A? MyCourses

CIV-E4100 - Stability of Structures L, 24.02.2020-09.04.2020

Content of the 2nd week lectures:

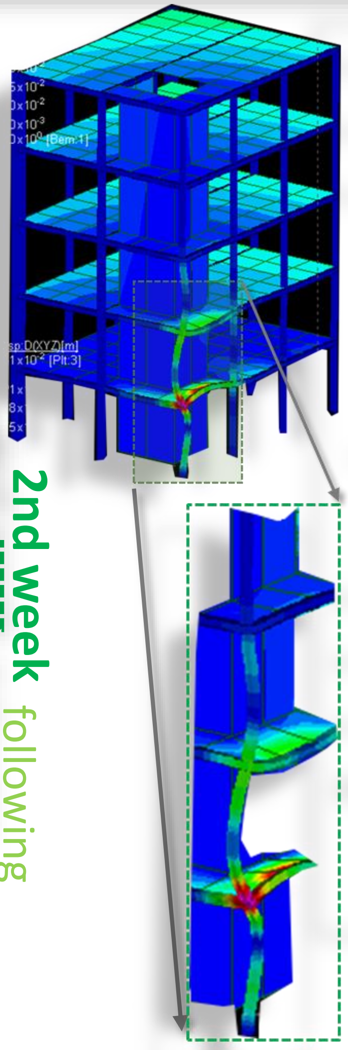
Content

0. Basic concepts
   Equilibrium, Stability
   The energy criterion of stability
1. **Flexural buckling (nurjahdus)**
2. Lateral-torsional buckling (kiepahdus)
3. Torsional buckling (vääntönurjahdus)
4. Buckling of thin plates
5. Buckling of shells (lommahdus)

First week

2nd week

following weeks



- General Energy criteria of loss of stability
- Trefftz stability loss criteria
- Flexural buckling
  - Buckling of beam-column
  - Timoshenko column
  - Buckling of beam-column on elastic foundation
- Effects of imperfections
  - ❑ Ayreton-Perry formula & Eurocode buckling curves
- Linear buckling analysis
- Post-buckling analysis
- Finite element method – a hand version for buckling analysis (= the slope deflection method)

Feb	24	25	26	27	28	29	1
March	2	3	4	5	6	7	8
	9	10	11	12	13	14	15
	16	17	18	19	20	21	22
	23	24	25	26	27	28	29
March	30	31	1	2	3	4	5

One topic per week

Lecturer
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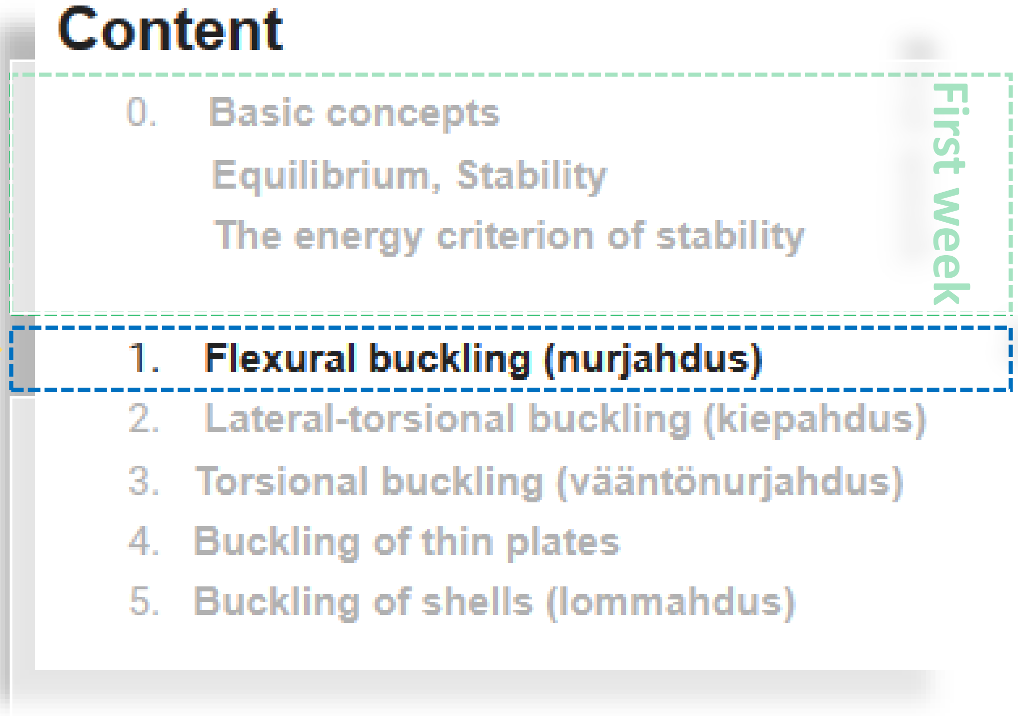
Civil Engineering Department
Aalto University

version 2.3.2020

Elastic Stability of Structures

Topics of the lectures and homework



Content

First week

0. Basic concepts
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   The energy criterion of stability

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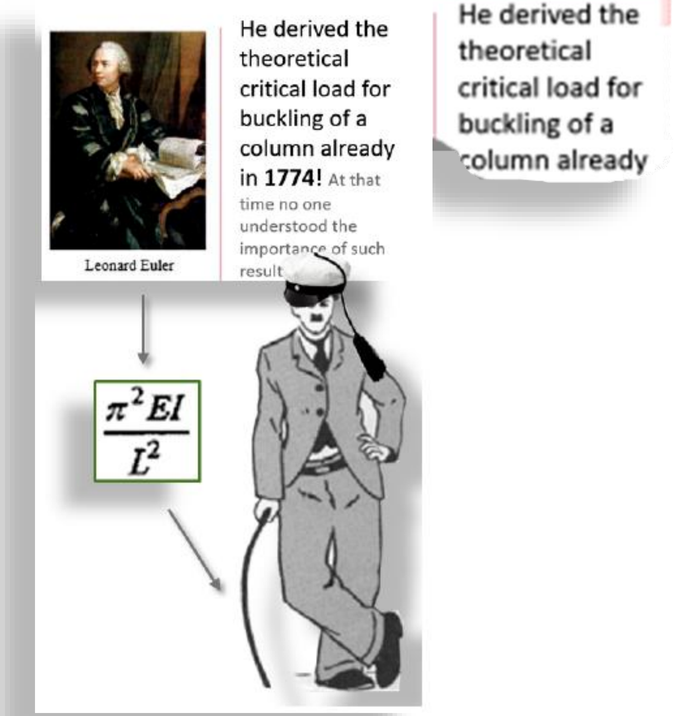
2nd week



He derived the theoretical critical load for buckling of a column already in **1774!** At that time no one understood the importance of such result

