potential energy

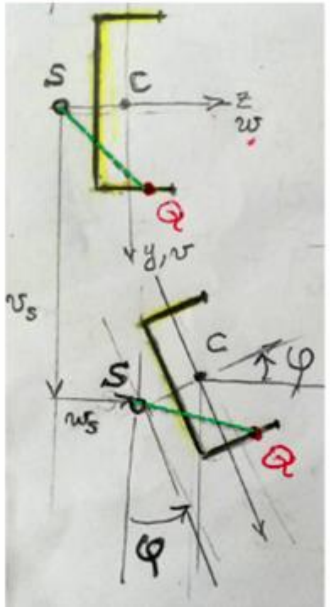
$$\begin{aligned}\Delta\Pi =&\frac{1}{2}\int_0^\ell EI_y w''^2\mathrm{d}x + \frac{1}{2}\int_0^\ell EI_z v''^2\mathrm{d}x +\\ &+\frac{1}{2}\int_0^\ell GI_t\phi'^2\mathrm{d}x + \frac{1}{2}\int_0^\ell EI_\omega \phi''^2\mathrm{d}x +\\ &+\int_0^\ell\int_A \sigma_x^0\frac{1}{2}[(w'_Q)^2+(v'_Q)^2]\mathrm{d}A\mathrm{d}x\end{aligned}$$



Restrained warping

warping

Geometry of the motion of a material point on the cross-section



Pure torsional buckling will be treated as a special case where no flexion occurs

Torsional buckling

Total potential energy

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell EI_z v''^2 \mathrm{d}x + \\ + \frac{1}{2}\int_0^\ell GI_t \phi'^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell EI_\omega \phi''^2 \mathrm{d}x + \\ + \int_0^\ell \int_A \sigma_x^0 \frac{1}{2}[\boxed{(w'_Q)^2} + \boxed{(v'_Q)^2}]\mathrm{d}A\mathrm{d}x$$

- this expression is more general & accounts for **combined torsional and flexural buckling**
- the loading is axial centric thrust

neglecting the work of shear stresses,

The kinematics

arbitrary material point $Q(y, z)$ of the cross-section

$$\begin{cases} u_Q(x) = u - yv' - zw' - \omega\phi', \\ \boxed{v_Q(x)} = v - (z - z_s)\phi, \\ \boxed{w_Q(x)} = w + (y - y_s)\phi, \end{cases}$$

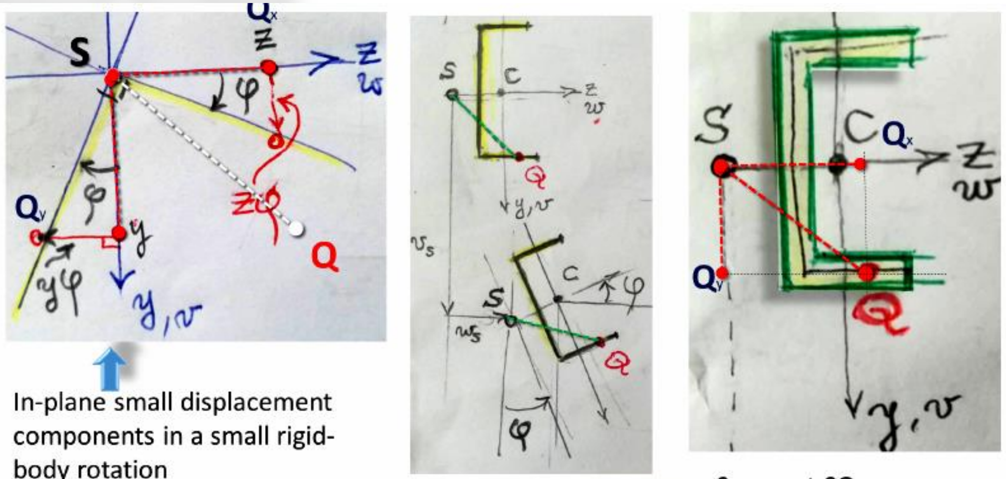
combined motion

translation

rigid body rotation $\phi(x)$

Centroid (C) translations

Kinematics for combined torsional and flexural buckling



In-plane small displacement components in a small rigid-body rotation

(the rotation direction in this subfigure is taken negative)

Segment SQ has only rigid-body translation and rotation around he shear center S

$$\begin{cases} w_Q = w_c + (y_Q - y_s)\underbrace{\sin\varphi}_{\approx\varphi} \\ v_Q = v_c - \underbrace{(z_Q - z_s)}_{SQ}\sin\varphi \end{cases}$$

Rotation in this subfigure is correctly positive

Segment SQ moves as a rigid-body in the cross-section plan.

$$\Downarrow$$

The increment of work due to initial stresses

$$\frac{1}{2}\int_V \sigma_x^0[(w'_Q)^2 + (v'_Q)^2]\mathrm{d}V = \\ = \frac{P}{2}\int_0^\ell [(w')^2 + (v')^2 + r^2(\phi')^2 - 2z_s v'\phi' + 2y_s w'\phi']\mathrm{d}x$$

$$r^2 = I_p/A = (I_z + I_y)/A + (y_s^2 + z_s^2)$$

$$\delta(\Delta\Pi) = 0 \Longrightarrow$$ Stability loss equations:

Stability criteria

$$EI_z v^{(4)} + P\,[v'' + (z_s - e_z)\phi''] = 0,$$
$$EI_y w^{(4)} + P\,[w'' - (y_s - e_y)\phi''] = 0,$$
$$EI_\omega \phi^{(4)} - GI_t\phi'' + P\,[(z_s - e_z)v'' - (y_s - e_y)w'' + \gamma\phi''] = 0,$$

The eccentricities being e_y and e_z

Pure torsional buckling will be treated as a special case where no flexion occurs

Torsional buckling

Total potential energy

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI_y w''^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell EI_z v''^2 \mathrm{d}x + \\ + \frac{1}{2}\int_0^\ell GI_t \phi'^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell EI_\omega \phi''^2 \mathrm{d}x + \\ + \int_0^\ell \int_A \sigma_x^0 \frac{1}{2}[(w_Q')^2 + (v_Q')^2]\mathrm{d}A\mathrm{d}x$$

The kinematics*

neglecting the work of shear stresses.

arbitrary material point $Q(y,z)$ of the cross-section

$$\begin{cases} u_Q(x) = u - yv' - zw' - \omega\phi', \\ v_Q(x) = v - (z - z_s)\phi, \\ w_Q(x) = w + (y - y_s)\phi, \end{cases}$$

combined motion

translation

rigid body rotation $\phi(x)$

Centroid (C) translations

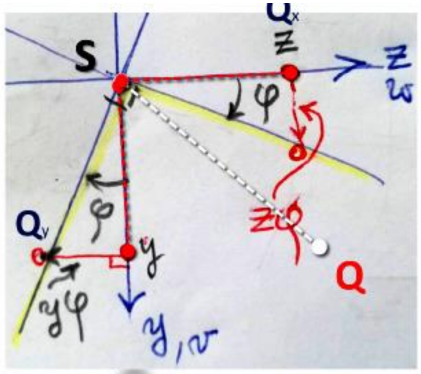
The increment of work due to initial stresses

$$\frac{1}{2}\int_V \sigma_x^0[(w_Q')^2 + (v_Q')^2]\mathrm{d}V = \\ = \frac{P}{2}\int_0^\ell [(w')^2 + (v')^2 + r^2(\phi')^2 - 2z_s v'\phi' + 2y_s w'\phi']\mathrm{d}x$$

$$r^2 = I_p/A = (I_z + I_y)/A + (y_s^2 + z_s^2)$$

Kinematics for combined torsional and flexural buckling

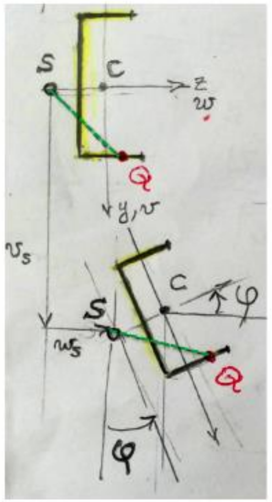
The kinematics* here we assume that distortional (=local plate buckling) does not occur first or we have enough plate stiffeners to avoid it.



In-plane small displacement components in a small rigid-body rotation

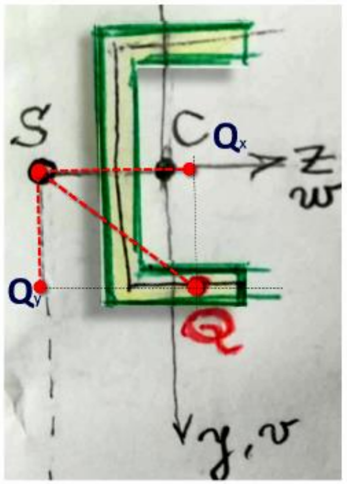
(the rotation direction in this subfigure is taken negative)

Segment SQ has only rigid-body translation and rotation around he shear center S



$$\begin{cases} w_Q = w_c + (y_Q - y_s)\overset{\approx\varphi}{\sin\varphi} \\ v_Q = v_c - \underbrace{(z_Q - z_s)}_{SQ}\sin\varphi \end{cases}$$

Rotation in this subfigure is correctly positive



Segment SQ moves as a rigid-body in the cros section plan.

Stability criteria

$$\delta(\Delta\Pi) = 0 \Longrightarrow$$ Stability loss equations

Solutions of these **PDEs** (Eigen-value problems) provides the **buckling load** and the corresponding mode

Next, we consider symmetry cases of cross-sections simplifications $\Longrightarrow$ *a)* **singly symmetric** cross-section ….

Torsional buckling

Combined torsional and flexural buckling

Singly symmetric cross-section

the loading acts in the plane of symmetry

$$e_z = 0,\ \beta_z = 0 \quad z_s = 0,$$

geometric factors of the cross-section

$$\begin{aligned} r^2 &= \frac{I_y + I_z}{A} + y_s^2 + z_s^2, \\ \beta_y &= \frac{1}{2I_z}\int_A y(y^2+z^2)\mathrm{d}A - y_s, \\ \beta_z &= \frac{1}{2I_y}\int_A z(y^2+z^2)\mathrm{d}A - z_s, \\ \gamma &= (r^2 + 2\beta_y e_y + 2\beta_z e_z). \end{aligned}$$

Stability criteria

$$\delta(\Delta\Pi) = 0 \quad \Downarrow$$

Stability loss equations

Combined torsional and flexural buckling

$$EI_z v^{(4)} + Pv'' = 0,$$

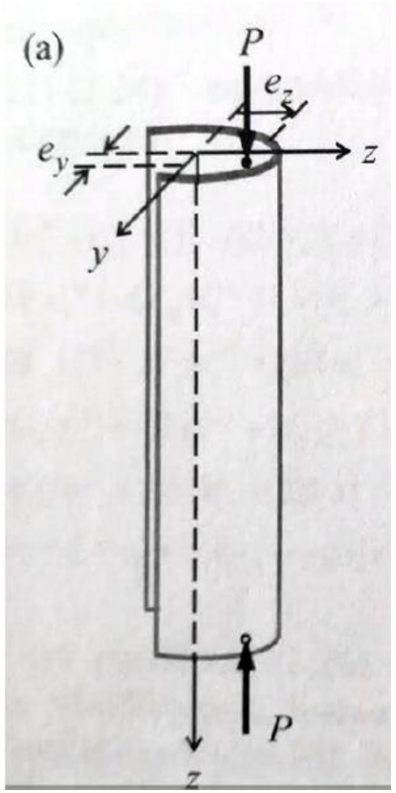
$$EI_y w^{(4)} + P\left[w'' - (y_s - \boxed{e_y})\phi''\right] = 0,$$

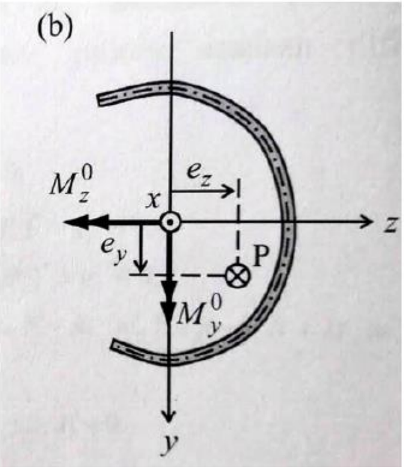
$$EI_\omega \phi^{(4)} - GI_t\phi'' + P\left[-(y_s - \boxed{e_y})w'' + (r^2 + 2\beta_y \boxed{e_y})\phi''\right] = 0,$$

Coordinates of the SC

For centric loading, put all the eccentricities equal to zero …
and … solve the problem

$$\Downarrow$$





General illustration
Eccentric loading in this figures

Torsional buckling

Combined torsional and flexural buckling

Singly symmetric cross-section

$$EI_z v^{(4)} + Pv'' = 0,$$
$$EI_y w^{(4)} + P\left[w'' - (y_s - e_y)\phi''\right] = 0,$$
$$EI_\omega \phi^{(4)} - GI_t\phi'' + P\left[-(y_s - e_y)w'' + (r^2 + 2\beta_y e_y)\phi''\right] = 0,$$

Now centric loading $\Downarrow$

Doubly symmetric cross-section & centric thrust

$y_s = z_s = 0,\ e_y = e_z = 0,\ \beta_y = \beta_z = 0$

$$EI_z v^{(4)} + Pv'' = 0,$$
$$EI_y w^{(4)} + P\left[w'' \qquad\qquad\right] = 0,$$

Flexural buckling

Torsional buckling

$$EI_\omega \phi^{(4)} - GI_t\phi'' + P\left[- \qquad\qquad + (r^2 \qquad\qquad)\phi''\right] = 0,$$

The **smallest** critical load is the buckling load

Decoupled torsion and bending $\Downarrow$

Pure torsional buckling

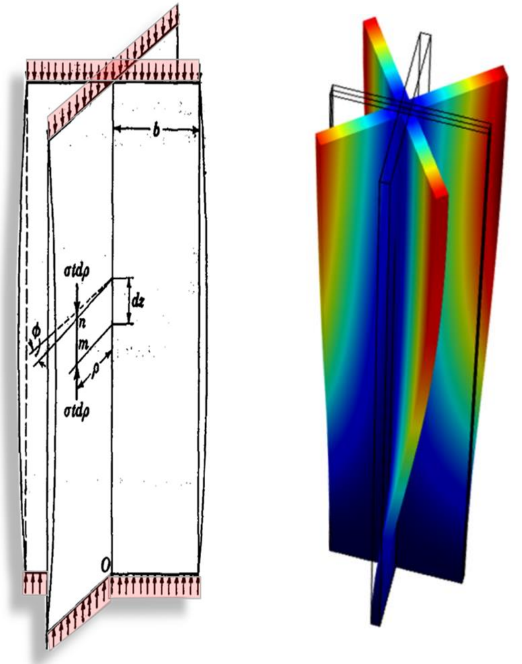
$$EI_\omega \phi^{(4)} + (Pr^2 - GI_t)\phi'' = 0. \Longrightarrow$$

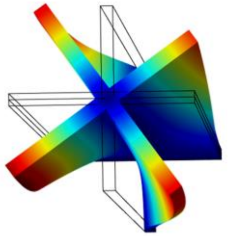
Buckling load in pure torsional mode

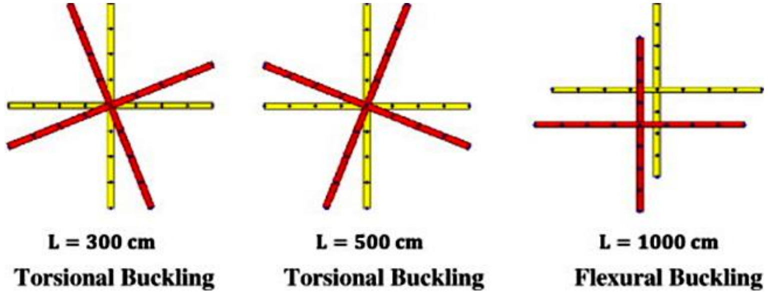
$$P_{cr} = \frac{1}{r^2}\left[\frac{\pi^2 EI_\omega}{L_\phi^2} + GI_t\right]$$

the buckling length L_ϕ^2 should be determined according to the boundary conditions.

Pure torsional buckling






L = 300 cm
Torsional Buckling

L = 500 cm
Torsional Buckling

L = 1000 cm
Flexural Buckling

Pure torsional buckling

Combined torsional and flexural buckling

Doubly symmetric cross-section & centric thrust

$$y_s = z_s = 0,\ e_y = e_z = 0,\ \beta_y = \beta_z = 0$$

$$v(x) = A + Bx + C\sin(kx) + D\cos(kx)$$

$$k^2 = P/EI$$

$$EI_z v^{(4)} + Pv'' = 0,$$

$$EI_y w^{(4)} + Pw'' = 0,$$

Flexural buckling

Decoupled

$$EI_\omega \phi^{(4)} - GI_t\phi'' + Pr^2\phi'' = 0.$$

Pure torsional buckling

$$\Downarrow$$

$$\begin{cases} P_{cr,v} = \pi^2 EI_z/L_v^2, \\ P_{cr,w} = \pi^2 EI_y/L_w^2, \\ P_{cr,\phi} = \frac{1}{r^2}[\pi^2 EI_\omega/L_\phi^2 + GI_t] \end{cases}$$

The **smallest** critical load is the **buckling load**

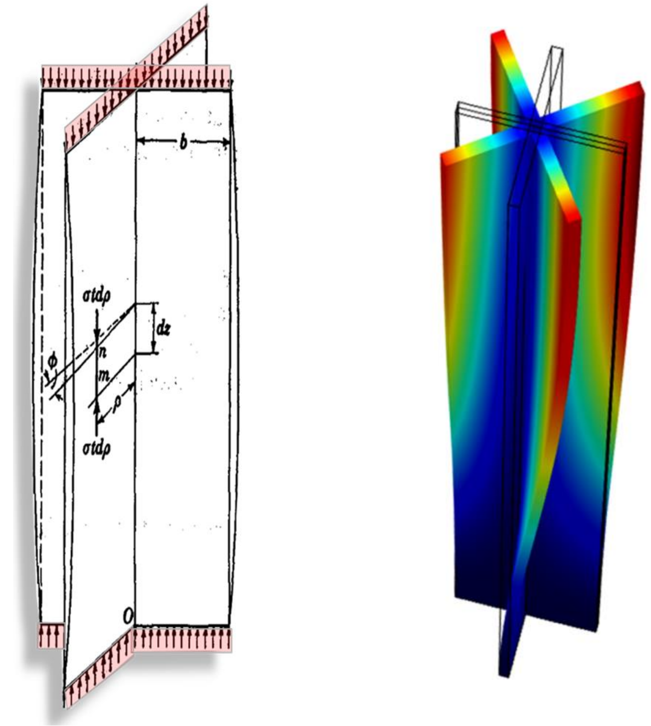
buckling length are L_v^2, L_w^2 and L_ϕ^2