He derived the theoretical critical load for buckling of a column already

The key stability question in structural design

CIV-E4100 - Stability of Structures L, 25.02.2019-11.04.2019

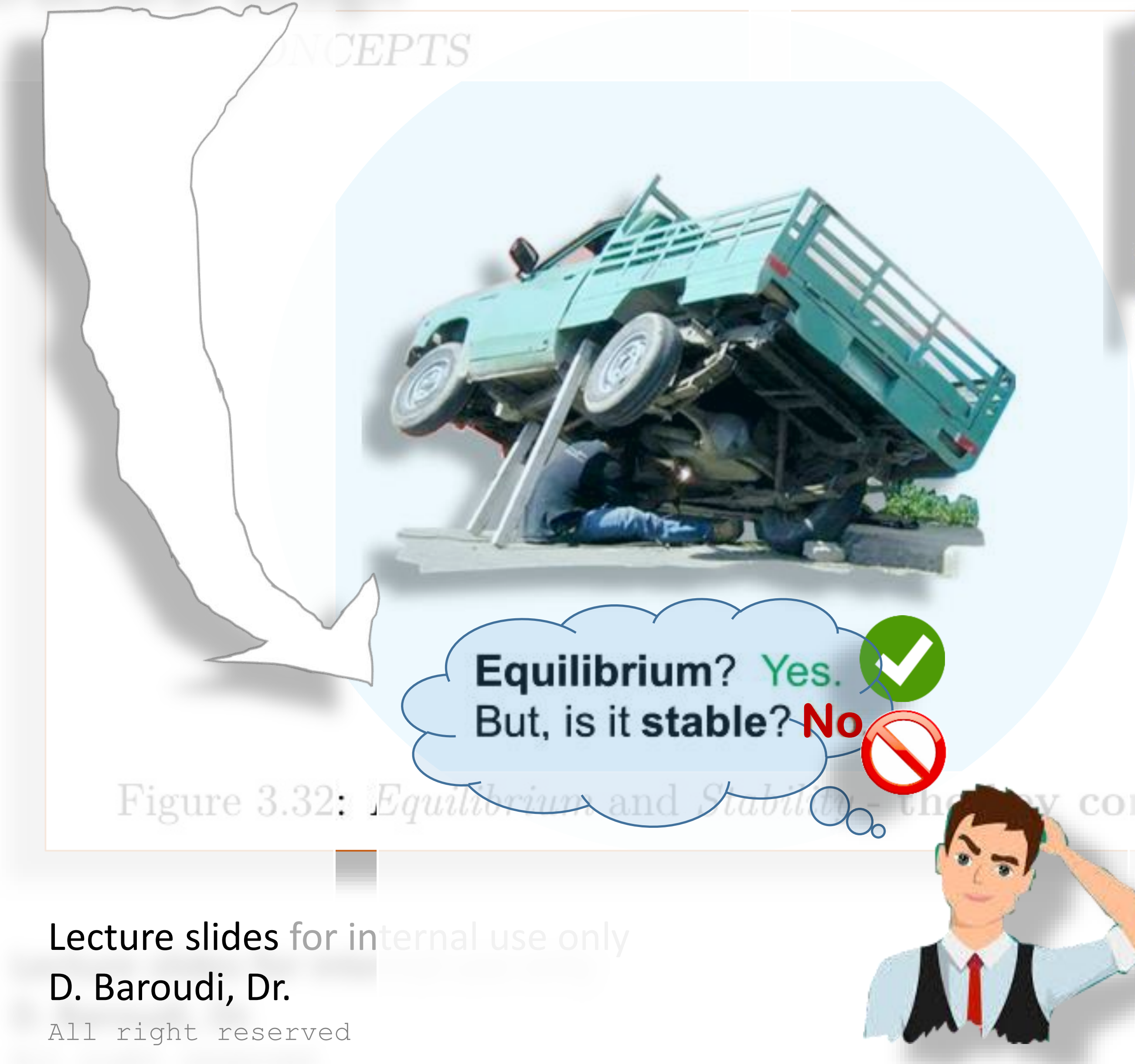
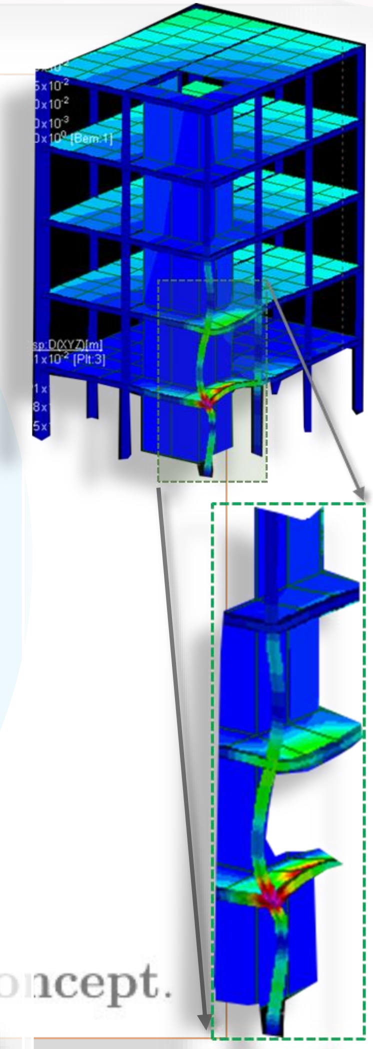
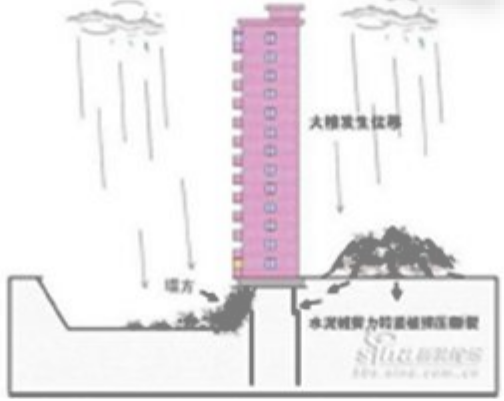
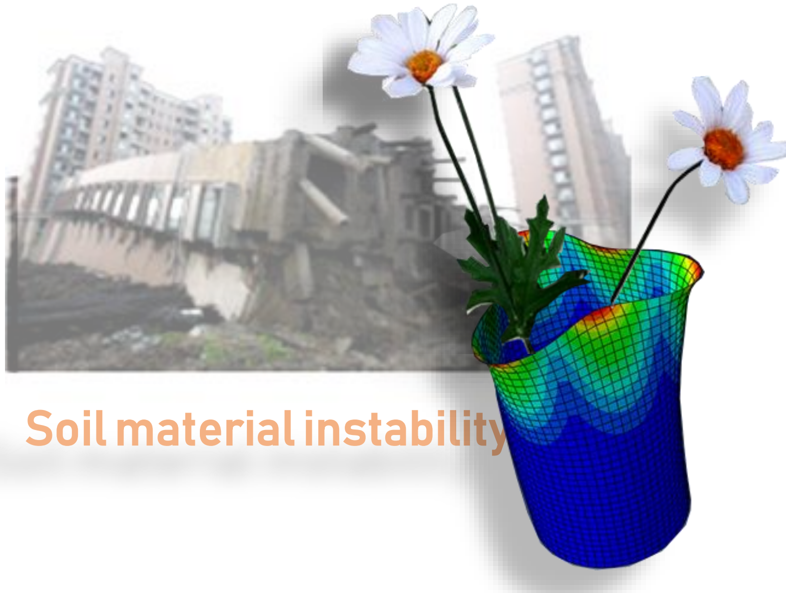


Figure 3.32: Equilibrium and Stability – the key concept.





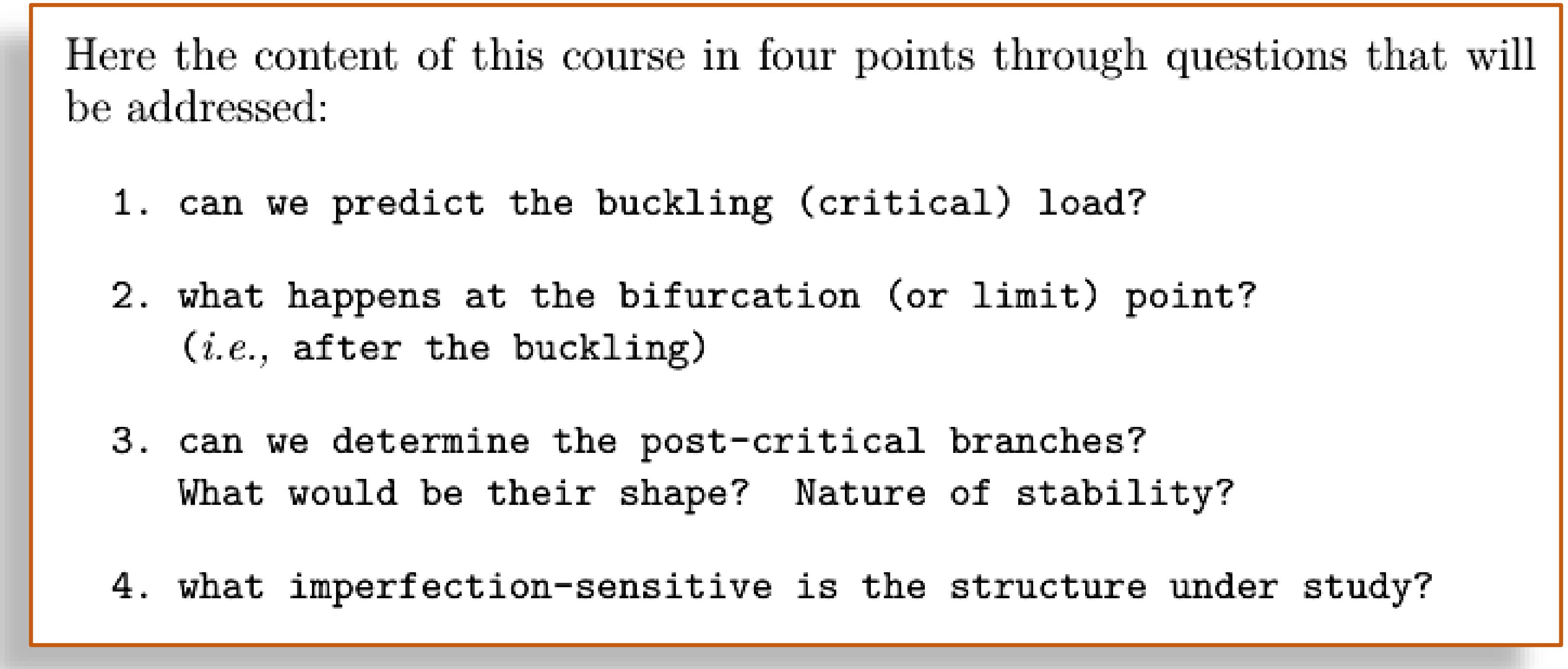


Soil material instability

Lecture slides for internal use only

D. Baroudi, Dr.

(recall) The Fundamental Question



Here the content of this course in four points through questions that will be addressed:

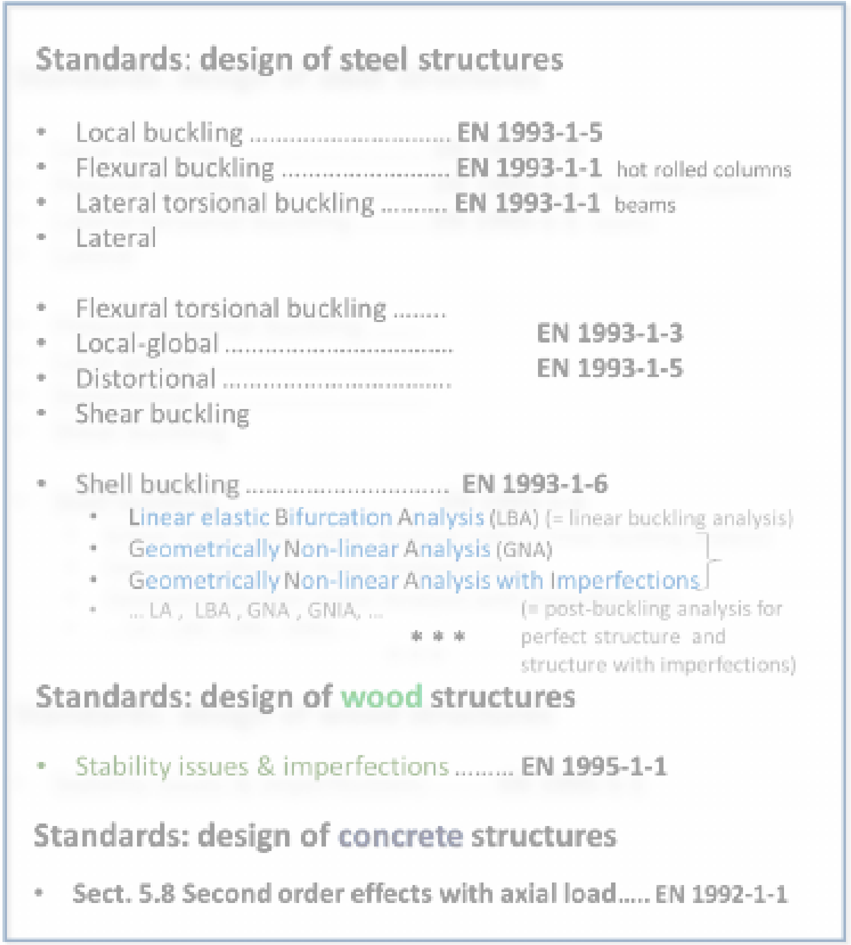
1. can we predict the buckling (critical) load?
2. what happens at the bifurcation (or limit) point? (*i.e.*, after the buckling)
3. can we determine the post-critical branches? What would be their shape? Nature of stability?
4. what imperfection-sensitive is the structure under study?

Effect of imperfections

All **real structural** systems are **imperfect**

- ✓ in form,
- ✓ in material properties,
- ✓ in the sense of residual stresses
- ✓ in the way the loads are applied

Structural design and stability



Standards: design of steel structures

- Local buckling .... **EN 1993-1-5**
- Flexural buckling .... **EN 1993-1-1** hot rolled columns
- Lateral torsional buckling .... **EN 1993-1-1** beams
- Lateral

- Flexural torsional buckling ....
- Local-global ....
- Distortional ....
- Shear buckling

EN 1993-1-3
EN 1993-1-5

- Shell buckling .... **EN 1993-1-6**
  - Linear elastic Bifurcation Analysis (LBA) (= linear buckling analysis)
  - Geometrically Non-linear Analysis (GNA)
  - Geometrically Non-linear Analysis with Imperfections
  - ... LA , LBA , GNA , GNIA, ...

(= post-buckling analysis for perfect structure and structure with imperfections)

* * *

Standards: design of wood structures

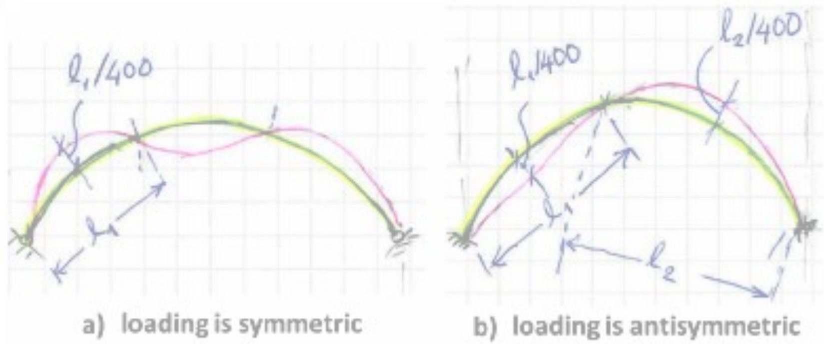
- Stability issues & imperfections .... **EN 1995-1-1**

Standards: design of concrete structures

- **Sect. 5.8 Second order effects with axial load..... EN 1992-1-1**

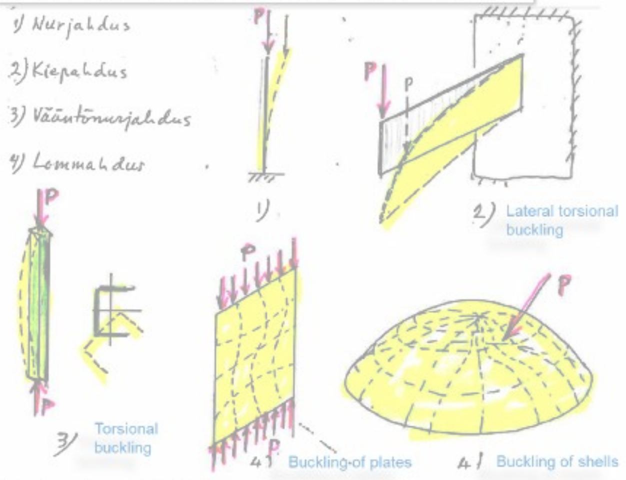
Some standards related to stability issues in structural design.