Buckling lengths depend on the specific boundary conditions

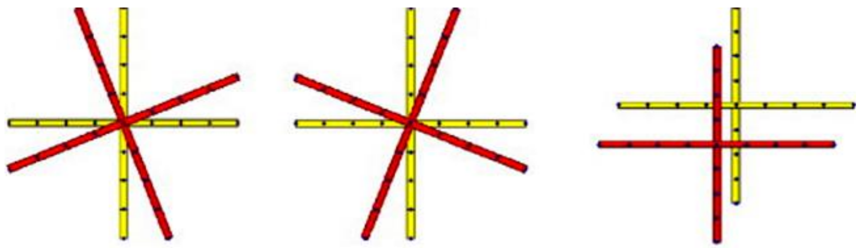
Pure torsional buckling

$$\phi^{(4)} + \underbrace{\frac{Pr^2 - GI_t}{EI_\omega}}_{k^2}\phi'' = 0,$$

General solution: $\phi(x) = A + Bx + C\sin kx + D\cos kx,$



Centric load with doubly symmetric X-section



L = 300 cm
Torsional Buckling

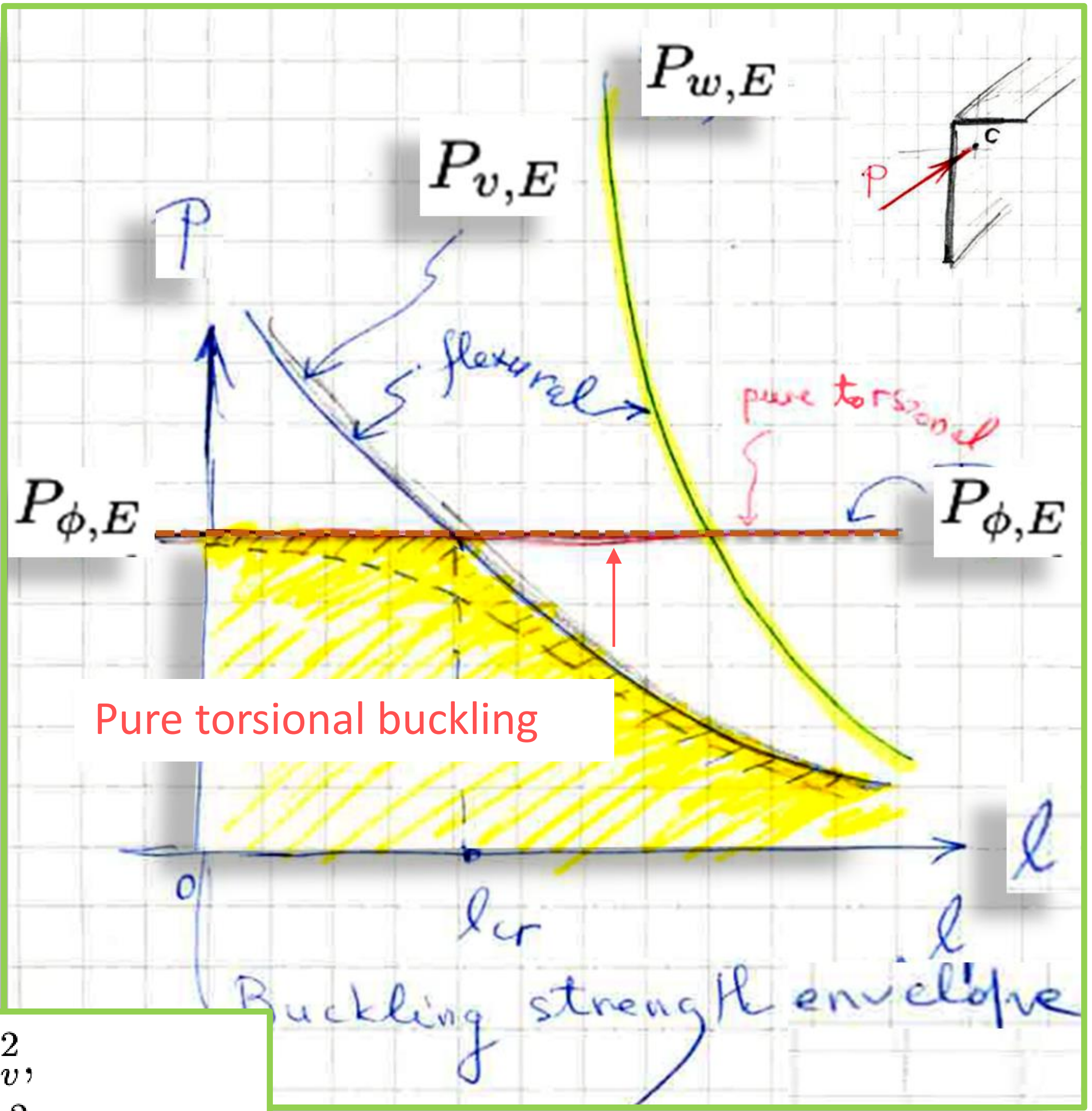
L = 500 cm
Torsional Buckling

L = 1000 cm
Flexural Buckling

Determine the critical length for the mode transition

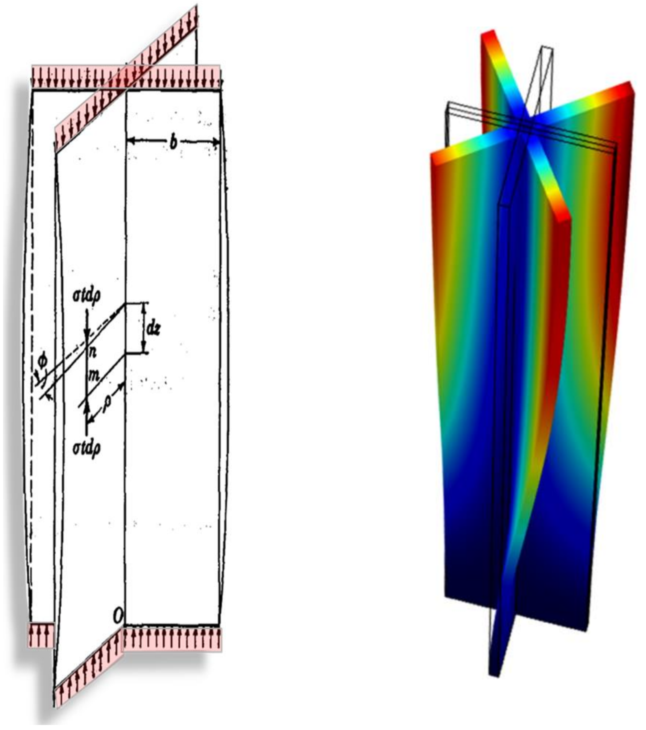
Pure torsional buckling

Doubly symmetric cross-section & centric thrust

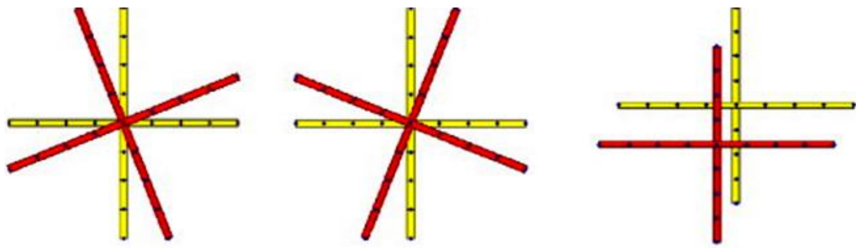


The **smallest** critical load is the **buckling load**

$$\begin{cases} P_{cr,v} = \pi^2 EI_z/L_v^2, \\ P_{cr,w} = \pi^2 EI_y/L_w^2, \\ P_{cr,\phi} = \frac{1}{r^2}[\pi^2 EI_\omega/L_\phi^2 + GI_t] \end{cases}$$



Centric load with doubly symmetric X-section



L = 300 cm **Torsional Buckling**

L = 500 cm **Torsional Buckling**

L = 1000 cm **Flexural Buckling**

Determine the critical length for the mode transition

Centric loading of beams having symmetric cross-section

Combined torsional and flexural buckling

$$
\begin{aligned}
&EI_z v^{(4)} + Pv'' + Pz_s\phi'' = 0,\\
&EI_y w^{(4)} + Pw'' - y_s\phi'' = 0,\\
&EI_\omega \phi^{(4)} - GI_t\phi'' + Pz_s v'' - Py_s w'' + Pr^2\phi'' = 0.
\end{aligned}
$$

$$P_{cr,v} = \pi^2 EI_z/L_v^2,$$

$$P_{cr,\phi} = \frac{1}{r^2}[\pi^2 EI_\omega/L_\phi^2 + GI_t],$$

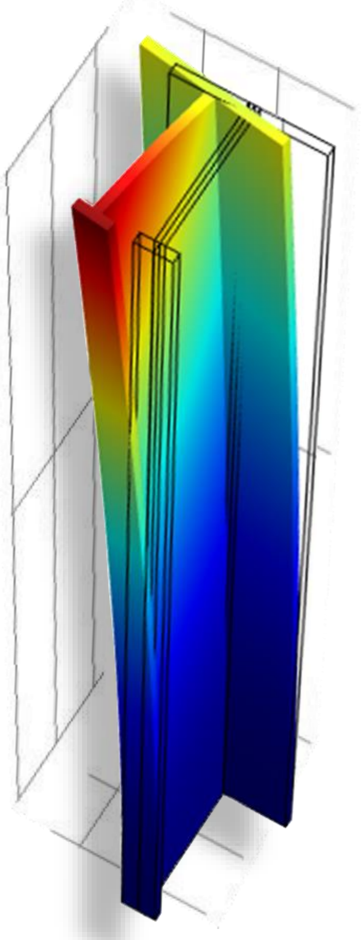
$$P_{cr,w} = \pi^2 EI_y/L_w^2,$$

$$
\begin{cases}
v(x) = A_1 + B_1 x + C_1 \sin[\pi/L_n(x - x_0)],\\
w(x) = A_2 + B_2 x + C_2 \sin[\pi/L_n(x - x_0)],\\
\phi(x) = A_3 + B_3 x + C_3 \sin[\pi/L_n(x - x_0)].
\end{cases}
$$

$$
\begin{bmatrix}
P_{cr,v} - P & 0 & -z_s P\\
0 & P_{cr,w} - P & y_s P\\
-z_s P & P\, y_s & r^2(P_{cr,\phi} - P)
\end{bmatrix}
\begin{bmatrix} C_1\\ C_2\\ C_3 \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}
$$

$$\mathbf{A} - P\,\mathbf{B} = \mathbf{0},$$

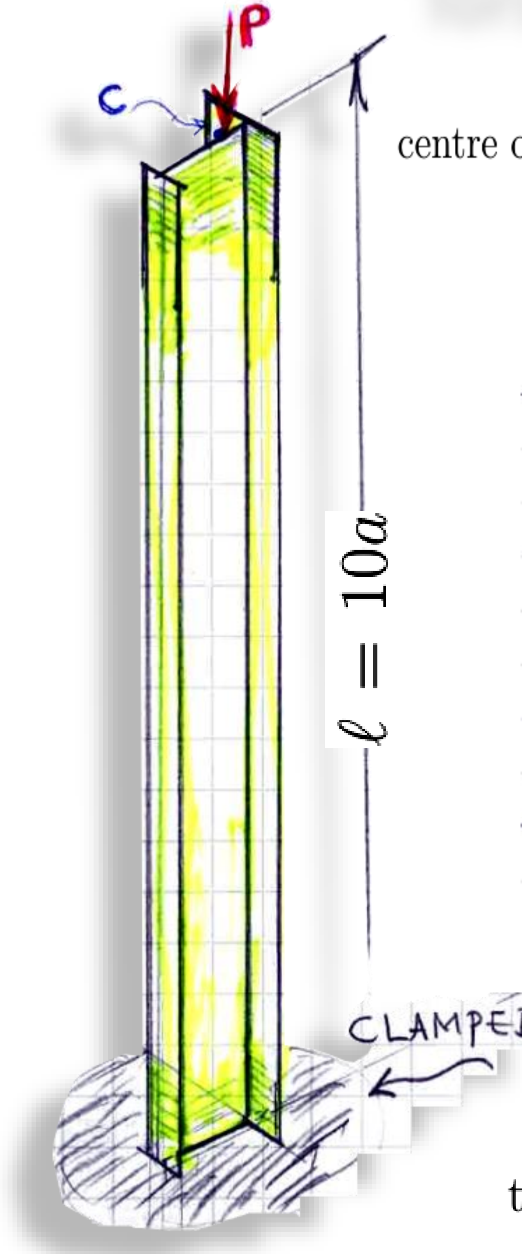
$$
\mathbf{A} = \begin{bmatrix}
P_{cr,v} & 0 & 0\\
0 & P_{cr,w} & 0\\
0 & 0 & r^2 P_{cr,\phi}
\end{bmatrix}
\qquad
\mathbf{B} = \begin{bmatrix}
1 & 0 & z_s\\
0 & 1 & -y_s\\
z_s & -y_s & r^2
\end{bmatrix}
$$



Combined flexural-torsional buckling of a cantilever-column loaded at its cross-section cenroid (FE-Linear Buckling Analysis.)

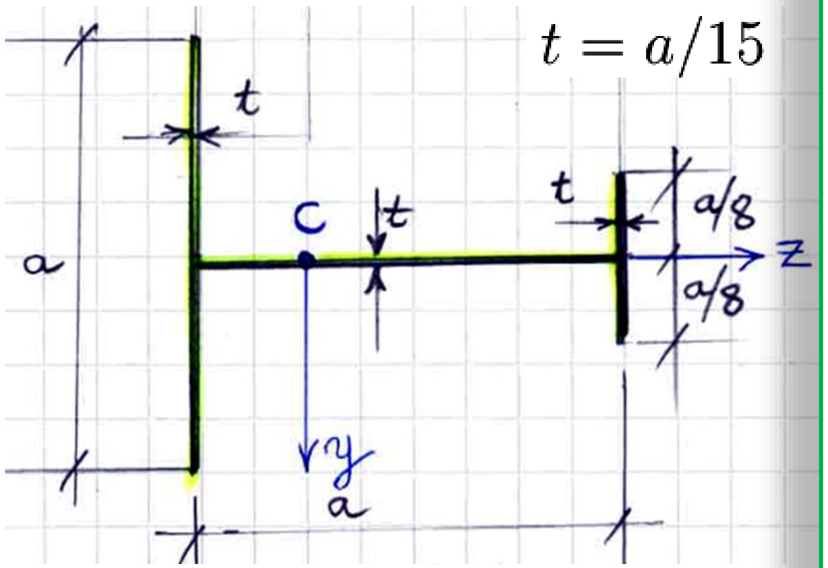
Numerical example - centric load column with singly symmetric T-section

Torsional-flexural buckling = Combined torsional and flexural buckling



centre of gravity (of area) of the cross-section is C.

$$G = 0.4E$$



$$e_c = 2/3a,$$
$$e_s = 64/65a,$$
$$z_s = -62/195a = -0.318a$$
$$y_s = 0$$
$$I_z = 13/2304a^4 = 5.642 \times 10^{-3}a^4,$$
$$I_y = a^4/45 = 2.222 \times 10^{-2}a^4,$$
$$I_t = a^4/4500 = 2.222 \times 10^{-4}a^4(= I_y/100),$$
$$I_\omega = a^6/11700 = 8.570 \times 10^{-5}a^6,$$
$$r^2 = (I_y + I_z)/A + y_s^2 + z_s^2 = 0.287a^2,$$
$$L_n = L_v = L_\phi = L_w = 2\ell = 20a.$$

From the modes

thrust load P is centric and applied at C

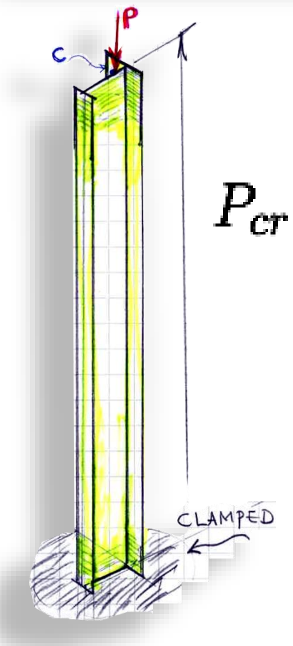
Analytical solution

Combined torsional and flexural buckling

$$\mathbf{A} = \begin{bmatrix} P_{cr,v} & 0 & 0 \\ 0 & P_{cr,w} & 0 \\ 0 & 0 & r^2 P_{cr,\phi} \end{bmatrix} = 10^{-3} \begin{bmatrix} 0.139 & 0 & 0 \\ 0 & 0.548 & 0 \\ 0 & 0 & 0.091 \end{bmatrix} \cdot Ea^2$$

and

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & z_s \\ 0 & 1 & -y_s \\ z_s & -y_s & r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.318a \\ 0 & 1 & 0 \\ -0.318a & 0 & 0.287a^2 \end{bmatrix}.$$



Programming the problem[202] in MATLAB leads to `[C, P] = eig(A, B`

$$\Downarrow$$

$$P_{cr} = 10^{-4} \begin{bmatrix} 1.158 \\ 5.483 \\ 5.886 \end{bmatrix} \cdot Ea^2,$$

The **smallest** critical load is the **buckling load**

$$\boxed{P_{cr} = 1.158 \times 10^{-4} Ea^2}$$

For $E = 70$ GPa and $a = 10$ cm,

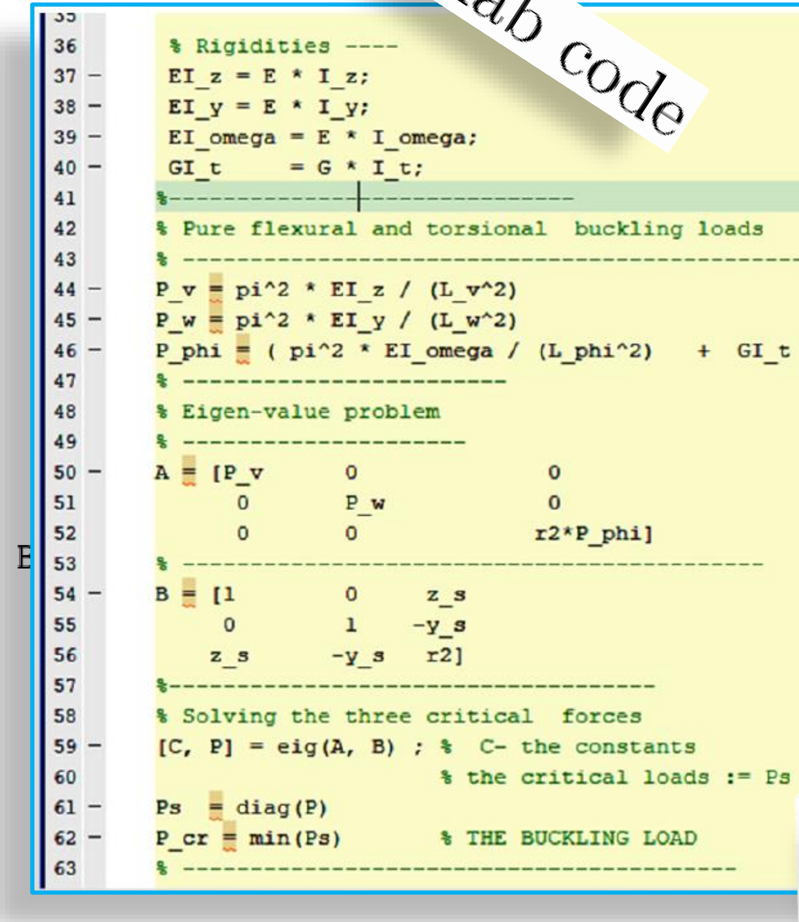
$$\boxed{P_{cr} = 11.6 \cdot 10^{-3} Ea^2 = 81 \text{ kN}}$$

(this analytical: 1-D Vlassov beam theory)

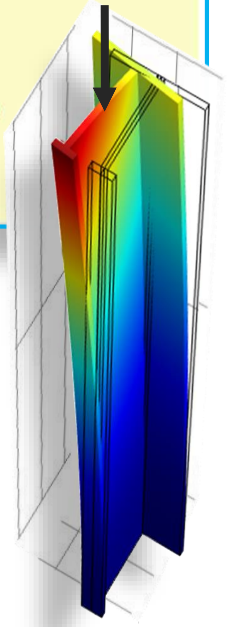
$P_{cr} = 75$ kN (Full 3D FEM)

Think: why the 1D is stiffer than 3D FEM solution?

Matlab code

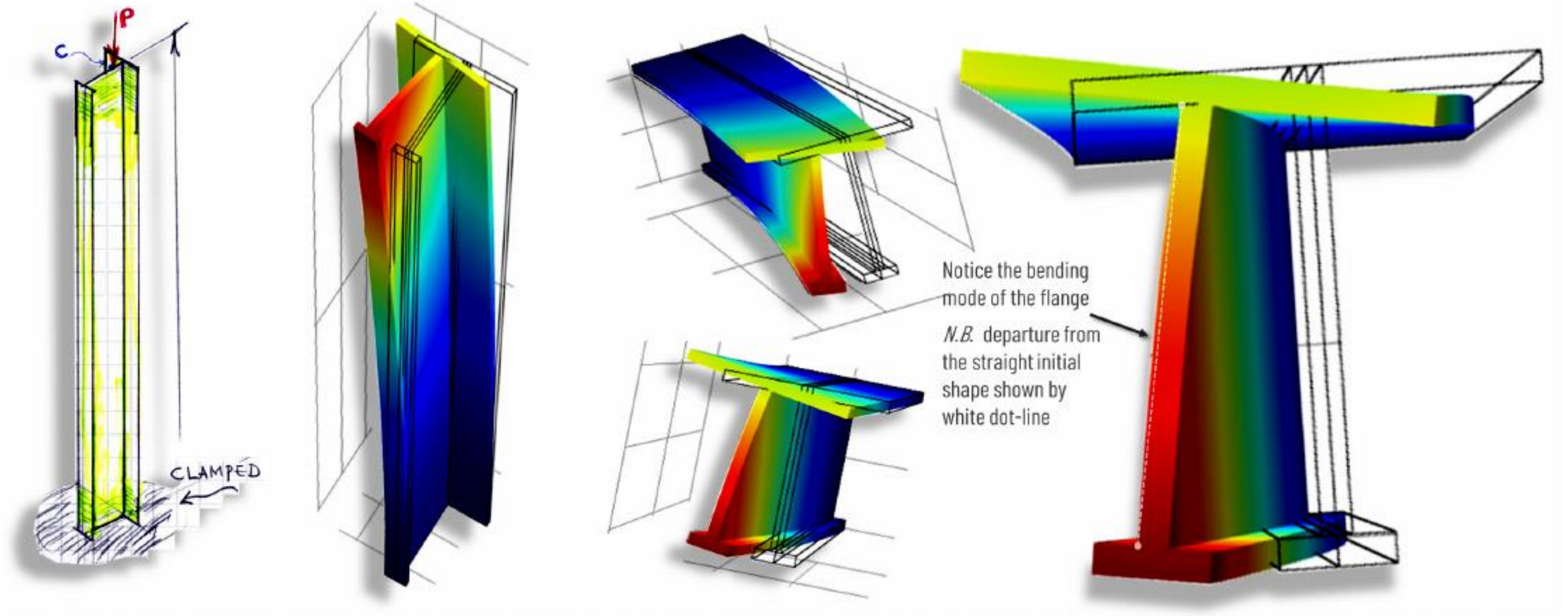


```
 % Rigidities ----
 EI_z = E * I_z;
 EI_y = E * I_y;
 EI_omega = E * I_omega;
 GI_t     = G * I_t;
%------------------------------
% Pure flexural and torsional  buckling loads
% ---------------------------------------------
P_v = pi^2 * EI_z / (L_v^2)
P_w = pi^2 * EI_y / (L_w^2)
P_phi = ( pi^2 * EI_omega / (L_phi^2)   +  GI_t ) /r2
% ------------------------
% Eigen-value problem
% ---------------------
A = [P_v      0              0
      0       P_w            0
      0       0            r2*P_phi]
% ---------------------------------------
B = [1        0     z_s
     0        1    -y_s
   z_s      -y_s   r2]
%-----------------------------------
% Solving the three critical  forces
[C, P] = eig(A, B) ; %  C- the constants
                     % the critical loads := Ps
Ps   = diag(P)
P_cr = min(Ps)       % THE BUCKLING LOAD
% ----------------------------------------
```



Torsional-flexural buckling

Computational linear buckling analysis (3D).