+ **Eurocode 7, geotechnical design**
- Slope stability
- Pile stability (foundations)
- …



Example of initial shape imperfections in an arch (Standards: design of wood structures - EN 1995-1-1)

Some typical loss of stability in structures

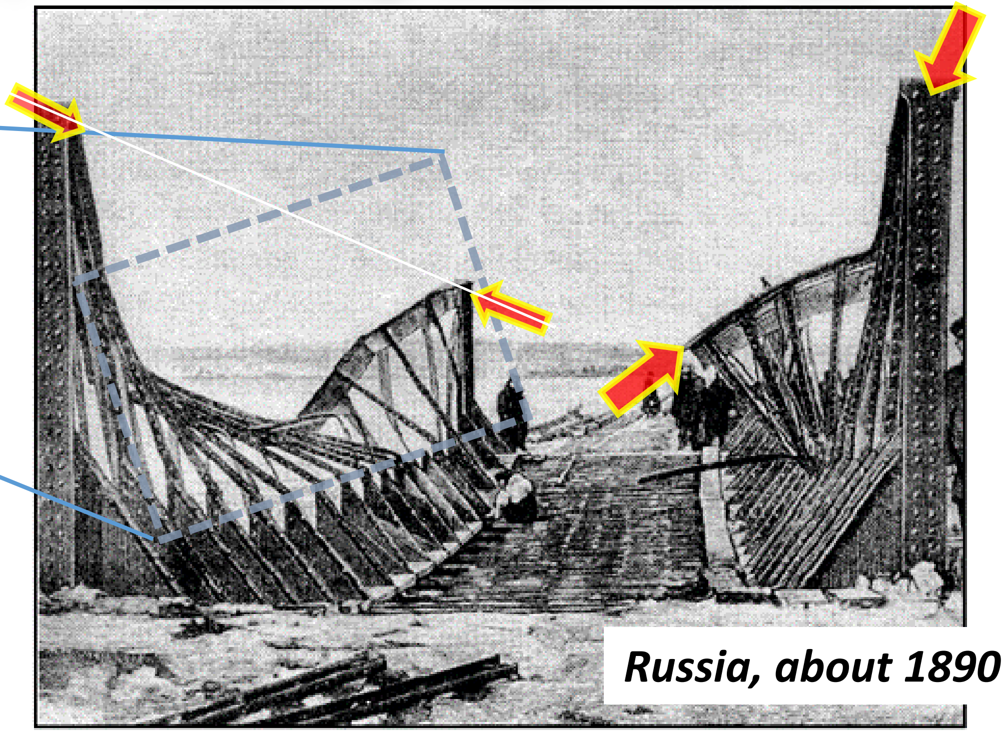


Drawings by: prof. M. Mikkola

Example of initial shape imperfections in wooden arches to be accounted in the structural analysis.

Foot bridge (ramp) collapse in Jiujiang City (China's Jiangxi)

Railway bridge collapse, Russia ~1890

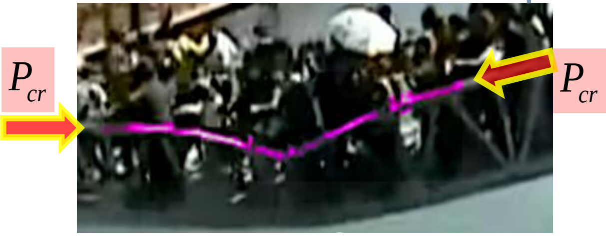


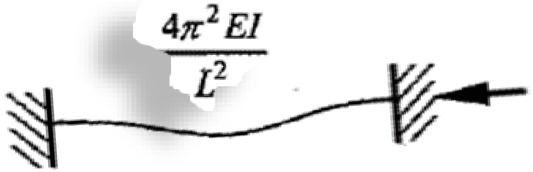
Russia, about 1890

fig. 8.3. - Flambement d'ensemble de la membrure supérieure des poutres en treillis d'un pont de chemin de fer (Russie, vers 1890).

buckling

Flambement d'ensemble de la membrure supérieure des poutres en treillis d'un pont de chemin de fer (Russie, vers 1890).



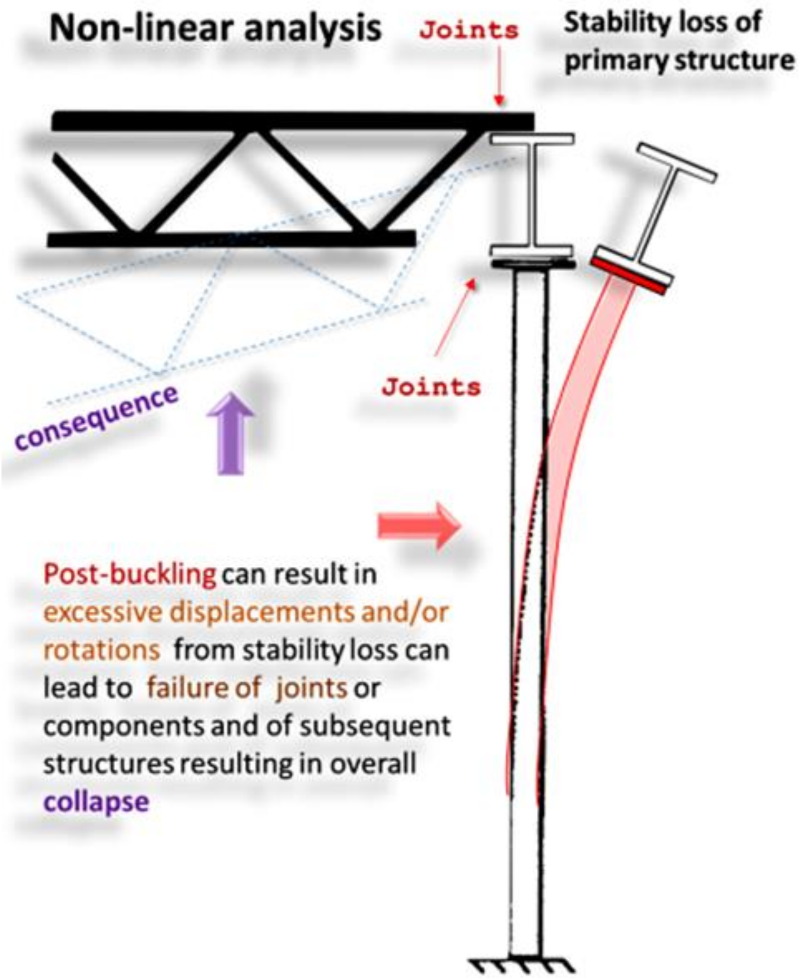


$$P_{cr} = \mu\pi^2 \frac{EI}{L^2}$$

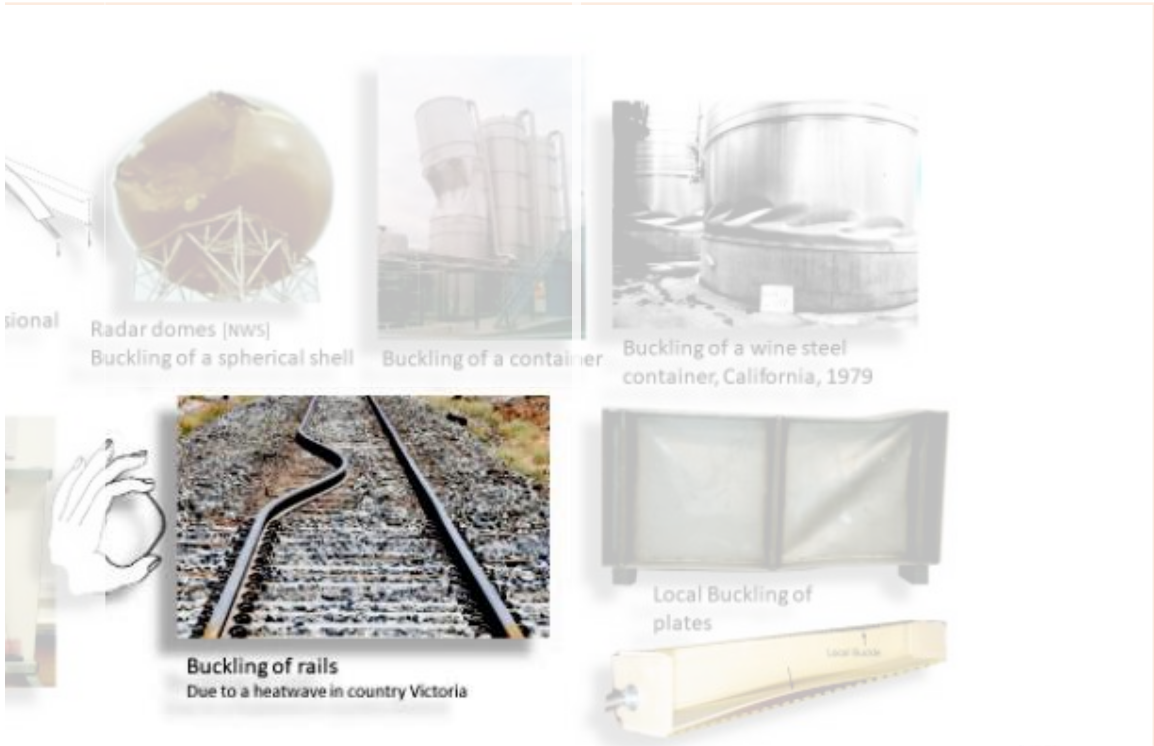
The mechanical cause of the collapse is the same: flexural buckling of compressed upper chord of the truss (yläpaarteen nurjahdus)