Computational linear buckling analysis (3D). Flexural-torsional buckling of axially loaded column at centroid. The cross-section is simply symmetric thin-walled `T` cross section. The thrust load P is centric and applied at the centre of mass C. The obtained $P_{cr} = 75$ kN.

Let's illustrate the Eigen-value problem (Eq. 1.548) above with an application and solve for the critical load (Figure 1.119). Here are the geometry-data: length of the column is $\ell = 10a$, $G = 0.4E$, $t = a/15$. The centre of gravity (of area) of the cross-section is C. The thrust load P is centric and applied at centroid C of the cross-section.

Appendix

In a bit disorder now … will be updated

- geometric properties of some open cross-sections
  (center of shear and warping moment of inertia)
- and many other things …

Energy criteria for determination of instability of elastic structures

Change of total potential energy **between which** two **states**?

?

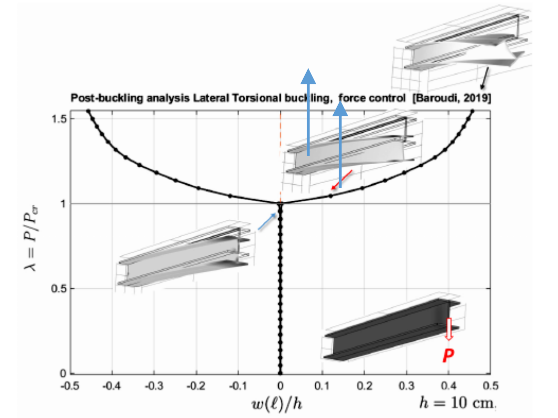
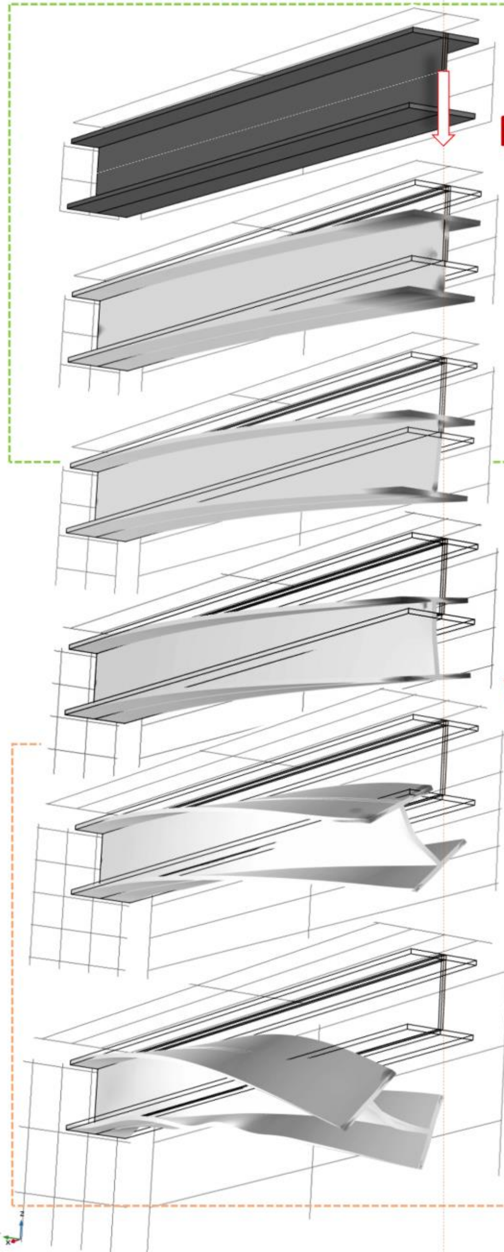


Figure 3.122: Equilibrium paths. FE-post-buckling analysis of an aluminium I-beam cantilever. The transversal tip-load is at the centroid.

Load increases



Primary *pre-buckling* state

No buckling yet

Π^0

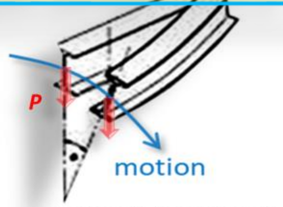
Buckling just started

Π^*

Lateral-torsional buckling occurs

***Post-buckling* behavior**

N.B. The perturbed configuration [.]* can be thought achieved keeping the load constant and for instance, giving a tiny kinematical (virtual) perturbation to a an adjacent equilibrium configuration v*



Lateral torsional buckling

$$\Delta\Pi = \Pi^* - \Pi^0$$

$$\Delta\Pi = \Pi(u^0 + \delta u) - \Pi(u^0) = \underbrace{\delta\Pi|_{u_0}}_{=0} + \frac{1}{2}\delta^2\Pi|_{u_0} + \dots$$

$$\delta(\Delta\Pi) = 0 \implies \delta\left(\frac{1}{2}\delta^2\Pi|_{u_0} + \dots\right) = 0$$

General *stability loss* criterion

Keeping up to quadratic terms

$$\implies \delta(\delta^2\Pi) = 0$$

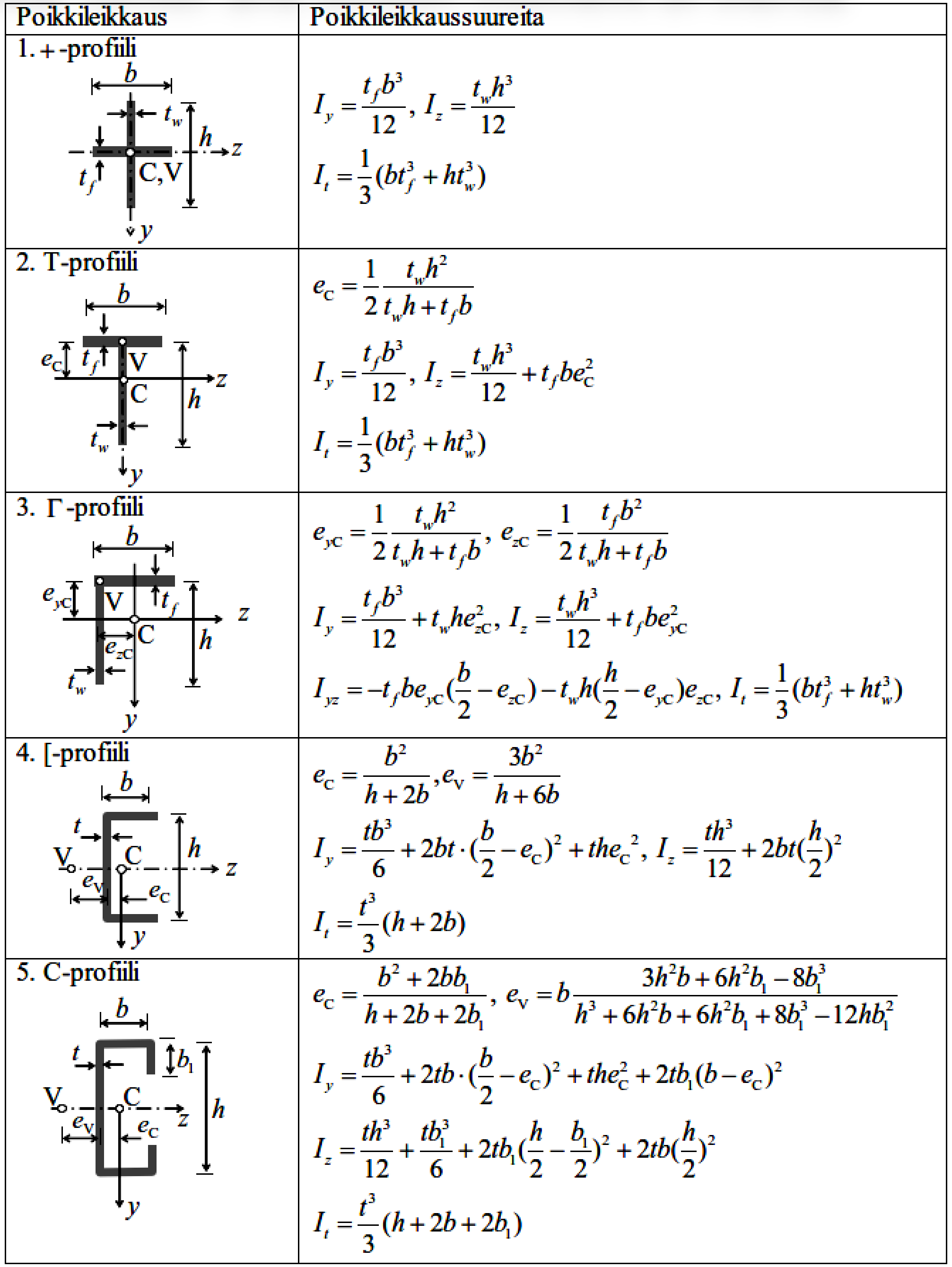
Trefftz *stability loss* criterion

$\Downarrow$

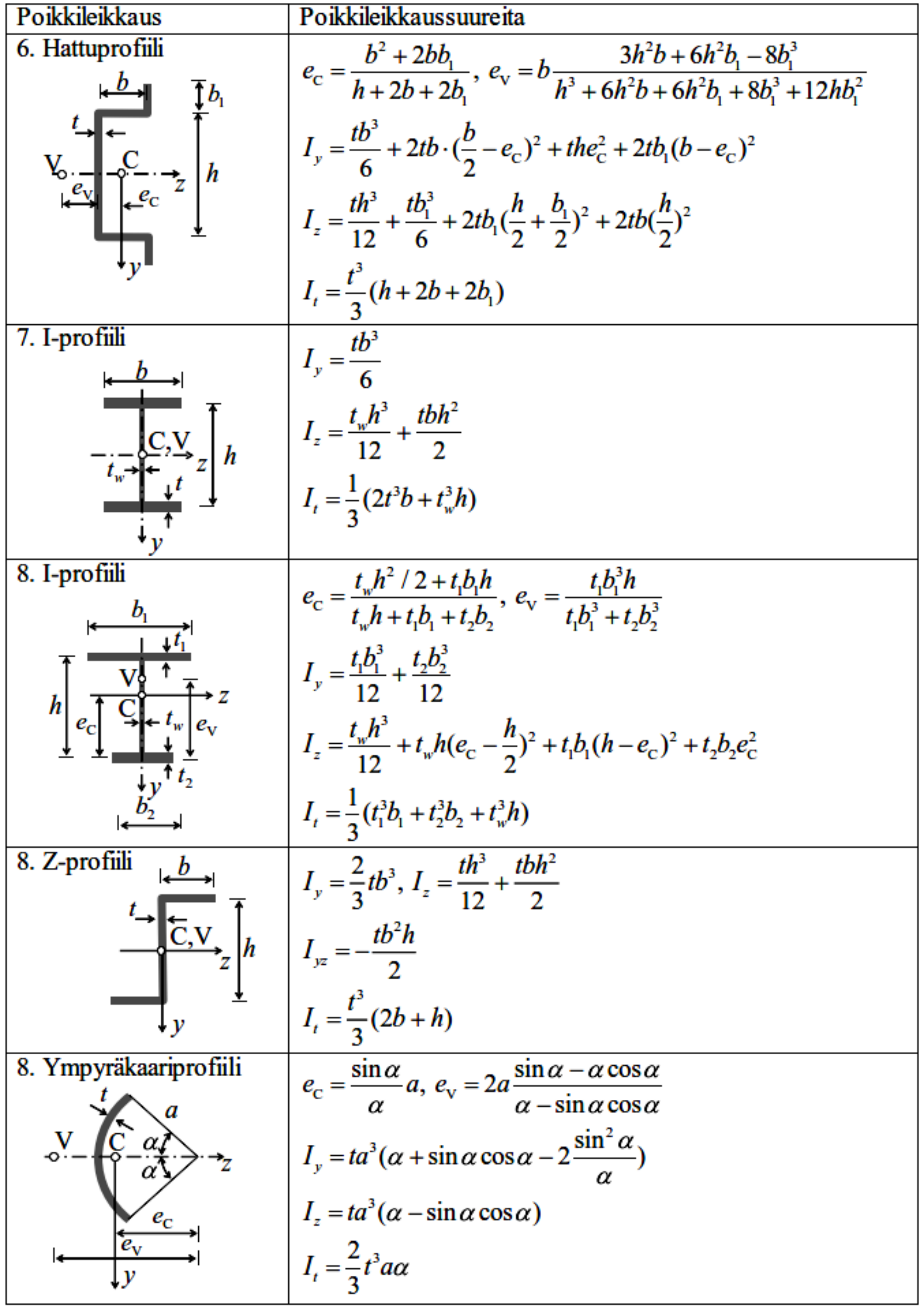
This criticality condition for bifurcation provides the Buckling Equations

Torsional buckling

Shear center and torsion moment of inertia



Poikkileikkaus	Poikkileikkaussuureita
1. +-profiili	$I_y = \frac{t_f b^3}{12},\ I_z = \frac{t_w h^3}{12}$ $I_t = \frac{1}{3}(bt_f^3 + ht_w^3)$
2. T-profiili	$e_C = \frac{1}{2}\frac{t_w h^2}{t_w h + t_f b}$ $I_y = \frac{t_f b^3}{12},\ I_z = \frac{t_w h^3}{12} + t_f b e_C^2$ $I_t = \frac{1}{3}(bt_f^3 + ht_w^3)$
3. Γ-profiili	$e_{yC} = \frac{1}{2}\frac{t_w h^2}{t_w h + t_f b},\ e_{zC} = \frac{1}{2}\frac{t_f b^2}{t_w h + t_f b}$ $I_y = \frac{t_f b^3}{12} + t_w h e_{zC}^2,\ I_z = \frac{t_w h^3}{12} + t_f b e_{yC}^2$ $I_{yz} = -t_f b e_{yC}(\frac{b}{2} - e_{zC}) - t_w h(\frac{h}{2} - e_{yC})e_{zC},\ I_t = \frac{1}{3}(bt_f^3 + ht_w^3)$
4. [-profiili	$e_C = \frac{b^2}{h+2b}, e_V = \frac{3b^2}{h+6b}$ $I_y = \frac{tb^3}{6} + 2bt\cdot(\frac{b}{2} - e_C)^2 + the_C{}^2,\ I_z = \frac{th^3}{12} + 2bt(\frac{h}{2})^2$ $I_t = \frac{t^3}{3}(h+2b)$
5. C-profiili	$e_C = \frac{b^2 + 2bb_1}{h + 2b + 2b_1},\ e_V = b\frac{3h^2 b + 6h^2 b_1 - 8b_1^3}{h^3 + 6h^2 b + 6h^2 b_1 + 8b_1^3 - 12hb_1^2}$ $I_y = \frac{tb^3}{6} + 2tb\cdot(\frac{b}{2} - e_C)^2 + the_C^2 + 2tb_1(b - e_C)^2$ $I_z = \frac{th^3}{12} + \frac{tb_1^3}{6} + 2tb_1(\frac{h}{2} - \frac{b_1}{2})^2 + 2tb(\frac{h}{2})^2$ $I_t = \frac{t^3}{3}(h + 2b + 2b_1)$

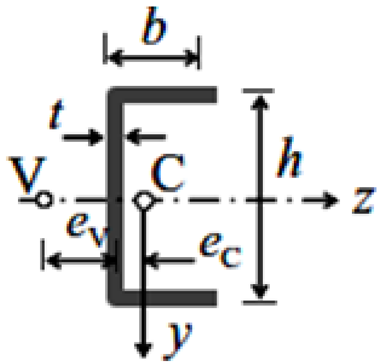


Poikkileikkaus	Poikkileikkaussuureita
6. Hattuprofiili	$e_C = \frac{b^2 + 2bb_1}{h + 2b + 2b_1},\ e_V = b\frac{3h^2 b + 6h^2 b_1 - 8b_1^3}{h^3 + 6h^2 b + 6h^2 b_1 + 8b_1^3 + 12hb_1^2}$ $I_y = \frac{tb^3}{6} + 2tb\cdot(\frac{b}{2} - e_C)^2 + the_C^2 + 2tb_1(b - e_C)^2$ $I_z = \frac{th^3}{12} + \frac{tb_1^3}{6} + 2tb_1(\frac{h}{2} + \frac{b_1}{2})^2 + 2tb(\frac{h}{2})^2$ $I_t = \frac{t^3}{3}(h + 2b + 2b_1)$
7. I-profiili	$I_y = \frac{tb^3}{6}$ $I_z = \frac{t_w h^3}{12} + \frac{tbh^2}{2}$ $I_t = \frac{1}{3}(2t^3 b + t_w^3 h)$
8. I-profiili	$e_C = \frac{t_w h^2 / 2 + t_1 b_1 h}{t_w h + t_1 b_1 + t_2 b_2},\ e_V = \frac{t_1 b_1^3 h}{t_1 b_1^3 + t_2 b_2^3}$ $I_y = \frac{t_1 b_1^3}{12} + \frac{t_2 b_2^3}{12}$ $I_z = \frac{t_w h^3}{12} + t_w h(e_C - \frac{h}{2})^2 + t_1 b_1 (h - e_C)^2 + t_2 b_2 e_C^2$ $I_t = \frac{1}{3}(t_1^3 b_1 + t_2^3 b_2 + t_w^3 h)$
8. Z-profiili	$I_y = \frac{2}{3}tb^3,\ I_z = \frac{th^3}{12} + \frac{tbh^2}{2}$ $I_{yz} = -\frac{tb^2 h}{2}$ $I_t = \frac{t^3}{3}(2b + h)$
8. Ympyräkaariprofiili	$e_C = \frac{\sin\alpha}{\alpha}a,\ e_V = 2a\frac{\sin\alpha - \alpha\cos\alpha}{\alpha - \sin\alpha\cos\alpha}$ $I_y = ta^3(\alpha + \sin\alpha\cos\alpha - 2\frac{\sin^2\alpha}{\alpha})$ $I_z = ta^3(\alpha - \sin\alpha\cos\alpha)$ $I_t = \frac{2}{3}t^3 a\alpha$

The shear center (SC or V) is *the instantaneous center of rotation* for a section under pure torsion or when the resultant of loading does not pass through this center

- V = shear center = SC (vääntökeskiö)

G = center of gravity



Ref: Emir prof. J. Aalto lectures

Shear center and warping moment of inertia

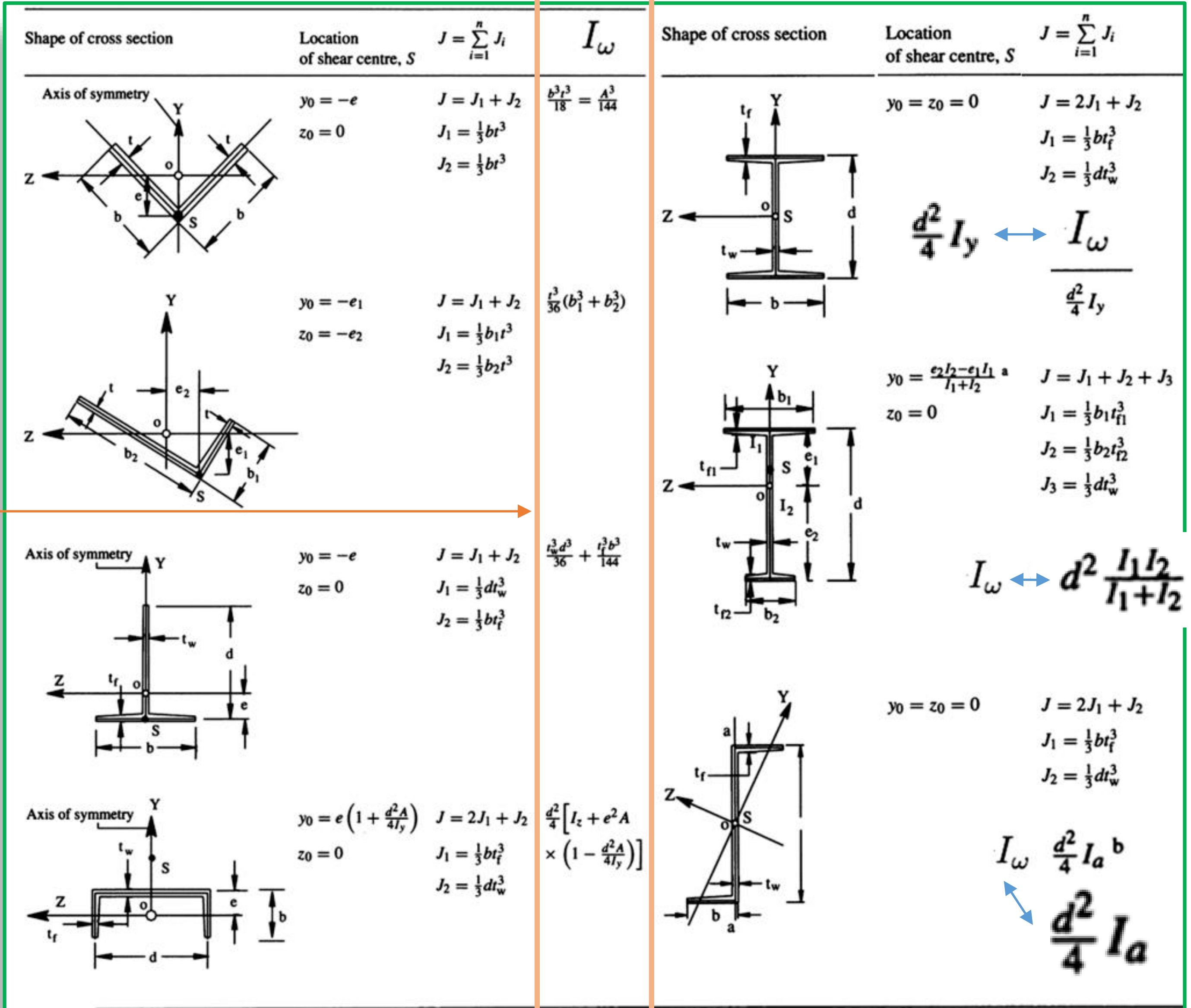
I_ω

$\frac{b^3t^3}{18} = \frac{A^3}{144}$

$\frac{t^3}{36}(b_1^3 + b_2^3)$

$\frac{t_w^3 d^3}{36} + \frac{t_f^3 b^3}{144}$

$\frac{d^2}{4}\left[I_z + e^2 A \times \left(1 - \frac{d^2 A}{4 I_y}\right)\right]$



Shape of cross section	Location of shear centre, S	$J = \sum_{i=1}^{n} J_i$	I_ω
Axis of symmetry	$y_0 = -e$ $z_0 = 0$	$J = J_1 + J_2$ $J_1 = \frac{1}{3}bt^3$ $J_2 = \frac{1}{3}bt^3$	$\frac{b^3t^3}{18} = \frac{A^3}{144}$
	$y_0 = -e_1$ $z_0 = -e_2$	$J = J_1 + J_2$ $J_1 = \frac{1}{3}b_1t^3$ $J_2 = \frac{1}{3}b_2t^3$	$\frac{t^3}{36}(b_1^3 + b_2^3)$
Axis of symmetry	$y_0 = -e$ $z_0 = 0$	$J = J_1 + J_2$ $J_1 = \frac{1}{3}dt_w^3$ $J_2 = \frac{1}{3}bt_f^3$	$\frac{t_w^3d^3}{36} + \frac{t_f^3b^3}{144}$
Axis of symmetry	$y_0 = e\left(1 + \frac{d^2A}{4I_y}\right)$ $z_0 = 0$	$J = 2J_1 + J_2$ $J_1 = \frac{1}{3}bt_f^3$ $J_2 = \frac{1}{3}dt_w^3$	$\frac{d^2}{4}\left[I_z + e^2A \times \left(1 - \frac{d^2A}{4I_y}\right)\right]$

Shape of cross section	Location of shear centre, S	$J = \sum_{i=1}^{n} J_i$
	$y_0 = z_0 = 0$	$J = 2J_1 + J_2$ $J_1 = \frac{1}{3}bt_f^3$ $J_2 = \frac{1}{3}dt_w^3$
	$y_0 = \frac{e_2I_2 - e_1I_1}{I_1+I_2}$ [a] $z_0 = 0$	$J = J_1 + J_2 + J_3$ $J_1 = \frac{1}{3}b_1t_{f1}^3$ $J_2 = \frac{1}{3}b_2t_{f2}^3$ $J_3 = \frac{1}{3}dt_w^3$
	$y_0 = z_0 = 0$	$J = 2J_1 + J_2$ $J_1 = \frac{1}{3}bt_f^3$ $J_2 = \frac{1}{3}dt_w^3$

$\frac{d^2}{4} I_y \leftrightarrow I_\omega$

I_ω

$\frac{d^2}{4} I_y$

$I_\omega \leftrightarrow d^2 \frac{I_1 I_2}{I_1 + I_2}$

I_ω $\frac{d^2}{4} I_a$ [b]

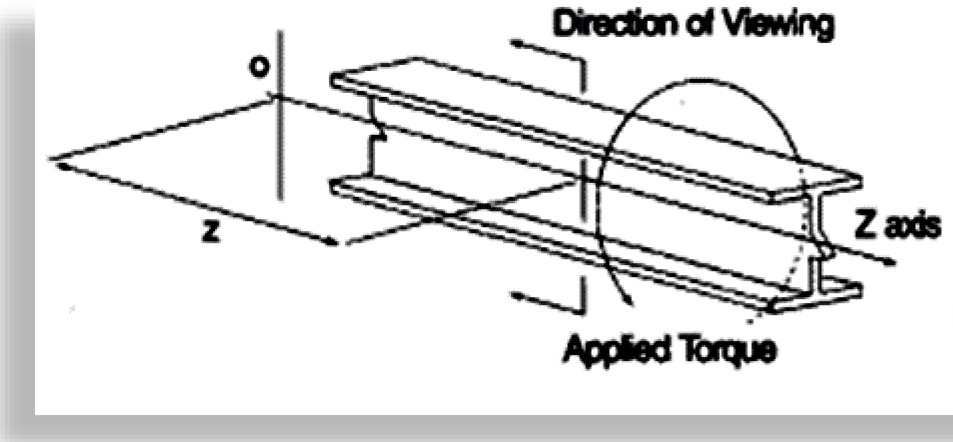
$\frac{d^2}{4} I_a$

---

[a] I_1 and I_2 are the moments of inertia of the top and the bottom flanges, respectively, with respect to the Y-axis

[b] I_a is the moment of inertia of the cross-section with respect to the centerline a-a of the web

Torsional stresses

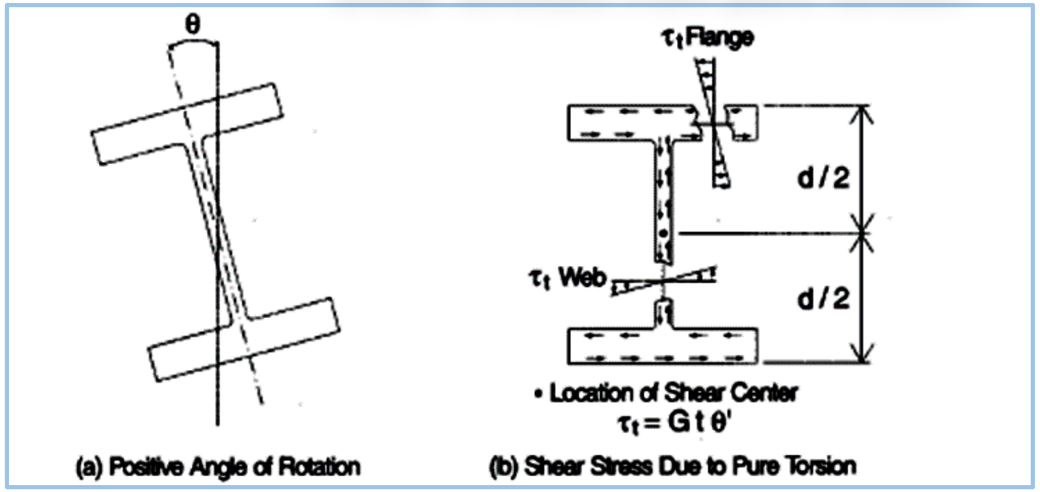


Pure torsion
Puhdas vääntö
Saint Venant’s torsion

warping torsion
Non-uniform torsion

Estetty vääntö
Vlassov’s torsion

Shear stresses from pure torsion

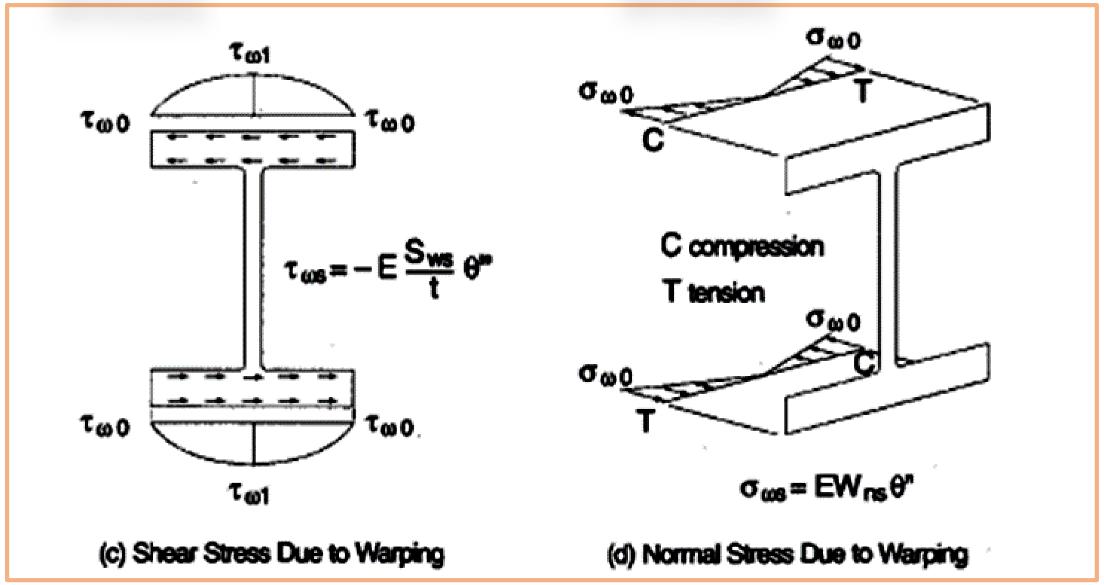


(a) Positive Angle of Rotation

(b) Shear Stress Due to Pure Torsion

Warping shear stresses

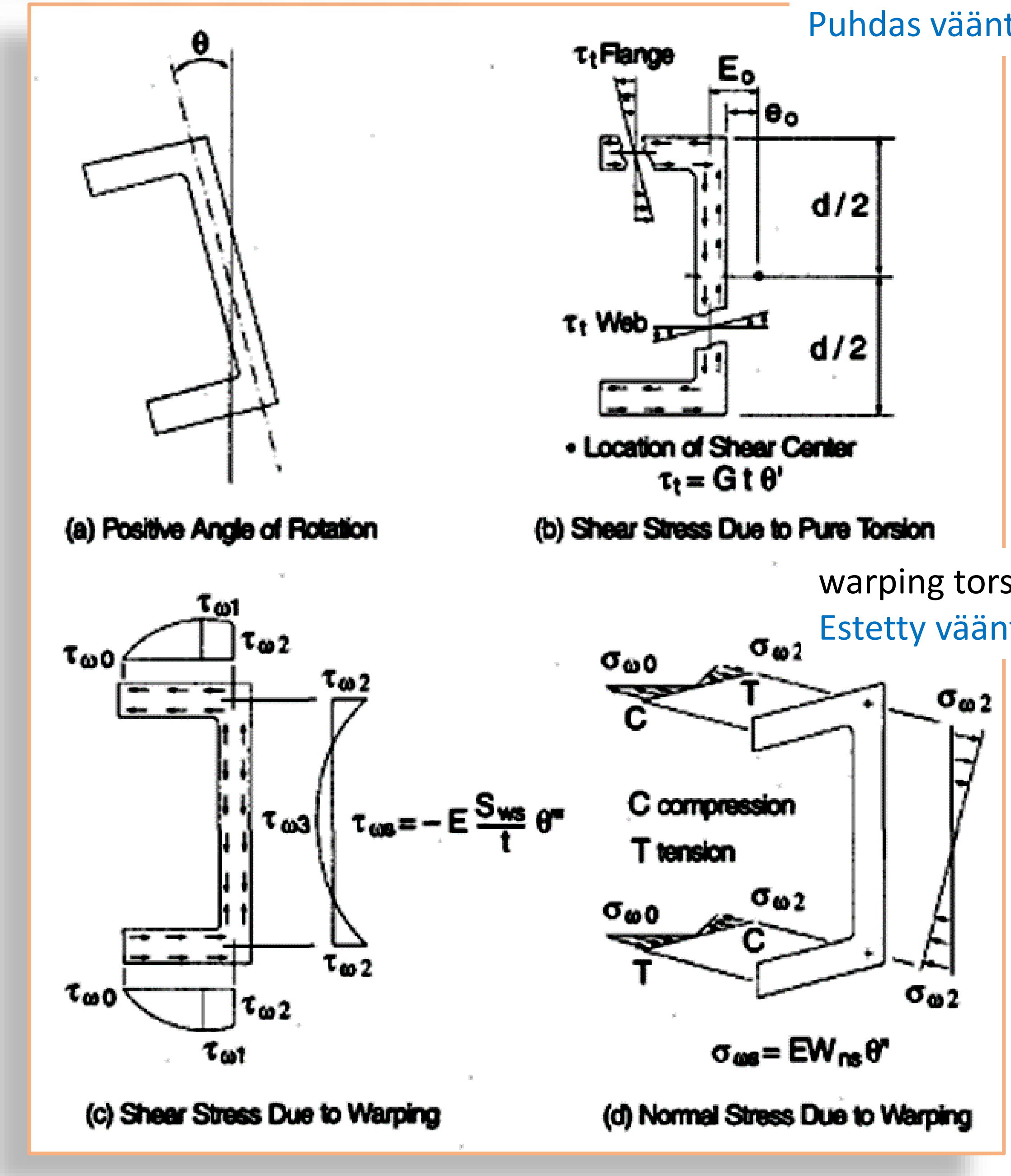
Warping normal stresses



(c) Shear Stress Due to Warping

(d) Normal Stress Due to Warping

Torsional stresses

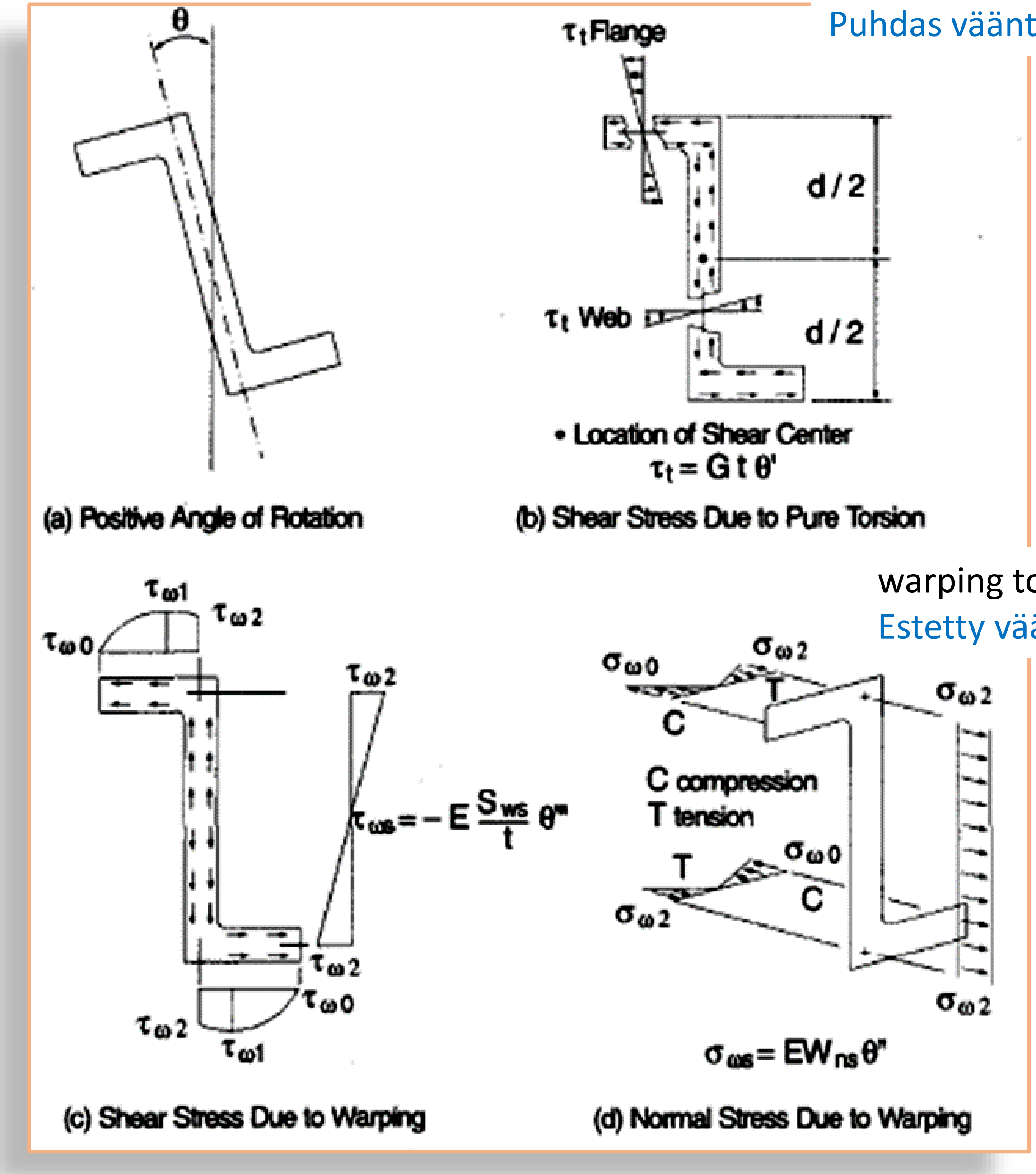


Pure torsion

Puhdas vääntö

warping torsion

Estetty vääntö



Pure torsion

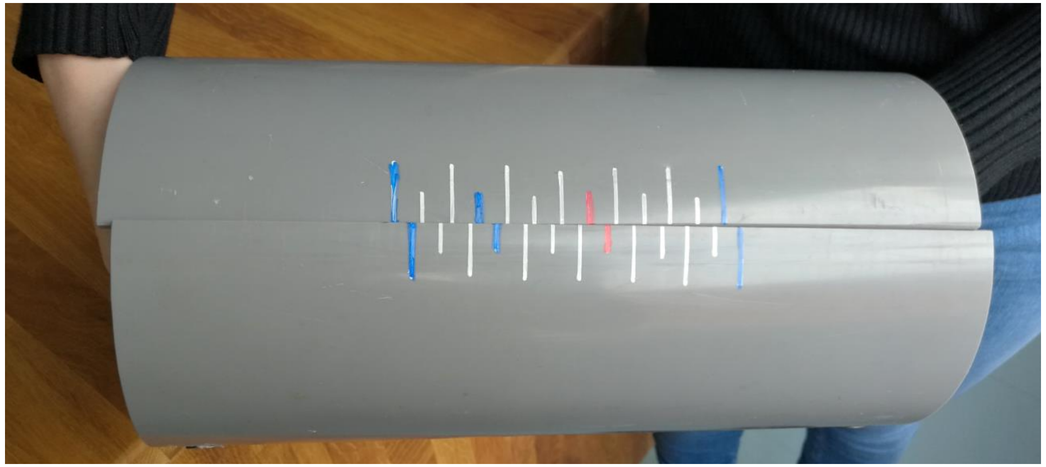
Puhdas vääntö

warping torsion

Estetty vääntö

warping

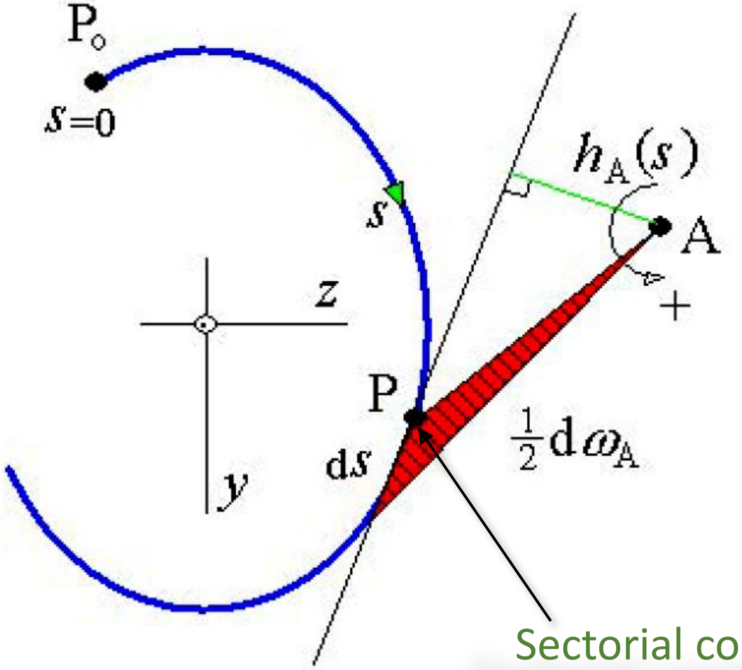
warping



Deplanation = out-of-plane motion (means the plane of the cross-section)