Structural design and stability

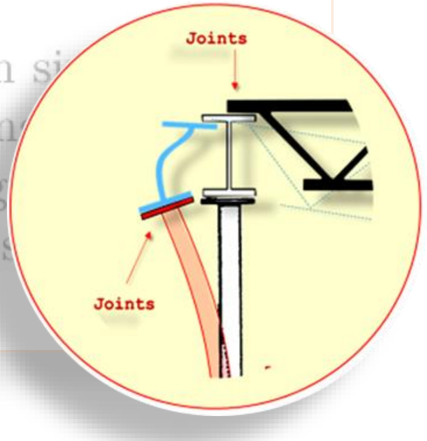
Flexural buckling

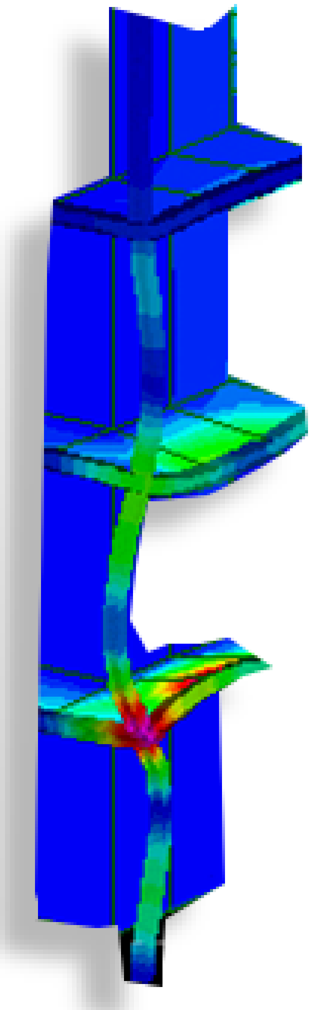


Post-buckling can result in excessive displacements and/or rotations from stability loss can lead to failure of joints or components and of subsequent structures resulting in overall collapse



es of various types of loss of stability in si
ight: lateral-torsional buckling, bucklin
and cylindrical shells, buckling of slender columns, buckling
rail bonded to a support and plate buckling represented by s
of the flanges and compressive buckling of web.





Despite our interest for post-buckling behavior, in *structural design*, **stability loss** is an unwanted event.

However, bifurcational buckling exists only for a non-existing perfect structure and thus GNA should be performed to find the limit-load, if any.

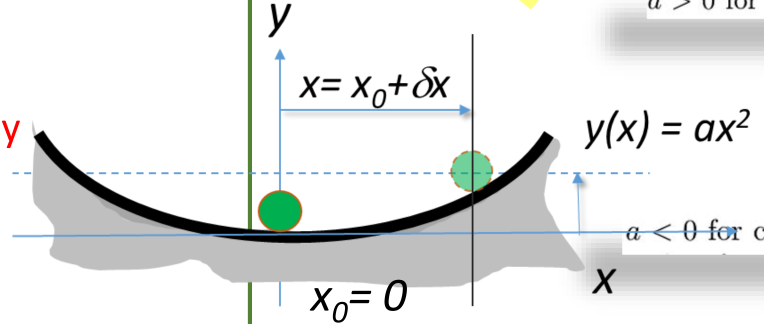
Energy criteria for determination of instability of elastic structures

Self-reading

RECALL

Let's illustrate mathematically the basic **stability types**

- stable
- Indifferent $\boxed{\Delta\Pi = 0.}$ ← this will be one condition for loss of stability
- unstable
- keeping a simplified example of the rigid ball (null strain energy)

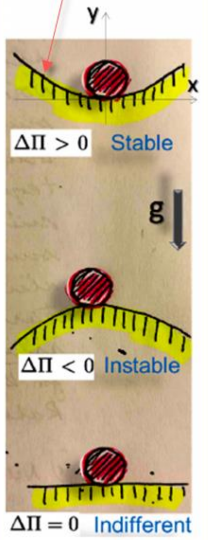


Geometry locally approximated $y(x) = ax^2$

$a > 0$ for convex,

$a < 0$ for concave

$a = 0$ for the neutral



Three various types of equilibrium configurations.

The total potential energy of the system $\Pi(x) = \Pi_0 + mgax^2$

Initial total potential energy

potential energy of gravitation

perturbed equilibrium position

$$\Pi(x_0 + \delta x) = \Pi(x_0) + \underbrace{\frac{\mathrm{d}\Pi(x)}{\mathrm{d}x}|_{x_0}\delta x}_{\delta\Pi|_{x_0}} + \frac{1}{2}\underbrace{\boxed{\frac{\mathrm{d}^2\Pi(x)}{\mathrm{d}x^2}|_{x_0}}(\delta x)^2}_{\delta^2\Pi|_{x_0}} + \frac{1}{3!}\underbrace{\frac{\mathrm{d}^3\Pi(x)}{\mathrm{d}x^3}|_{x_0}(\delta x)^3}_{\delta^3\Pi|_{x_0}} + \ldots$$

Initial equilibrium position

$$\equiv \Pi(x_0) + \delta\Pi|_{x_0} + \frac{1}{2}\delta^2\Pi|_{x_0} + \frac{1}{3!}\delta^3\Pi|_{x_0} + \ldots$$

Since x_0 is an equilibrium then $\delta\Pi|_{x_0} = 0.$

Equilibrium? Yes.
But, is it **stable**? No.

$\Pi'' = 2mga.$

… or equivalently

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$

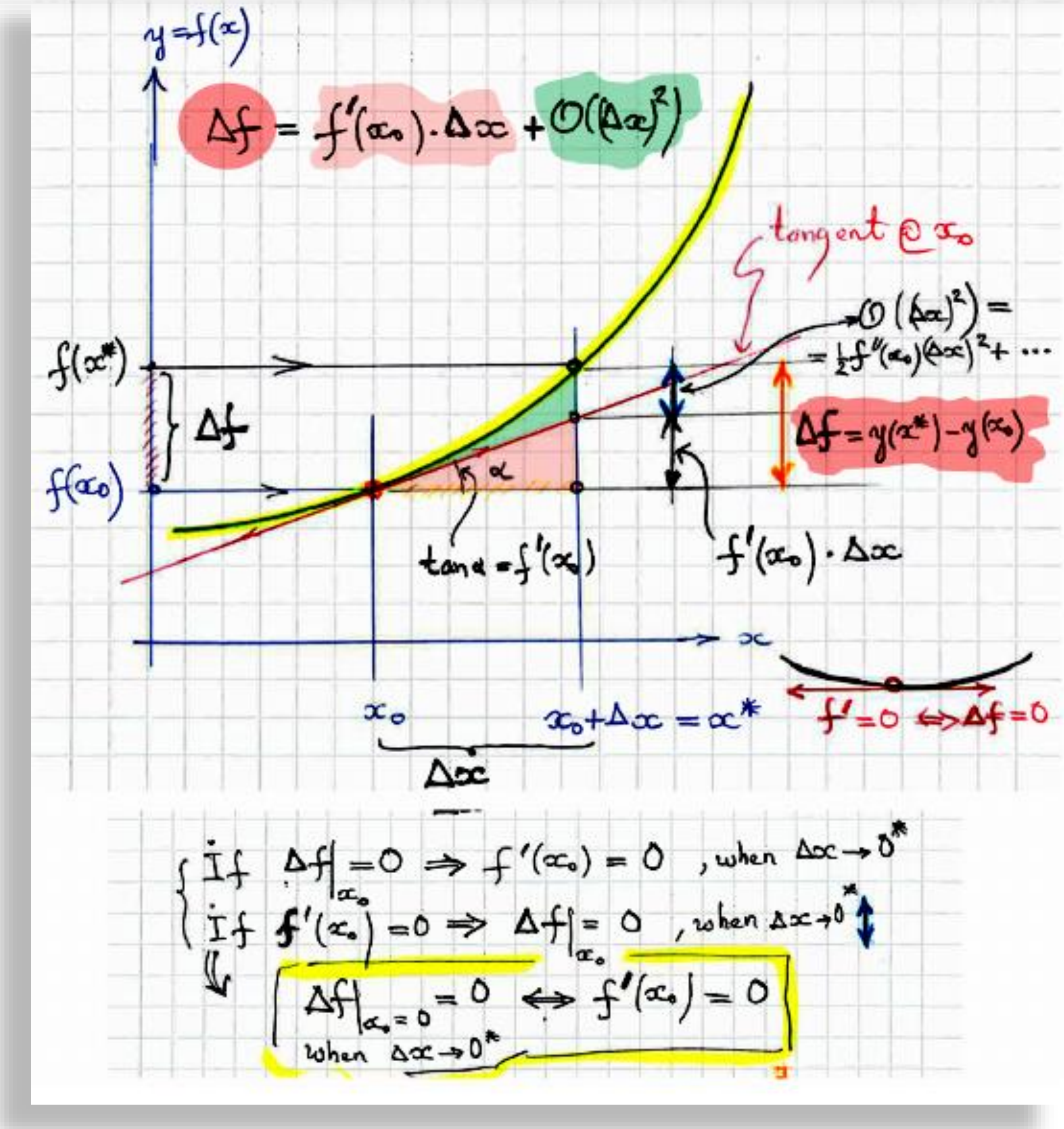
The sign of $\Delta\Pi$ **gives the full information about the <u>stability</u> behavior**