Open thin-walled cross-sections

The Sectorial Coordinate



Sectorial coordinate with respect to *A*

$$\mathrm{d}\hat{\omega}_{\mathrm{A}} = \pm h_{\mathrm{A}}(s)\mathrm{d}s$$

$$\mathrm{d}\vec{\hat{\omega}}_{\mathrm{A}} = (\vec{r}_{\mathrm{P}} - \vec{r}_{\mathrm{A}}) \times \mathrm{d}\vec{s}$$

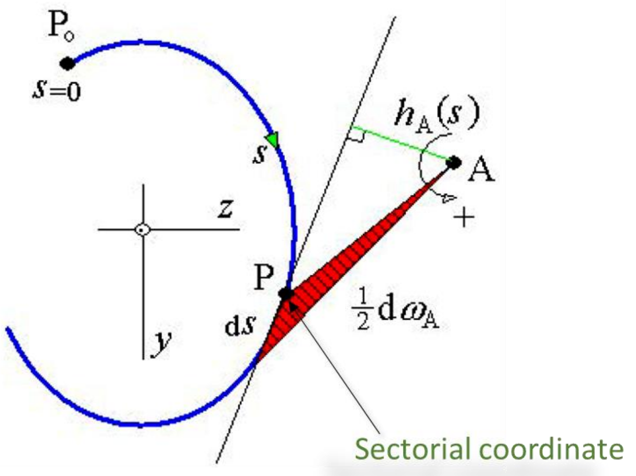
$$\mathrm{d}\hat{\omega}_{\mathrm{A}} = -(z - z_{\mathrm{A}})\mathrm{d}y + (y - y_{\mathrm{A}})\mathrm{d}z$$

Open thin-walled cross-sections

The Sectorial Coordinate ω

Example: determine the sectorial coordinate, the shear center and I_ω

Definition

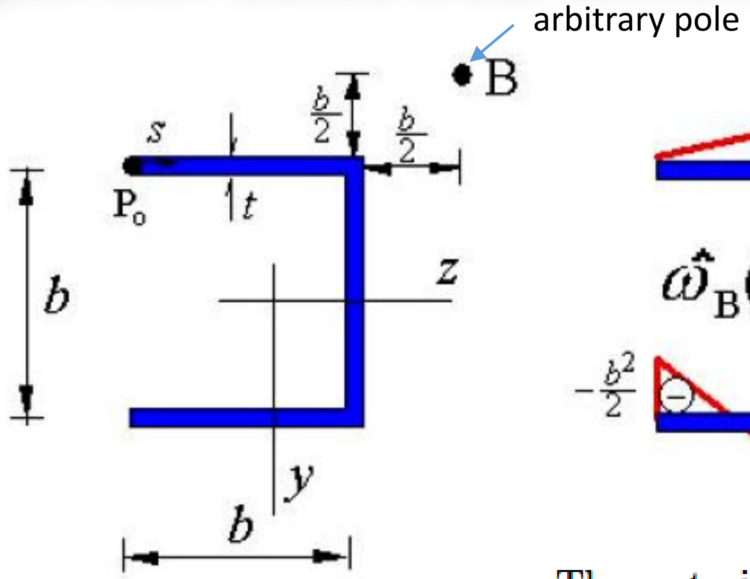


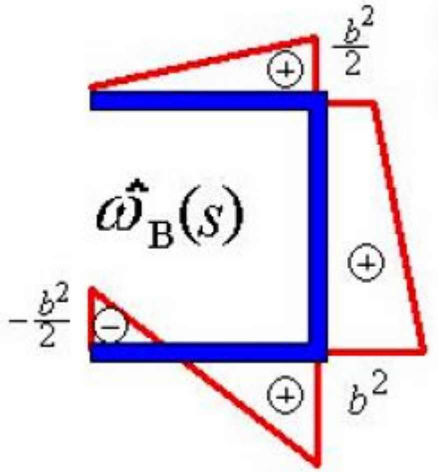
Sectorial coordinate
with respect to *A*

$$d\hat{\omega}_A = \pm h_A(s)ds$$

$$d\vec{\omega}_A = (\vec{r}_P - \vec{r}_A) \times d\vec{s}$$

$$d\hat{\omega}_A = -(z - z_A)dy + (y - y_A)dz$$





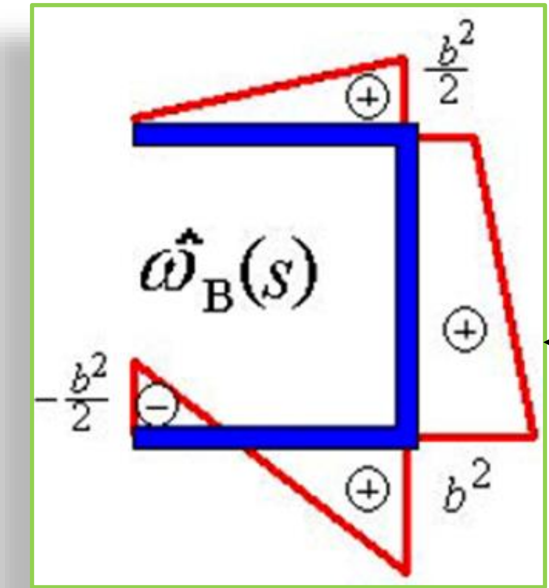
The sectorial coordinate $\hat{\omega}_B(s)$.

Let's use the arbitrary point *B* as a pole (You will find that, it is computationally wiser to chose *a corner point* the cross-section as an initial pole)

$$\hat{\omega}_A = \int_{P_o}^{P} d\hat{\omega}_A = \pm \int_{P_o}^{P} h_A(s)ds = -\int_{P_o}^{P} [(z - z_A)dy - (y - y_A)dz]$$

The sectorial coordinate with respect to *B* as determined from the definition is

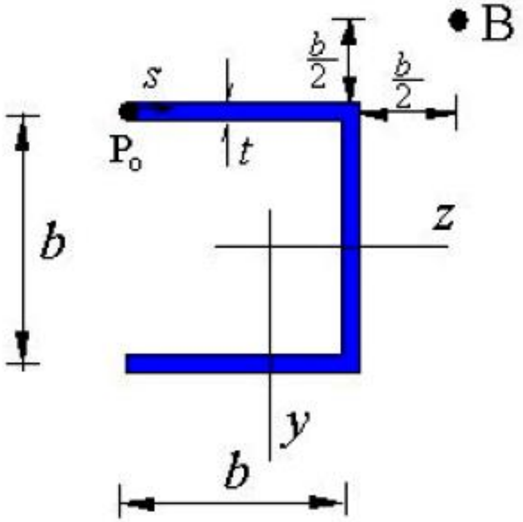
$$\hat{\omega}_B(s) = \begin{cases} \int_0^s \frac{b}{2}ds = \frac{b}{2}s, & \text{kun} \quad 0 \leq s \leq b \qquad\qquad 0 \leq s \leq 3b \\ \hat{\omega}_B(b) + \int_b^s \frac{b}{2}ds = \frac{b}{2}s, & \text{kun} \quad b \leq s \leq 2b \\ \hat{\omega}_B(2b) - \int_{2b}^s \frac{3b}{2}ds = 4b^2 - \frac{3b}{2}s, & \text{kun} \quad 2b \leq s \leq 3b \end{cases}$$

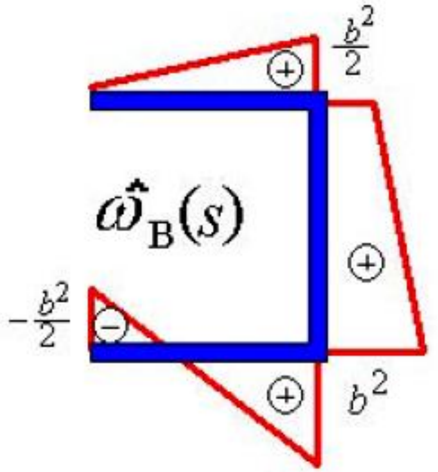


To be useable, It should be normalised such that its static moment vanishes (read the lecture notes)

The sectorial coordinate graph

A





The sectorial coordinate $\hat{\omega}_{\mathrm{B}}(s)$.

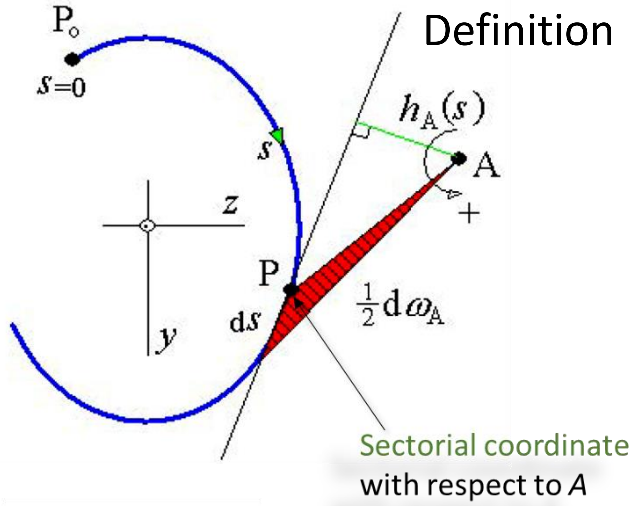
Let’s shift or re-allocate the pole *B* to an other point *A.*
How the coordinate- ω is then transformed?

$$\hat{\omega}_{\mathrm{A}} = -\int_{\mathrm{P}_o}^{\mathrm{P}} [(z - z_{\mathrm{A}})\mathrm{d}y - (y - y_{\mathrm{A}})\mathrm{d}z]$$

$$= -\int_{\mathrm{P}_o}^{\mathrm{P}} [(z - z_{\mathrm{B}} + z_{\mathrm{B}} - z_{\mathrm{A}})\mathrm{d}y - (y - y_{\mathrm{B}} + y_{\mathrm{B}} - y_{\mathrm{A}})\mathrm{d}z]$$

$$= -\int_{\mathrm{P}_o}^{\mathrm{P}} [(z - z_{\mathrm{B}})\mathrm{d}y - (y - y_{\mathrm{B}})\mathrm{d}z] - \int_{\mathrm{P}_o}^{\mathrm{P}} [(z_{\mathrm{B}} - z_{\mathrm{A}})\mathrm{d}y - (y_{\mathrm{B}} - y_{\mathrm{A}})\mathrm{d}z]$$

$$= \hat{\omega}_{\mathrm{B}} - (z_{\mathrm{B}} - z_{\mathrm{A}})(y - y_o) + (y_{\mathrm{B}} - y_{\mathrm{A}})(z - z_o)$$



Definition

Sectorial coordinate with respect to *A*

$$\mathrm{d}\hat{\omega}_{\mathrm{A}} = \pm h_{\mathrm{A}}(s)\mathrm{d}s$$

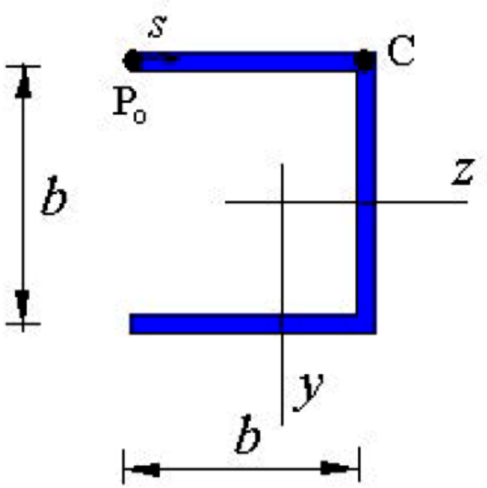
$$\mathrm{d}\vec{\omega}_{\mathrm{A}} = (\vec{r}_{\mathrm{P}} - \vec{r}_{\mathrm{A}}) \times \mathrm{d}\vec{s}$$

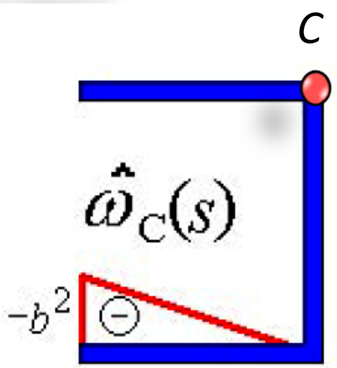
$$\mathrm{d}\hat{\omega}_{\mathrm{A}} = -(z - z_{\mathrm{A}})\mathrm{d}y + (y - y_{\mathrm{A}})\mathrm{d}z$$

$$z_{\mathrm{B}} - z_{\mathrm{C}} = \frac{b}{2}$$

$$y_{\mathrm{B}} - y_{\mathrm{C}} = -\frac{b}{2}$$

The sectorial coordinate graph





Sectorial coordinate $\hat{\omega}_C$.

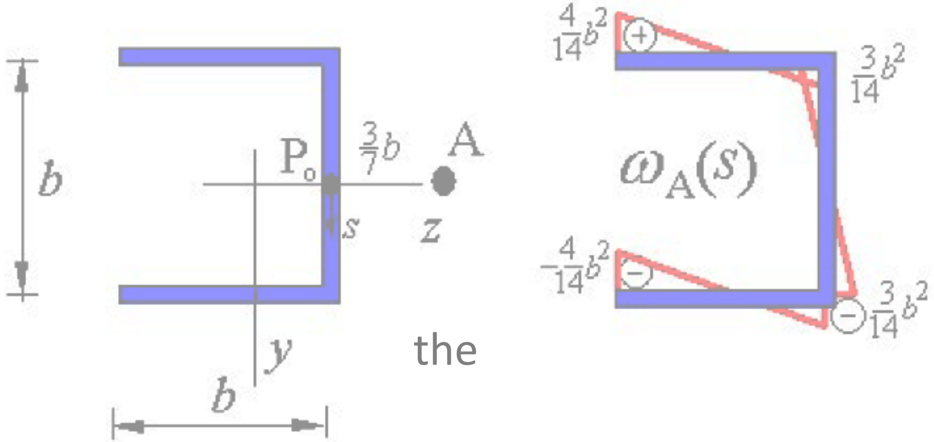
$$y_B - y_C = -\frac{b}{2}$$
$$z_B - z_C = \frac{0}{2}$$

Let’s re-allocate the pole to the corner point *C* of the U-profile.
How the coordinate- ω is then transformed?

$$\hat{\omega}_A = -\int_{P_o}^{P}[(z-z_A)dy-(y-y_A)dz]$$
$$= -\int_{P_o}^{P}[(z-z_B+z_B-z_A)dy-(y-y_B+y_B-y_A)dz]$$
$$= -\int_{P_o}^{P}[(z-z_B)dy-(y-y_B)dz]-\int_{P_o}^{P}[(z_B-z_A)dy-(y_B-y_A)dz]$$
$$= \hat{\omega}_B-(z_B-z_A)(y-y_o)+(y_B-y_A)(z-z_o)$$

$$\hat{\omega}_C = \begin{cases} \frac{b}{2}s-\frac{b}{2}0+(-\frac{b}{2})s=0, & \text{kun} \quad 0 \le s \le b \\ \frac{b}{2}s-\frac{b}{2}(s-b)+(-\frac{b}{2})b=0, & \text{kun} \quad b \le s \le 2b \\ 4b^2-\frac{3b}{2}s-\frac{b}{2}b+(-\frac{b}{2})(3b-s)=2b^2-bs, & \text{kun} \quad 2b \le s \le 3b \end{cases}$$

normalized sectorial coordinate

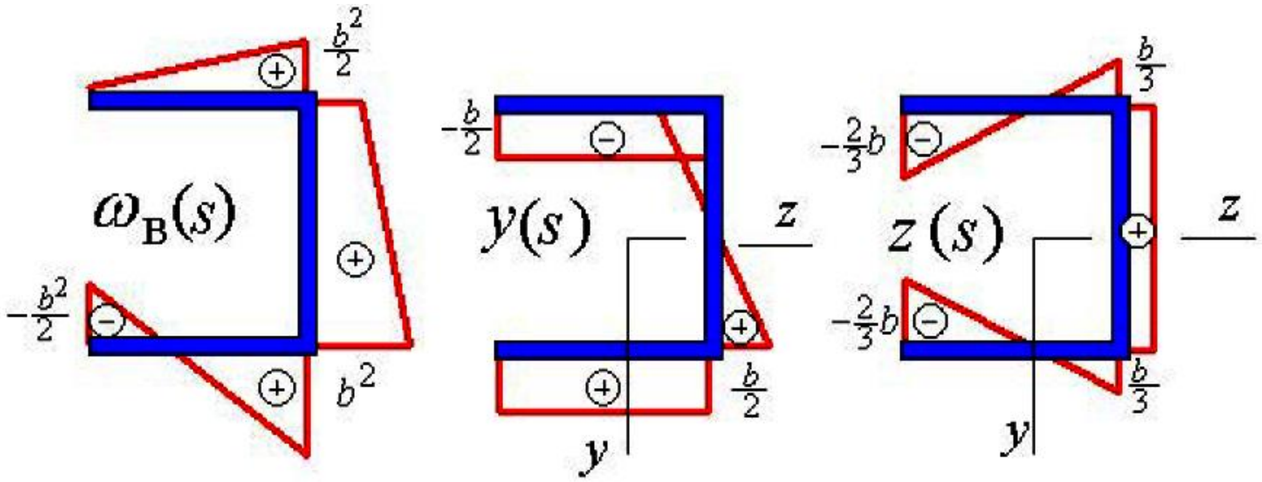


the

Coordinates of SC:

$$y_A = y_B + \frac{I_{\hat{\omega}_B z}}{I_y}$$
$$z_A = z_B - \frac{I_{\hat{\omega}_B y}}{I_z}$$

The pdf-material by emeritus prof. J. Paavola provides detailed illustrative examples.



$$y_{\mathrm{A}} = y_{\mathrm{B}} + \frac{I_{\hat{\omega}_{\mathrm{B}} z}}{I_y}$$

$$z_{\mathrm{A}} = z_{\mathrm{B}} - \frac{I_{\hat{\omega}_{\mathrm{B}} y}}{I_z}$$

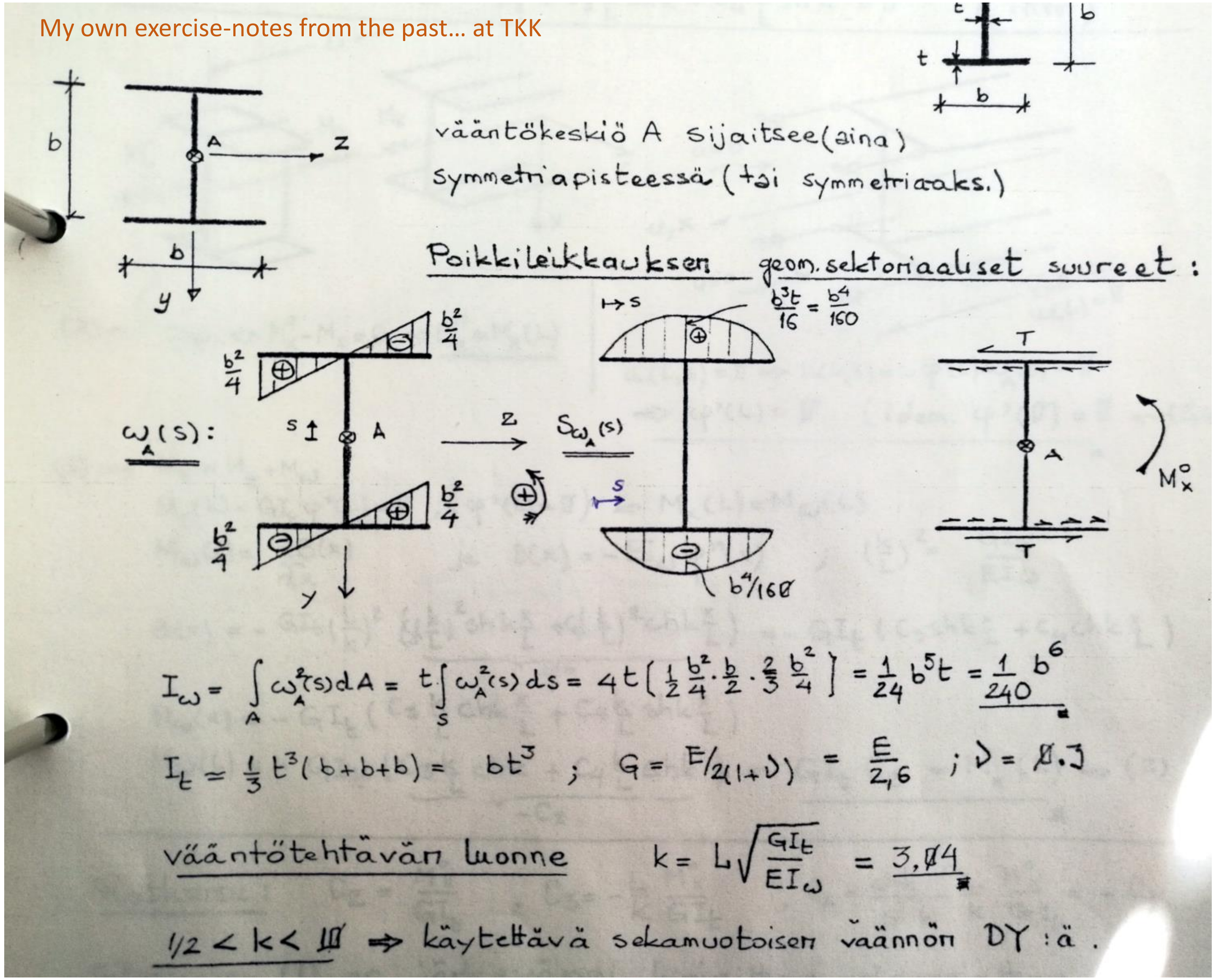
Normalization of the sectorial coordinate

$$S_{\omega_{\mathrm{A}}} = \int_A \omega_{\mathrm{A}} \mathrm{d}A = 0$$

$$\omega_{\mathrm{A}}(s) = \int_{\mathrm{P}_o}^{\mathrm{P}} \mathrm{d}\omega_{\mathrm{A}} = \int_{\mathrm{P}_o'}^{\mathrm{P}} \mathrm{d}\omega_{\mathrm{A}} - \int_{\mathrm{P}_o'}^{\mathrm{P}_o} \mathrm{d}\omega_{\mathrm{A}} = \hat{\omega}_{\mathrm{A}}(s) - \hat{\omega}_{\mathrm{A}}(s_o)$$

$$\omega_{\mathrm{A}}(s) = \hat{\omega}_{\mathrm{A}}(s) - \frac{S_{\omega_{\mathrm{A}}}}{A}$$

Example from the past: sectorial coordinate distribution and …



vääntökeskiö A sijaitsee(aina)
symmetriapisteessä (tai symmetriaaks.)

Poikkileikkauksen geom. sektoriaaliset suureet:

$\frac{b^3t}{16} = \frac{b^4}{160}$

$$I_\omega = \int_A \omega_A^2(s)\,dA = t\int_s \omega_A^2(s)\,ds = 4t\left(\frac{1}{2}\frac{b^2}{4}\cdot\frac{b}{2}\cdot\frac{2}{3}\frac{b^2}{4}\right] = \frac{1}{24}b^5t = \frac{1}{240}b^6$$

$$I_t \simeq \frac{1}{3}t^3(b+b+b) = bt^3 \;;\; G = E/2(1+\nu) = \frac{E}{2{,}6} \;;\; \nu = 0.3$$

Vääntötehtävän luonne $k = L\sqrt{\frac{GI_t}{EI_\omega}} = 3{,}04$

$1/2 < k < 10 \Rightarrow$ käytettävä sekamuotoisen väännön DY:ä.

Homework: a) analytically, b) Rayleigh-Ritz, c) FEA – buckling analysis and post-buckling analysis

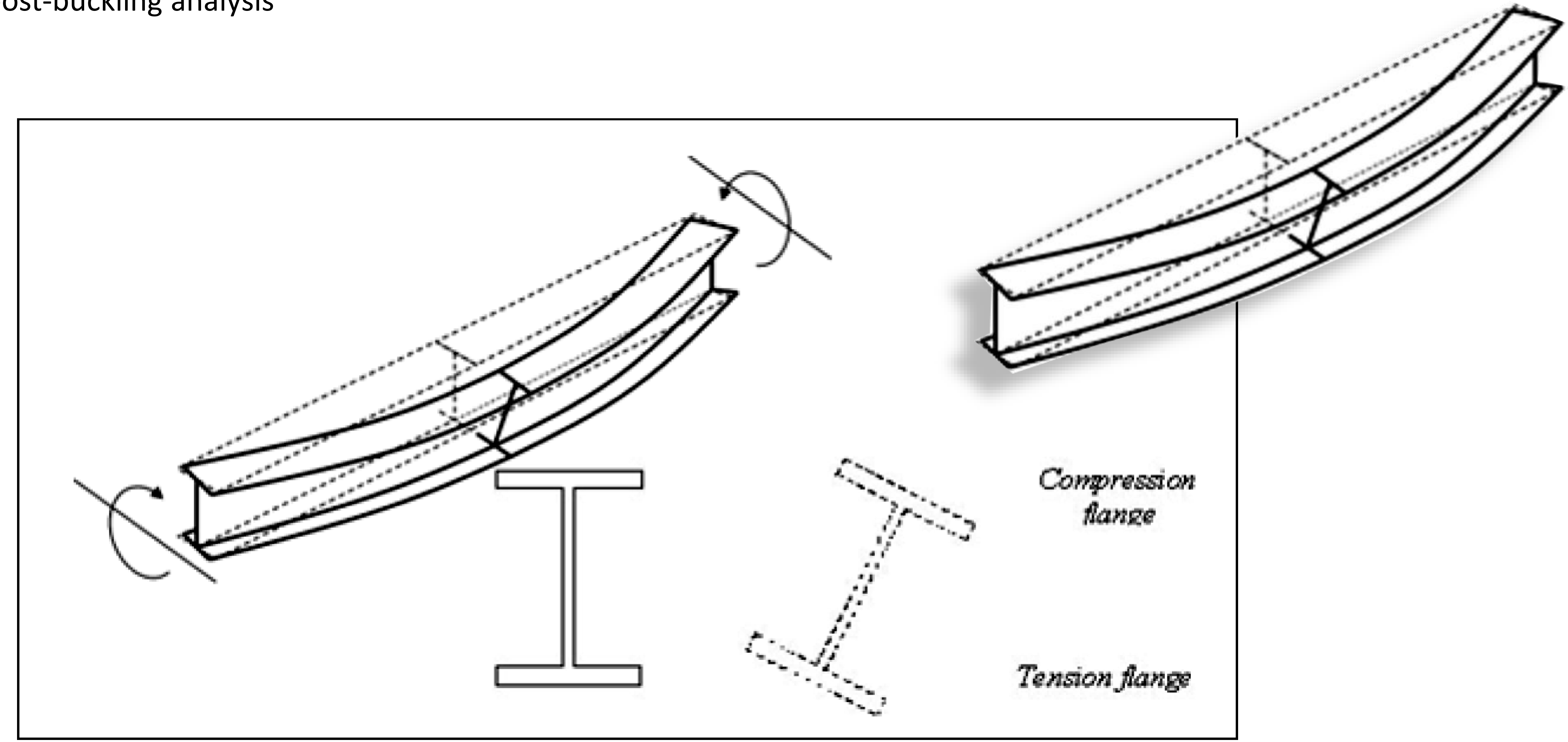
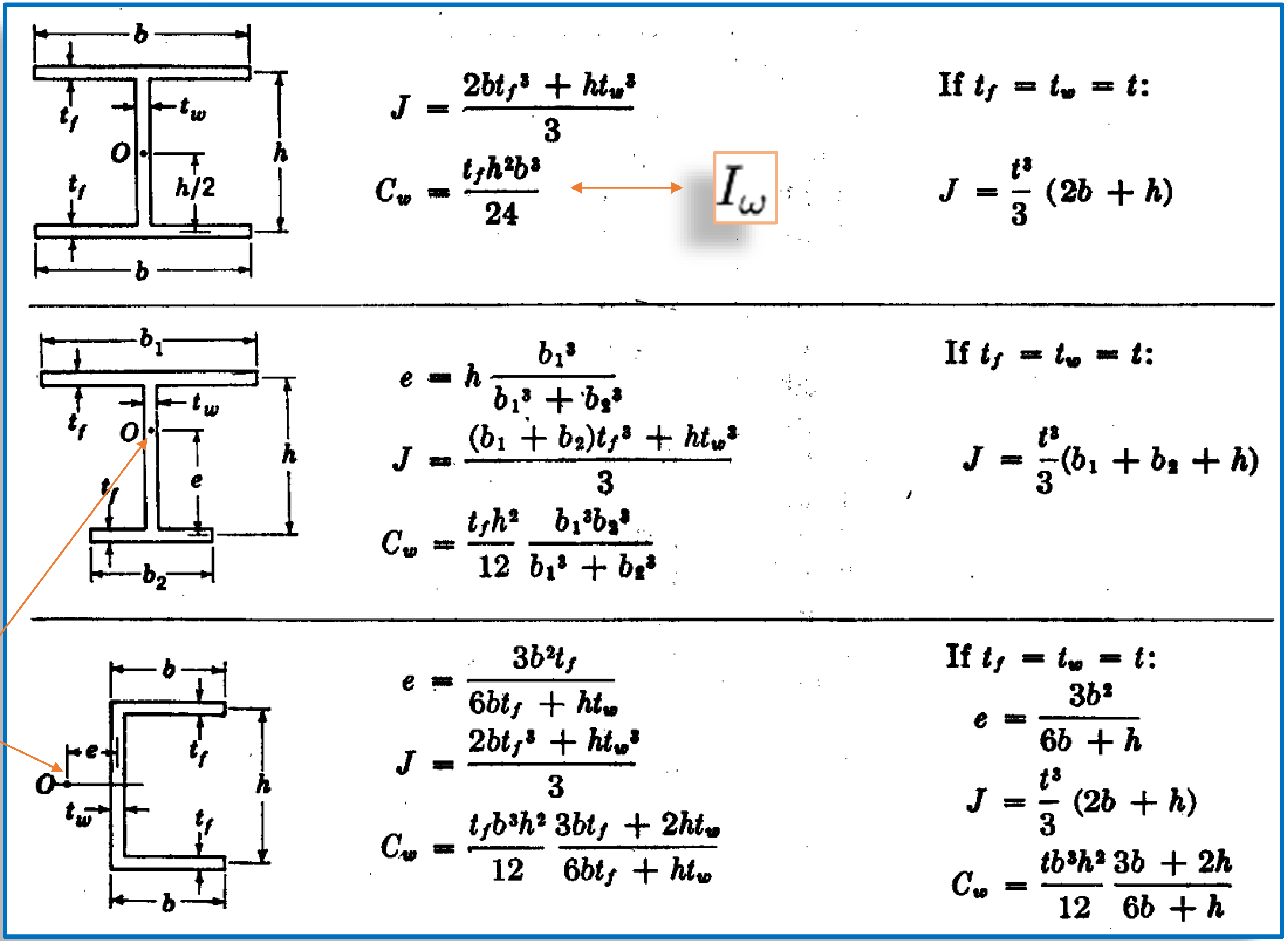


Figure 2.8 Lateral torsional buckling due to bending

Example of table giving shear center and the warping inertia moment I_ω

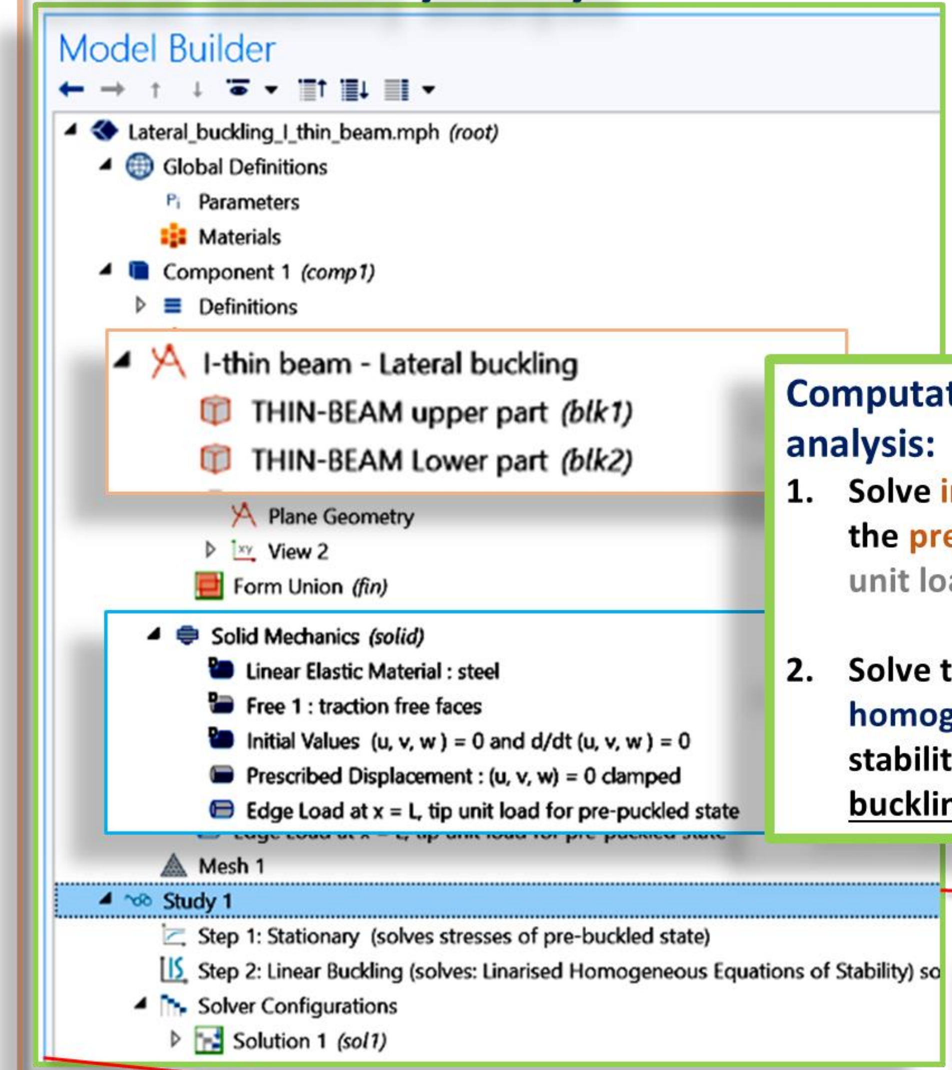


$$J = \frac{2bt_f^3 + ht_w^3}{3}$$ $$C_w = \frac{t_f h^2 b^3}{24} \longleftrightarrow \boxed{I_\omega}$$	If $t_f = t_w = t$: $$J = \frac{t^3}{3}(2b + h)$$
$$e = h\frac{b_1^3}{b_1^3 + b_2^3}$$ $$J = \frac{(b_1 + b_2)t_f^3 + ht_w^3}{3}$$ $$C_w = \frac{t_f h^2}{12}\frac{b_1^3 b_2^3}{b_1^3 + b_2^3}$$	If $t_f = t_w = t$: $$J = \frac{t^3}{3}(b_1 + b_2 + h)$$
$$e = \frac{3b^2 t_f}{6bt_f + ht_w}$$ $$J = \frac{2bt_f^3 + ht_w^3}{3}$$ $$C_w = \frac{t_f b^3 h^2}{12}\frac{3bt_f + 2ht_w}{6bt_f + ht_w}$$	If $t_f = t_w = t$: $$e = \frac{3b^2}{6b + h}$$ $$J = \frac{t^3}{3}(2b + h)$$ $$C_w = \frac{tb^3h^2}{12}\frac{3b + 2h}{6b + h}$$

Shear Center

- Now to stay realistic (6 weeks stability course) we will use tables for theses cross-section constants
- **Torsion topic** is a wide subject. Torsion of beams with thin-walled open-cross sections deserves, at least, a full three-weeks course by itself

Linear stability analysis



Model Builder

- Lateral_buckling_I_thin_beam.mph *(root)*
  - Global Definitions
    - Parameters
    - Materials
  - Component 1 *(comp1)*
    - Definitions
    - I-thin beam - Lateral buckling
      - THIN-BEAM upper part *(blk1)*
      - THIN-BEAM Lower part *(blk2)*
      - Plane Geometry
      - View 2
      - Form Union *(fin)*
    - Solid Mechanics *(solid)*
      - Linear Elastic Material : steel
      - Free 1 : traction free faces
      - Initial Values (u, v, w) = 0 and d/dt (u, v, w) = 0
      - Prescribed Displacement : (u, v, w) = 0 clamped
      - Edge Load at x = L, tip unit load for pre-puckled state
    - Mesh 1
  - Study 1
    - Step 1: Stationary (solves stresses of pre-buckled state)
    - Step 2: Linear Buckling (solves: Linarised Homogeneous Equations of Stability) so
    - Solver Configurations
      - Solution 1 *(sol1)*

Example using COMSOL

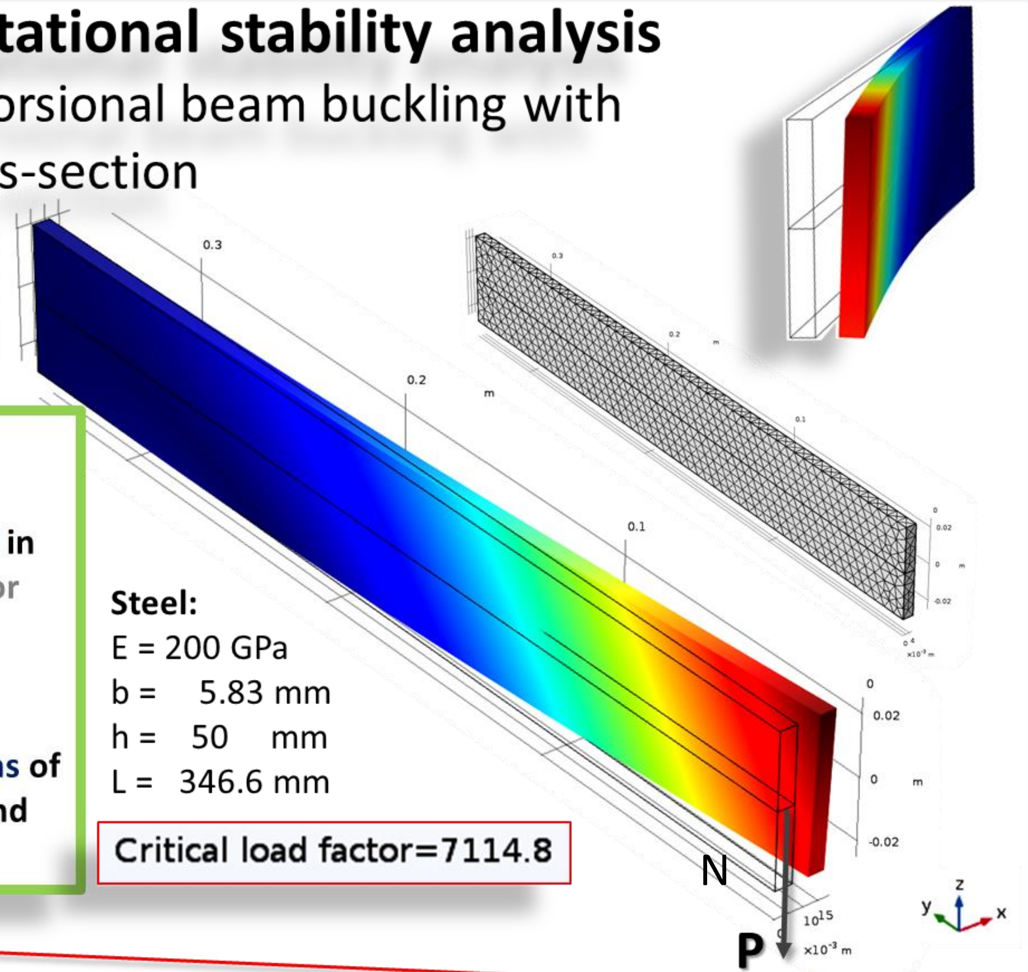
Computational stability analysis:

1. **Solve initial stress state in the pre-buckled state for unit loading**

2. **Solve the linearized homogeneous equations of stability: Critical load and buckling mode**