$$\Longrightarrow \quad \Delta\Pi = \Pi(x_0 + \delta x) - \Pi(x_0) = \frac{1}{2}\delta^2\Pi|_{x_0} + \frac{1}{3!}\delta^3\Pi|_{x_0} + \ldots$$

<u>The sign</u> will provides us the nature of **stability**

The idea is the make the study of stability in terms of *variational calculus*

Why saying that $f'(x_0) = 0$ is equivalent to say that $\Delta f|_{x_0} = 0$.



If $\Delta f|_{x_0} = 0 \Rightarrow f'(x_0) = 0$, when $\Delta x \to 0^*$

If $f'(x_0) = 0 \Rightarrow \Delta f|_{x_0} = 0$, when $\Delta x \to 0^*$

$$\boxed{\Delta f|_{x_0=0} = 0 \Leftrightarrow f'(x_0) = 0}$$

when $\Delta x \to 0^*$

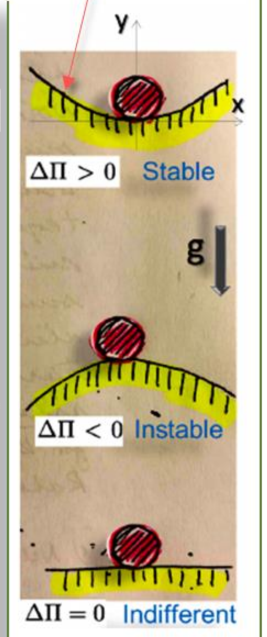
$\Delta\Pi = 0$.

Energy criteria for determination of instability of elastic structures

Self-reading

RECALL

Geometry locally approximated $y(x) = ax^2$



Three various types of equilibrium configurations.

First, keep only up-to the second order[21] term:

$$\Delta\Pi = \frac{1}{2}\frac{\mathrm{d}^2\Pi(x)}{\mathrm{d}x^2}|_{x_0}(\delta x)^2 = mga(\delta x)^2 + O(\delta x)^3.$$

Consequently, the initial equilibrium x_0 is stable when $a > 0$ (locally convex surface), unstable for $a < 0$ (locally concave surface) and indifferent when $a = 0$.

Bellow follows a résumé: At the critical points (equilibrium points), studying the sign of the increment of total potential energy $\Delta\Pi$, makes it possible to make statements on the nature of the actual equilibrium:

1. **stable**: (stabiili) $\Delta\Pi > 0$
2. **indifferent** : (indiferentti) $\Delta\Pi = 0$. Often, the total potential energy increment $\Delta\Pi$ is expanded to second order only (squares of small displacements). In this case, $\delta^2\Pi = 0$ and therefore, higher order terms should be included in the Taylor expansion to decide of the sign of $\Delta\Pi$ to disclose the character of indifferent equilibrium.
3. **unstable**: (labiili, epästabiili) $\Delta\Pi < 0$

Equilibrium? Yes.
But, is it **stable**? No.

So, the criticality condition: $\Delta\Pi = 0.$

Stability theorem of Lagrange-Dirichlet & **Trefftz** stability loss criteria

Lagrange-Dirichlet Theorem: *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total potential energy of the system has a strict minimum (i.e., is positive-definite).*

(This theorem is more general than **Trefftz** stability loss criteria)

$$\Longrightarrow$$

Trefftz *stability loss* criterion

$$\boxed{\delta(\delta^2\Pi) = 0.}$$

$$\Downarrow$$

stability loss criteria

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$

$$\Delta\Pi = \Pi(u^0 + \delta u) - \Pi(u^0) = \underbrace{\delta\Pi|_{u_0}}_{=0} + \frac{1}{2}\delta^2\Pi|_{u_0} + \frac{1}{3!}\delta^3\Pi|_{u_0} + \ldots$$

stability loss criteria

$$\Pi'' = 0 \longleftrightarrow \delta(\Delta\Pi) = 0 \Longrightarrow \delta\left(\frac{1}{2}\delta^2\Pi|_{u_0} + \frac{1}{3!}\delta^3\Pi|_{u_0} + \ldots\right) = 0$$

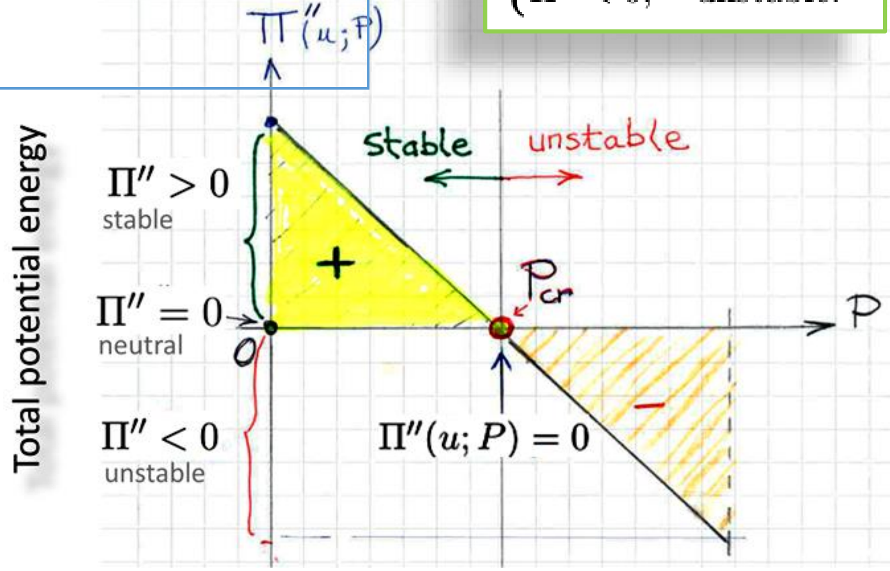
$$\delta\Pi^0 = \delta\Pi|_{u_0} = 0 \ (u^0 \text{ -equilibrium initial state})$$

keeping only the quadratic terms, one obtains the energy criterion

$$\Longrightarrow \quad \delta(\Delta\Pi) = 0 \Longrightarrow \delta(\delta^2\Pi) = 0,$$

More general criterion than Trefftz

Trefftz *stability loss* criterion

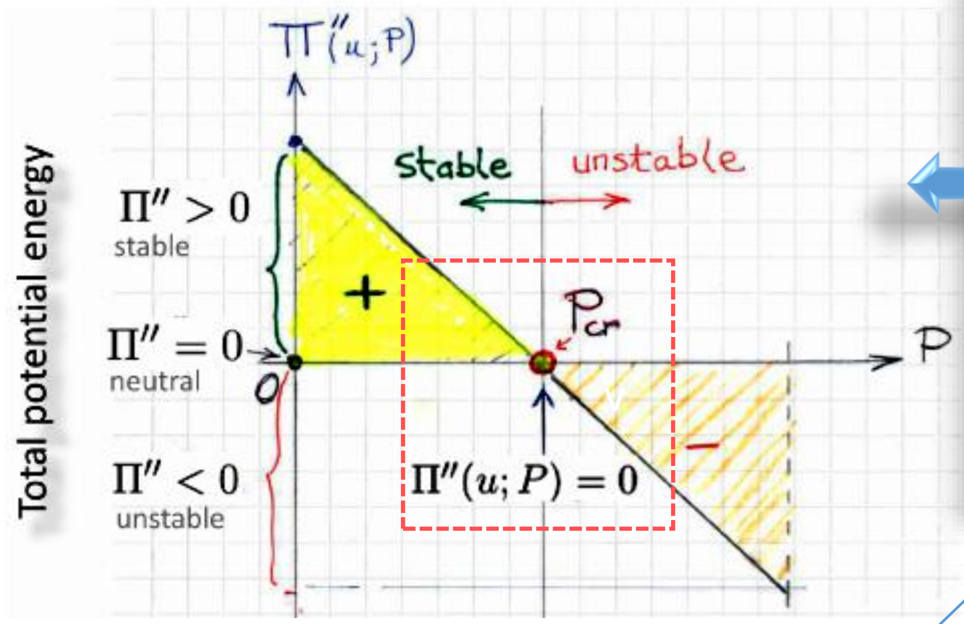


Trefftz is a particular case where the total potential energy increment is expanded only up-to its quadratic terms between the initial and perturbed states

It is tis form of criticality condition that will be used systematically thorough this course to derive the stability loss equations for all our structures

The criteria of loss of stability

RECALL



Lagrange-Dirichlet Theorem: *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total po*

A **Taylor expansion** of a function

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

This is a **Taylor expansion** of a function

$$\Pi(\underbrace{\mathbf{q}^0 + \delta\mathbf{q}}_{\mathbf{q}}) = \Pi(\mathbf{q}^0) + \sum_{i=1}^{N} \frac{\partial \Pi}{\partial q_i}\Big|_{\mathbf{q}^0} \cdot \delta q_i + \frac{1}{2!}\sum_{i,j=1}^{N} \underbrace{\frac{\partial^2 \Pi}{\partial q_i \partial q_j}\Big|_{\mathbf{q}^0}}_{\equiv \mathbf{H}(\mathbf{q}^0)} \cdot \delta q_i \delta q_j + \ldots$$

$$\approx \Pi(\mathbf{q}^0) + \underbrace{\underbrace{[\nabla \Pi(\mathbf{q}^0)]}_{=0,\ \text{equilibrium}}^{\mathrm{T}} \delta\mathbf{q}}_{\equiv \delta\Pi} + \underbrace{\boxed{\frac{1}{2!}\delta\mathbf{q}^{\mathrm{T}}[\mathbf{H}(\mathbf{q}^0)]\delta\mathbf{q}}}_{\equiv \delta^2\Pi} + \mathcal{O}(\|\delta\mathbf{q}\|^3),$$

at equilibrium $(\delta\Pi = 0)$.

$$\boxed{\Delta\Pi = \boxed{\delta^2\Pi} + \mathcal{O}(\|\delta\mathbf{q}\|^3) \sim \frac{1}{2!}\delta\mathbf{q}^{\mathrm{T}}[\mathbf{H}(\mathbf{q}^0)]\delta\mathbf{q}}$$

More suitable form for finite number of dofs and continuous case

Leading term for sign change in the increment of total potential energy

$$\boxed{\Pi''(u;P) = 0 \text{ or more generally, } \delta(\Delta\Pi) = 0,}$$

$$\boxed{\boxed{\delta(\Delta\Pi) = 0} \Longrightarrow \delta(\delta^2\Pi) = 0,}$$

Trefftz condition

It is tis form of criticality condition that will be used systematically thorough this course to derive the stability loss equations for all our structures

Physically speaking, this condition means simply that the perturbed state is also an equilibrium state; thus an neighboring equilibrium exists

Linear buckling analysis

About the criteria of loss of stability – Example with two dofs

$$\Delta\Pi(\epsilon_1,\epsilon_2)=\frac{1}{2}k\ell^2(\epsilon_1^2+\epsilon_2^2)-P\ell\cdot\left(\left[1-\sqrt{1-\epsilon_1^2}\right]+\left[1-\sqrt{1-(\epsilon_2-\epsilon_1)^2}\right]+\left[1-\sqrt{1-\epsilon_2^2}\right]\right)$$

the relative shortenings are defined as $\epsilon_1 = v_1/\ell$ and $\epsilon_2 = v_2/\ell$.

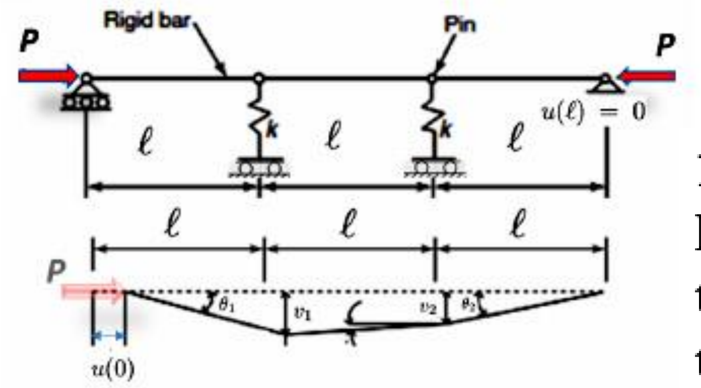


Figure 1.42: A simple system having two degrees of

1) Linear buckling analysis: We want to determine the Euler buckling load. In such analysis we have, by definition, both relative shortening of the column $\epsilon_1 \ll 1$ and $\epsilon_2 \ll 1$, so as the reader may recall, one expands the total potential energy increment into *Taylor expansion up-to quadratic* terms in v_1/ℓ and v_2/ℓ (or ϵ_1 and ϵ_2). So,

$$\Downarrow$$

$$\Delta\Pi(v_1,v_2)=\frac{1}{2}k(v_1^2+v_2^2)-P\ell\left[\frac{1}{2}\left(\frac{v_1}{\ell}\right)^2+\frac{1}{2}\left(\frac{v_2-v_1}{\ell}\right)^2+\frac{1}{2}\left(\frac{v_2}{\ell}\right)^2\right]$$

the loss of stability condition in its *variational*

form $\delta(\Delta\Pi) = 0$

Requiring the neutral equilibrium condition $\delta(\Delta\Pi) = 0$ (for loss of stability)

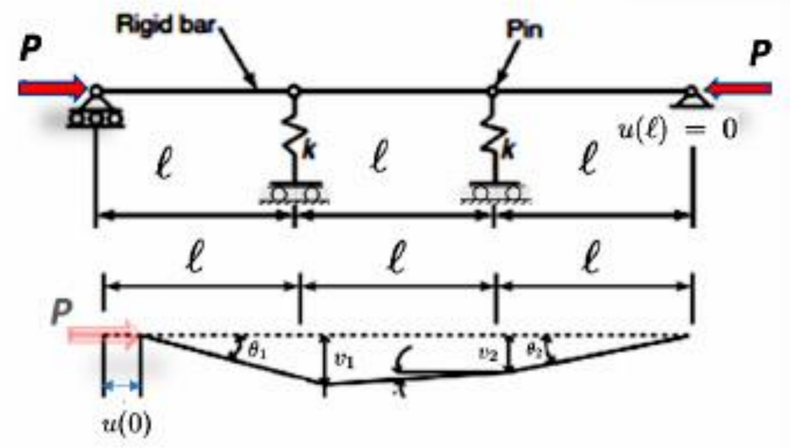
Self-reading

Linear buckling analysis

About the criteria of loss of stability –
Example with two dofs

RECALL

$$\Delta\Pi(v_1,v_2)=\frac{1}{2}k(v_1^2+v_2^2)-P\ell\left[\frac{1}{2}\left(\frac{v_1}{\ell}\right)^2+\frac{1}{2}\left(\frac{v_2-v_1}{\ell}\right)^2+\frac{1}{2}\left(\frac{v_2}{\ell}\right)^2\right]$$



re 1.42: A simple system having two degrees of freedom.