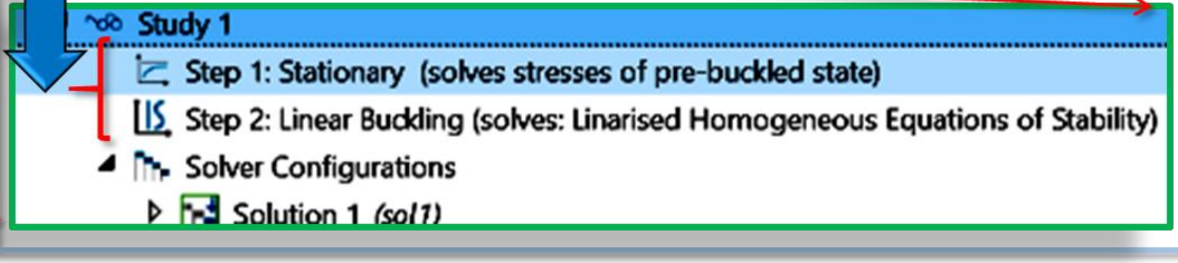
Computational stability analysis

Lateral torsional beam buckling with thin cross-section



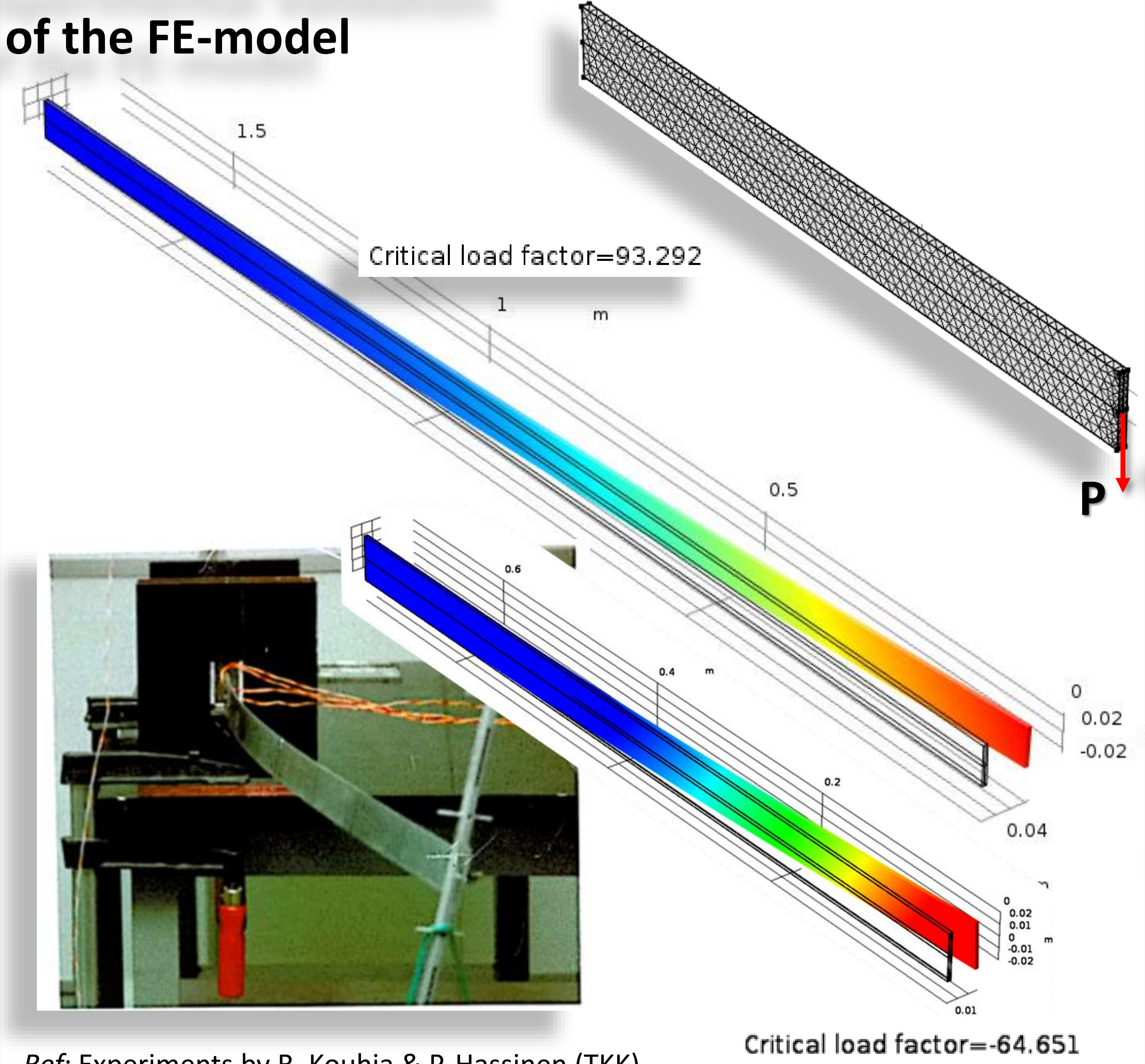
Steel:
E = 200 GPa
b = 5.83 mm
h = 50 mm
L = 346.6 mm

Critical load factor=7114.8



- Study 1
  - Step 1: Stationary (solves stresses of pre-buckled state)
  - Step 2: Linear Buckling (solves: Linarised Homogeneous Equations of Stability)
  - Solver Configurations
    - Solution 1 *(sol1)*

Experimental Validation of the FE-model

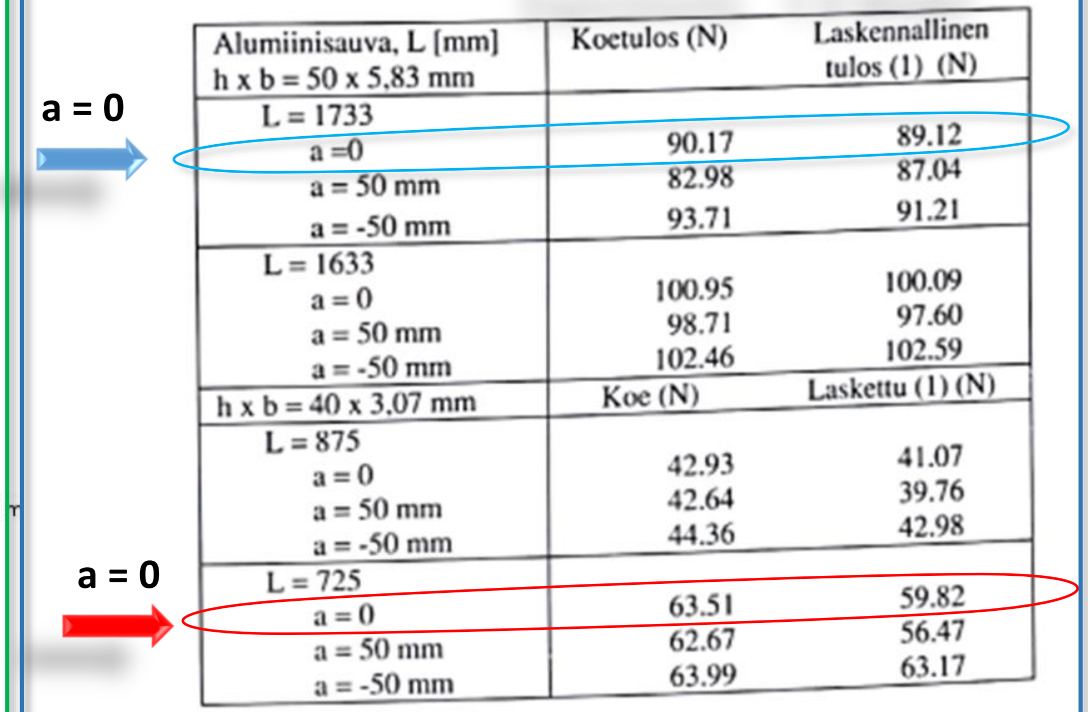


Ref: Experiments by R. Kouhia & P. Hassinen (TKK)

Material Aluminum: E = 70 GPa, ν = 0.33

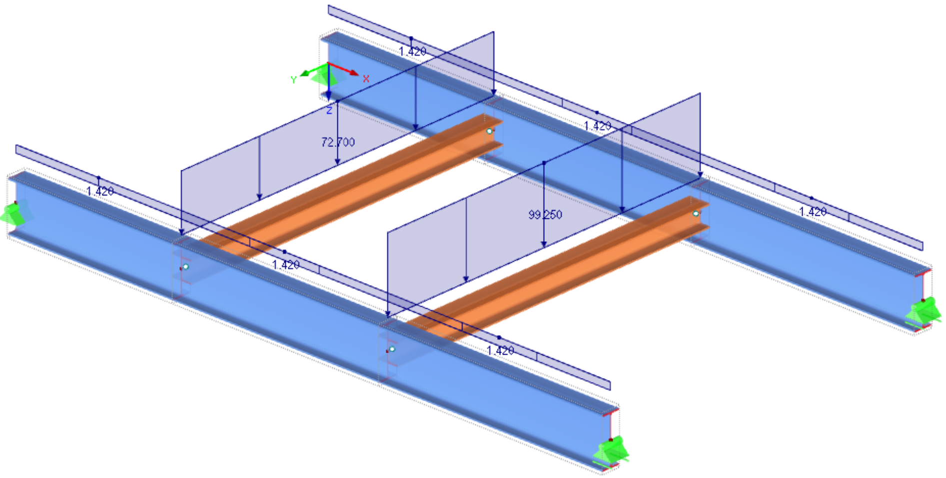
- **Experiment:** **63.5 N and 90.2 N**
  (Southwell-plot)
- **FE-model (3-D): 64.6 N and 93.3 N**
- **Analytical (beam model): 59.8 N and 89.1 N**

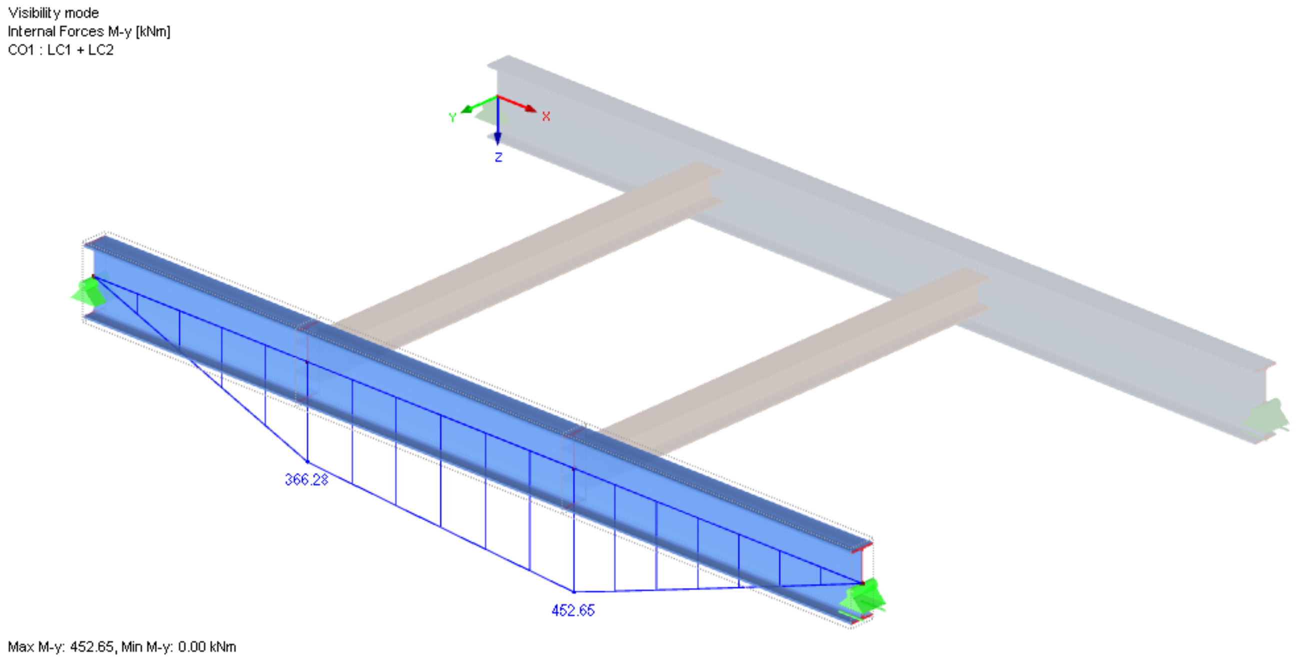
Experiments **1-D Model**



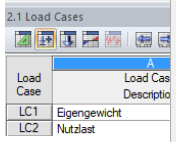
Alumiinisauva, L [mm] h x b = 50 x 5,83 mm	Koetulos (N)	Laskennallinen tulos (1) (N)
L = 1733		
a =0	90.17	89.12
a = 50 mm	82.98	87.04
a = -50 mm	93.71	91.21
L = 1633		
a = 0	100.95	100.09
a = 50 mm	98.71	97.60
a = -50 mm	102.46	102.59
h x b = 40 x 3,07 mm	Koe (N)	Laskettu (1) (N)
L = 875		
a = 0	42.93	41.07
a = 50 mm	42.64	39.76
a = -50 mm	44.36	42.98
L = 725		
a = 0	63.51	59.82
a = 50 mm	62.67	56.47
a = -50 mm	63.99	63.17

$$P_{cr} = \frac{4.013}{\ell^2}\sqrt{EI_yGI_t}\left[1+\frac{a}{L}\sqrt{\frac{EI_y}{GI_t}}\right]$$

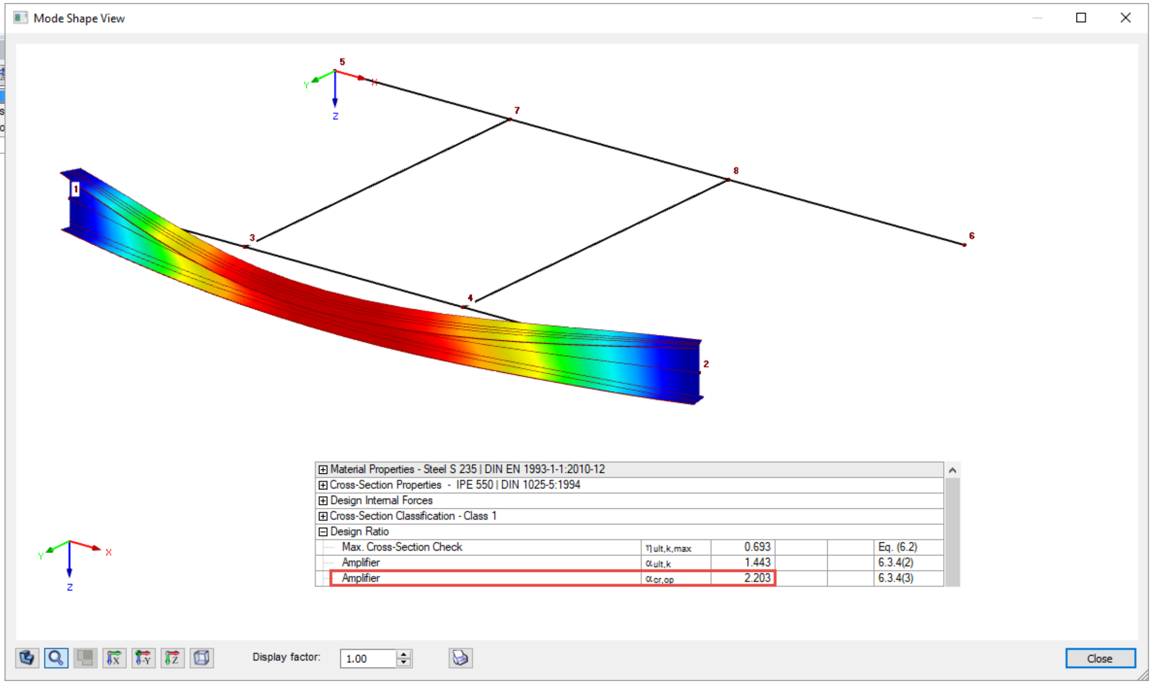



Visibility mode
Internal Forces M-y [kNm]
CO1 : LC1 + LC2

Max M-y: 452.65, Min M-y: 0.00 kNm

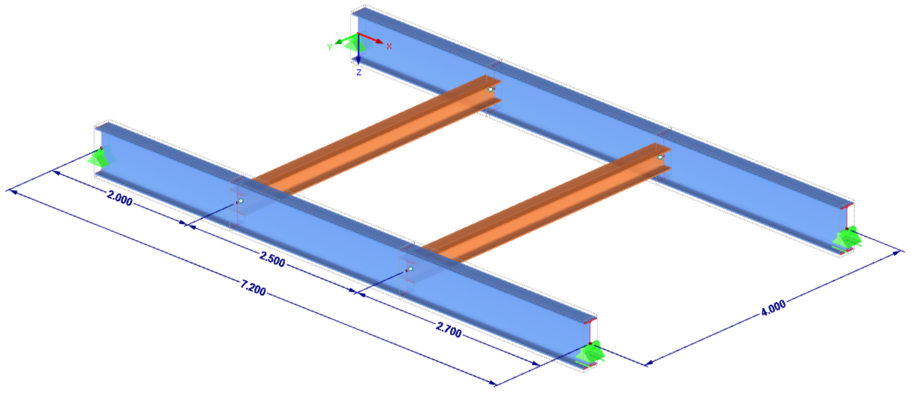

2.1 Load Cases

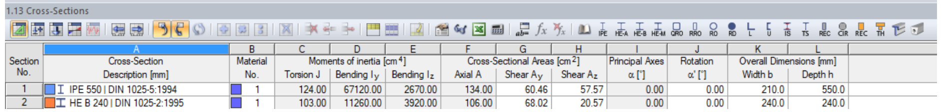
Load Case	Load Case Descriptio
LC1	Eigengewicht
LC2	Nutzlast



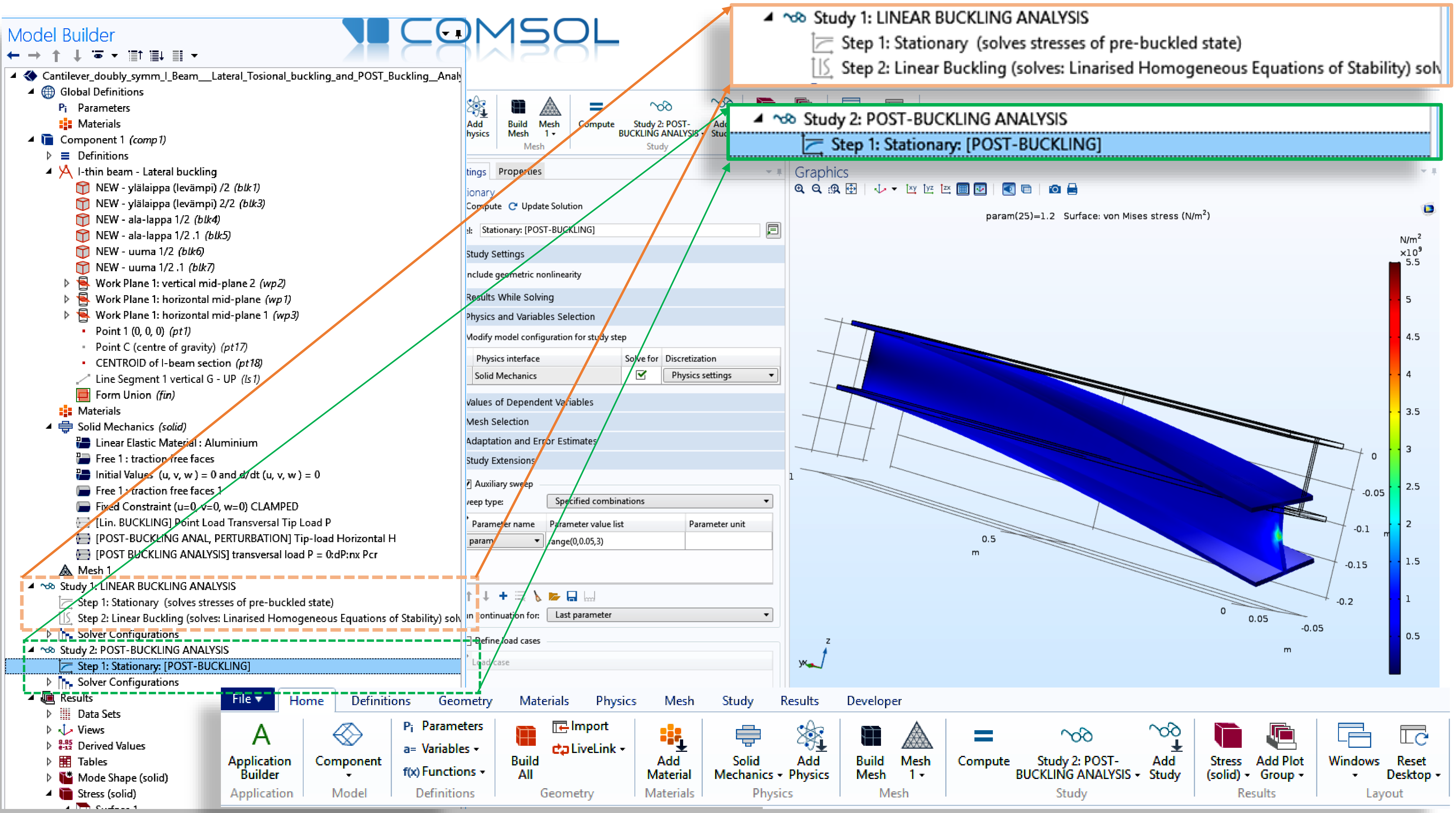
⊞ Material Properties - Steel S 235 | DIN EN 1993-1-1:2010-12
⊞ Cross-Section Properties - IPE 550 | DIN 1025-5:1994
⊞ Design Internal Forces
⊞ Cross-Section Classification - Class 1
⊟ Design Ratio

Max. Cross-Section Check	$\eta_{ult,k,max}$	0.693	Eq. (6.2)
Amplifier	$\alpha_{ult,k}$	1.443	6.3.4(2)
Amplifier	$\alpha_{cr,op}$	2.203	6.3.4(3)




1.13 Cross-Sections

Section No.	Cross-Section Description [mm]	Material No.	Moments of inertia [cm⁴] Torsion J	Bending I_y	Bending I_z	Cross-Sectional Areas [cm²] Axial A	Shear A_y	Shear A_z	Principal Axes α [°]	Rotation α' [°]	Overall Dimensions [mm] Width b	Depth h
1	IPE 550 | DIN 1025-5:1994	1	124.00	67120.00	2670.00	134.00	60.46	57.57	0.00	0.00	210.0	550.0
2	HE B 240 | DIN 1025-2:1995	1	103.00	11260.00	3920.00	106.00	68.02	20.57	0.00	0.00	240.0	240.0



Study 1: LINEAR BUCKLING ANALYSIS
Step 1: Stationary (solves stresses of pre-buckled state)
Step 2: Linear Buckling (solves: Linarised Homogeneous Equations of Stability) solv

Study 2: POST-BUCKLING ANALYSIS
Step 1: Stationary: [POST-BUCKLING]

- Cantilever_doubly_symm_I_Beam__Lateral_Tosional_buckling_and_POST_Buckling__Anal
  - Global Definitions
    - Parameters
    - Materials
  - Component 1 *(comp1)*
    - Definitions
    - I-thin beam - Lateral buckling
      - NEW - ylälaippa (levämpi) /2 *(blk1)*
      - NEW - ylälaippa (levämpi) 2/2 *(blk3)*
      - NEW - ala-lappa 1/2 *(blk4)*
      - NEW - ala-lappa 1/2 .1 *(blk5)*
      - NEW - uuma 1/2 *(blk6)*
      - NEW - uuma 1/2 .1 *(blk7)*
      - Work Plane 1: vertical mid-plane 2 *(wp2)*
      - Work Plane 1: horizontal mid-plane *(wp1)*
      - Work Plane 1: horizontal mid-plane 1 *(wp3)*
      - Point 1 (0, 0, 0) *(pt1)*
      - Point C (centre of gravity) *(pt17)*
      - CENTROID of I-beam section *(pt18)*
      - Line Segment 1 vertical G - UP *(ls1)*
      - Form Union *(fin)*
    - Materials
    - Solid Mechanics *(solid)*
      - Linear Elastic Material : Aluminium
      - Free 1 : traction free faces
      - Initial Values (u, v, w) = 0 and d/dt (u, v, w) = 0
      - Free 1 : traction free faces 1
      - Fixed Constraint (u=0, v=0, w=0) CLAMPED
      - [Lin. BUCKLING] Point Load Transversal Tip Load P
      - [POST-BUCKLING ANAL, PERTURBATION] Tip-load Horizontal H
      - [POST BUCKLING ANALYSIS] transversal load P = 0:dP:nx Pcr
    - Mesh 1
  - Study 1: LINEAR BUCKLING ANALYSIS
    - Step 1: Stationary (solves stresses of pre-buckled state)
    - Step 2: Linear Buckling (solves: Linarised Homogeneous Equations of Stability) solv
    - Solver Configurations
  - Study 2: POST-BUCKLING ANALYSIS
    - Step 1: Stationary: [POST-BUCKLING]
    - Solver Configurations
  - Results
    - Data Sets
    - Views
    - Derived Values
    - Tables
    - Mode Shape (solid)
    - Stress (solid)

ionary

Compute Update Solution

el: Stationary: [POST-BUCKLING]

Study Settings

nclude geometric nonlinearity

Results While Solving

Physics and Variables Selection

Modify model configuration for study step

Physics interface	Solve for	Discretization
Solid Mechanics	☑	Physics settings

Values of Dependent Variables

Mesh Selection

Adaptation and Error Estimates

Study Extensions

☑ Auxiliary sweep

eep type: Specified combinations

Parameter name	Parameter value list	Parameter unit
param	range(0,0.05,3)	

n continuation for: Last parameter

☐ Define load cases

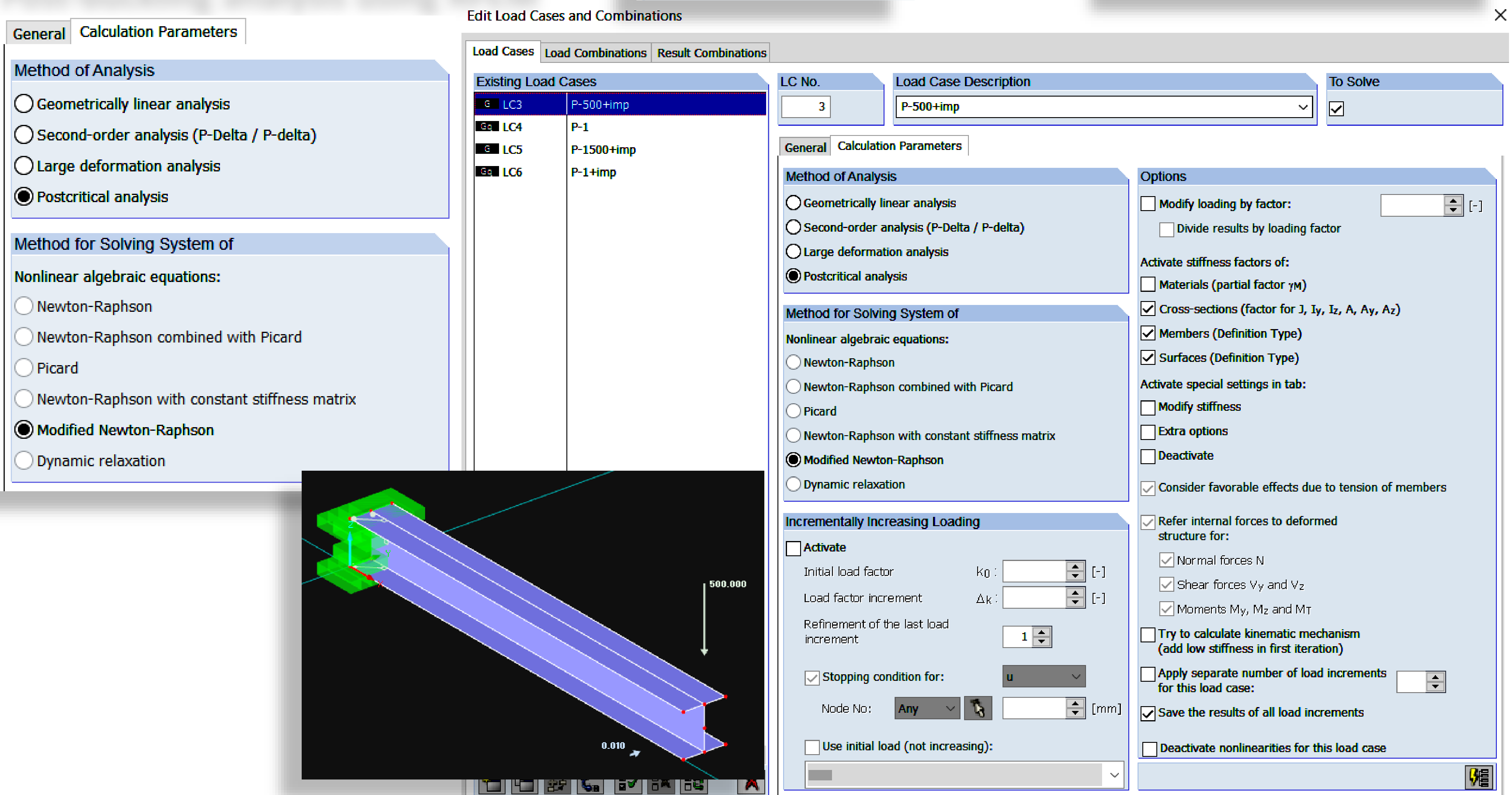
Load case

Post-buckling analysis using RFEM

● Postcritical analysis

RFEM

by DI Bahram S. using RFEM.



General | Calculation Parameters

Method of Analysis

- ○ Geometrically linear analysis
- ○ Second-order analysis (P-Delta / P-delta)
- ○ Large deformation analysis
- ● Postcritical analysis

Method for Solving System of

Nonlinear algebraic equations:

- ○ Newton-Raphson
- ○ Newton-Raphson combined with Picard
- ○ Picard
- ○ Newton-Raphson with constant stiffness matrix
- ● Modified Newton-Raphson
- ○ Dynamic relaxation

Edit Load Cases and Combinations

Load Cases | Load Combinations | Result Combinations

Existing Load Cases

LC3	P-500+imp
LC4	P-1
LC5	P-1500+imp
LC6	P-1+imp

LC No.: 3

Load Case Description: P-500+imp

To Solve: ☑

General | Calculation Parameters

Method of Analysis

- ○ Geometrically linear analysis
- ○ Second-order analysis (P-Delta / P-delta)
- ○ Large deformation analysis
- ● Postcritical analysis

Method for Solving System of

Nonlinear algebraic equations:

- ○ Newton-Raphson
- ○ Newton-Raphson combined with Picard
- ○ Picard
- ○ Newton-Raphson with constant stiffness matrix
- ● Modified Newton-Raphson
- ○ Dynamic relaxation

Incrementally Increasing Loading

- ☐ Activate
- Initial load factor k_0 : [-]
- Load factor increment Δk : [-]
- Refinement of the last load increment: 1
- ☑ Stopping condition for: u
- Node No: Any [mm]
- ☐ Use initial load (not increasing):

Options

- ☐ Modify loading by factor: [-]
  - ☐ Divide results by loading factor

Activate stiffness factors of:

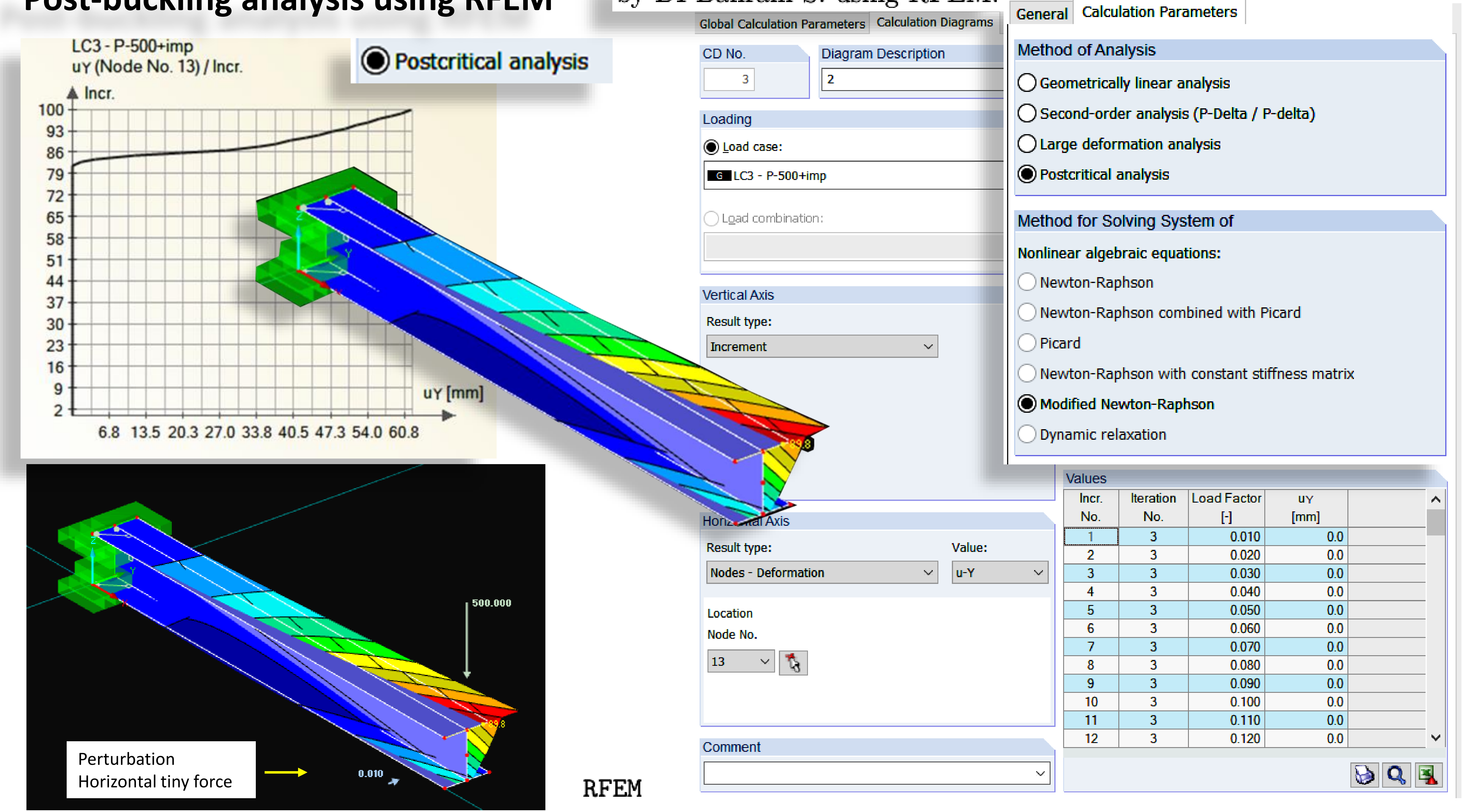
- ☐ Materials (partial factor γ_M)
- ☑ Cross-sections (factor for J, I_y, I_z, A, A_y, A_z)
- ☑ Members (Definition Type)
- ☑ Surfaces (Definition Type)

Activate special settings in tab:

- ☐ Modify stiffness
- ☐ Extra options
- ☐ Deactivate
- ☑ Consider favorable effects due to tension of members
- ☑ Refer internal forces to deformed structure for:
  - ☑ Normal forces N
  - ☑ Shear forces V_y and V_z
  - ☑ Moments M_y, M_z and M_T
- ☐ Try to calculate kinematic mechanism (add low stiffness in first iteration)
- ☐ Apply separate number of load increments for this load case:
- ☑ Save the results of all load increments
- ☐ Deactivate nonlinearities for this load case

Post-buckling analysis using RFEM

by DI Bahram S. using RFEM.



● Postcritical analysis

CD No.: 3

Diagram Description: 2

Loading

● Load case: G LC3 - P-500+imp

○ Load combination:

Vertical Axis

Result type: Increment

Horizontal Axis

Result type: Nodes - Deformation

Value: u-Y

Location

Node No. 13

Comment

Method of Analysis

○ Geometrically linear analysis

○ Second-order analysis (P-Delta / P-delta)

○ Large deformation analysis

● Postcritical analysis

Method for Solving System of

Nonlinear algebraic equations:

○ Newton-Raphson

○ Newton-Raphson combined with Picard

○ Picard

○ Newton-Raphson with constant stiffness matrix

● Modified Newton-Raphson

○ Dynamic relaxation

Values

Incr. No.	Iteration No.	Load Factor [-]	uY [mm]
1	3	0.010	0.0
2	3	0.020	0.0
3	3	0.030	0.0
4	3	0.040	0.0
5	3	0.050	0.0
6	3	0.060	0.0
7	3	0.070	0.0
8	3	0.080	0.0
9	3	0.090	0.0
10	3	0.100	0.0
11	3	0.110	0.0
12	3	0.120	0.0

Perturbation
Horizontal tiny force

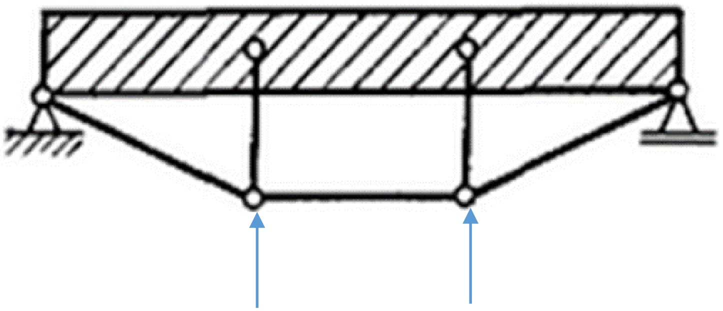
Lateral-torsional buckling

Application example: can you comment on lateral stability of the nodes of the stiffening truss

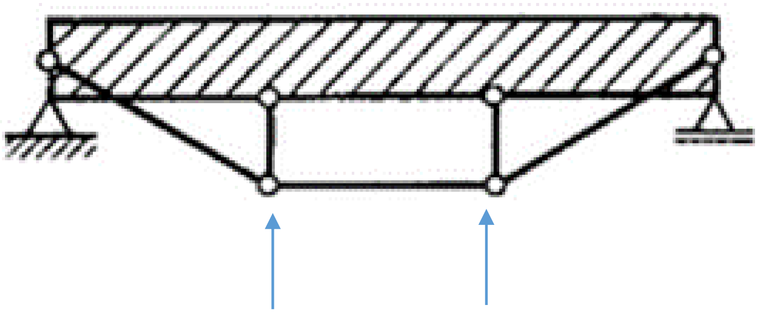
Two design solutions for the stiffened-beam (jäykistetty palkki)

- Which one is better?
- Which one need lateral supports for the nodes

Assume the hinge spherical

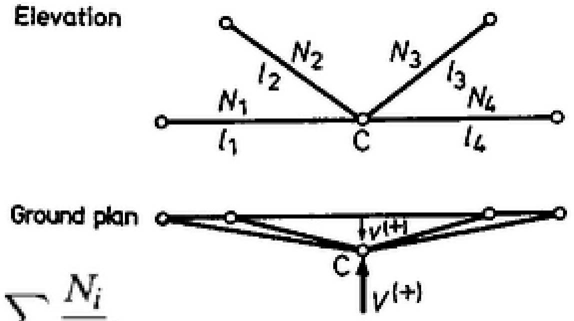


Stable nodes



Unstable nodes

Effect of location of the load



Kirste criterion: Tells when the node need lateral support against stability loss

$$V = v \sum_i \frac{N_i}{l_i},$$

We can also use the general stability criterion Trefftz or the sign of the variation of the change in total potential energy

Sign positive then stable of

Lateral-torsional buckling

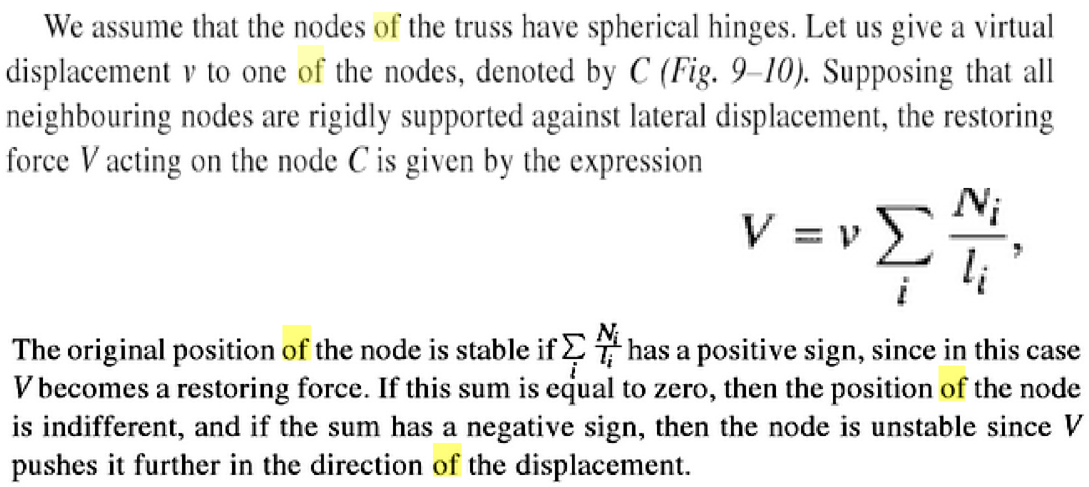
Application example: can you comment on lateral stability of the nodes of the stiffening truss

Two design solutions for the stiffened-beam (jäykistetty palkki)

- Which one is better?
- Which one need lateral supports for the nodes

Assume the hinge spherical

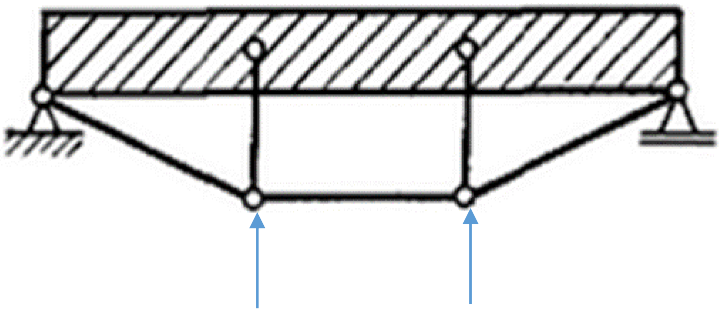
Kirste criterion:



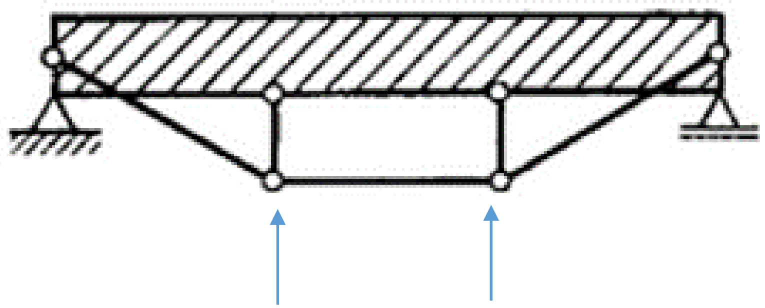
We assume that the nodes of the truss have spherical hinges. Let us give a virtual displacement v to one of the nodes, denoted by C (*Fig. 9–10*). Supposing that all neighbouring nodes are rigidly supported against lateral displacement, the restoring force V acting on the node C is given by the expression

$$V = v \sum_i \frac{N_i}{l_i},$$

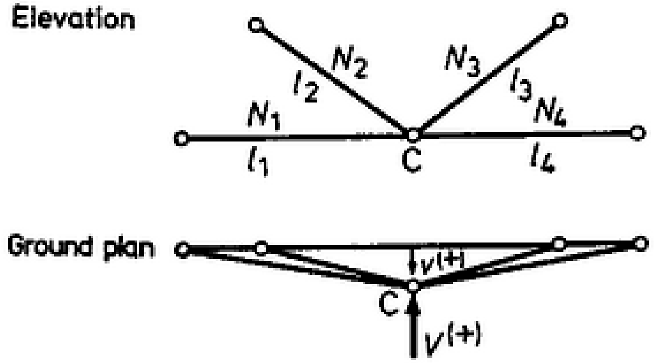
The original position of the node is stable if $\sum_i \frac{N_i}{l_i}$ has a positive sign, since in this case V becomes a restoring force. If this sum is equal to zero, then the position of the node is indifferent, and if the sum has a negative sign, then the node is unstable since V pushes it further in the direction of the displacement.



Stable nodes



Unstable nodes



Kirste criterion:

Kirste criterion: Tells when the node need lateral support against stability loss

We can also use the general stability criterion Trefftz