$$\Delta\Pi(v_1,v_2)=\frac{1}{2}\begin{bmatrix}v_1 & v_2\end{bmatrix}\underbrace{\left(\underbrace{\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}}_{\mathbf{K}}-\underbrace{\frac{P}{\ell}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}}_{\mathbf{S}(P)}\right)}_{\mathbf{H}(0,0)}\begin{bmatrix}v_1\\ v_2\end{bmatrix} \qquad (1.68)$$

o, one obtains the *quadratic form*

$$\Delta\Pi(\mathbf{q})=\frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{q}, \qquad (1.69)$$

here $\mathbf{q}$ being a tiny deviation from trivial equilibrium configuration $\mathbf{q}^0=\mathbf{0}$ nd

$$\mathbf{H}=\begin{bmatrix}\lambda-2P & P\\ P & \lambda-2P\end{bmatrix}. \qquad (1.70)$$

We can also write directly the loss of stability condition in its *variational form* $\delta(\Delta\Pi)=0$ and obtain

$$\delta(\Delta\Pi)=\frac{1}{2}\delta\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{q}+\frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{H}\delta\mathbf{q}=\delta\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{q}=0,\forall\delta\mathbf{q}\implies \qquad (1.71)$$

$$\implies \mathbf{H}\mathbf{q}=\mathbf{0}\text{, which is linear Eigen-value problem.} \qquad (1.72)$$

Note that the coefficient matrix of the associated Eigen-value problem (Equation 1.66) is the same[60] than our Hessian matrix So loss of stability occurs when

$$\Pi''=0\sim\det\{\mathbf{H}\}=0 \qquad (1.73)$$

Requiring the neutral equilibrium condition $\delta(\Delta\Pi)=0$ (for loss of stability)

Self-reading

Post-buckling analysis:

Post-buckling analysis: *What is the nature of the bifurcated branch just in the near neighbourhood of the bifurcation point $P_{1,E} = k\ell/3$?* For that, we do an asymptotic analysis and take up-to the fourth-order in the Taylor expansion of $\Delta\Pi$. In addition, since we are in the neighbourhood of the buckling load, the ratio $v_1 = -v_2$ as given by the corresponding buckling mode, remains unchanged if we limit ourselves to very small additional deflections v_1 and v_2 from the neutral configuration. (so ratios $v_1/\ell \ll 1$ and $v_2/\ell \ll 1$). Consequently,

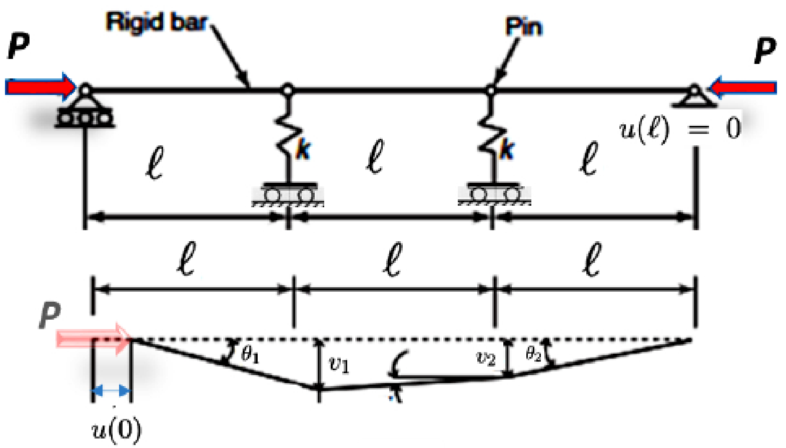
$$\Delta\Pi(v_1, v_2) = \frac{1}{2}k(v_1^2 + v_2^2) - P\ell\,[\,\frac{1}{2}\left(\frac{v_1}{\ell}\right)^2 + \frac{1}{8}\left(\frac{v_1}{\ell}\right)^4 + \\ +\frac{1}{2}\left(\frac{v_2 - v_1}{\ell}\right)^2 + \frac{1}{8}\left(\frac{v_2 - v_1}{\ell}\right)^4 + \\ +\frac{1}{2}\left(\frac{v_2}{\ell}\right)^2 + \frac{1}{8}\left(\frac{v_2}{\ell}\right)^4\,].$$

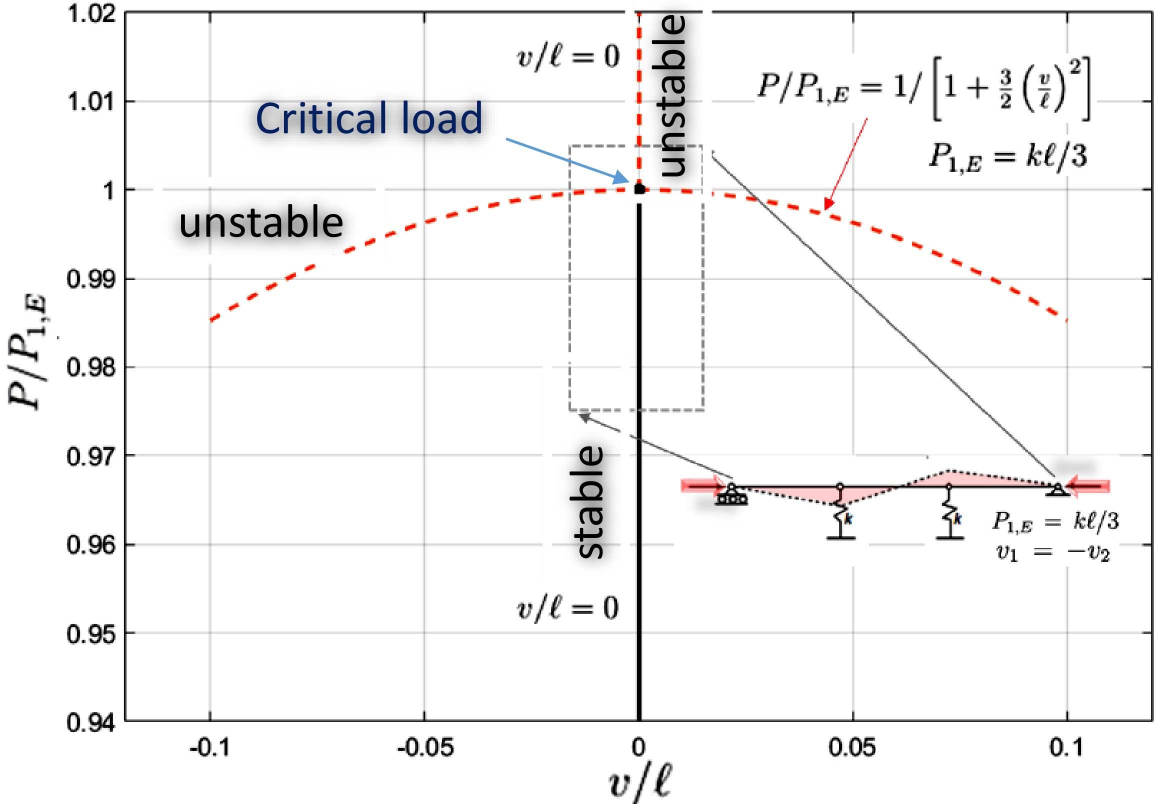
Inserting the relation $v \equiv v_1 = -v_2$, one finally obtains

$$\Delta\Pi(v) = k\ell^2\left(\frac{v}{\ell}\right)^2 - 3P\ell\left(\frac{v}{\ell}\right)^2 - \frac{9}{4}P\ell\left(\frac{v}{\ell}\right)^4$$

$$\boxed{\delta[\Delta\Pi(v)] = 0 \implies [\Delta\Pi]' = 0}$$

$$\boxed{\implies k\ell\left(\frac{v}{\ell}\right)\left[1 - \frac{P}{P_{1,E}}\left(1 + \frac{3}{2}\left(\frac{v}{\ell}\right)^2\right)\right] = 0}$$





Equilibrium path (asymptotic post-buckling analysis)

NEW Material starts from here …

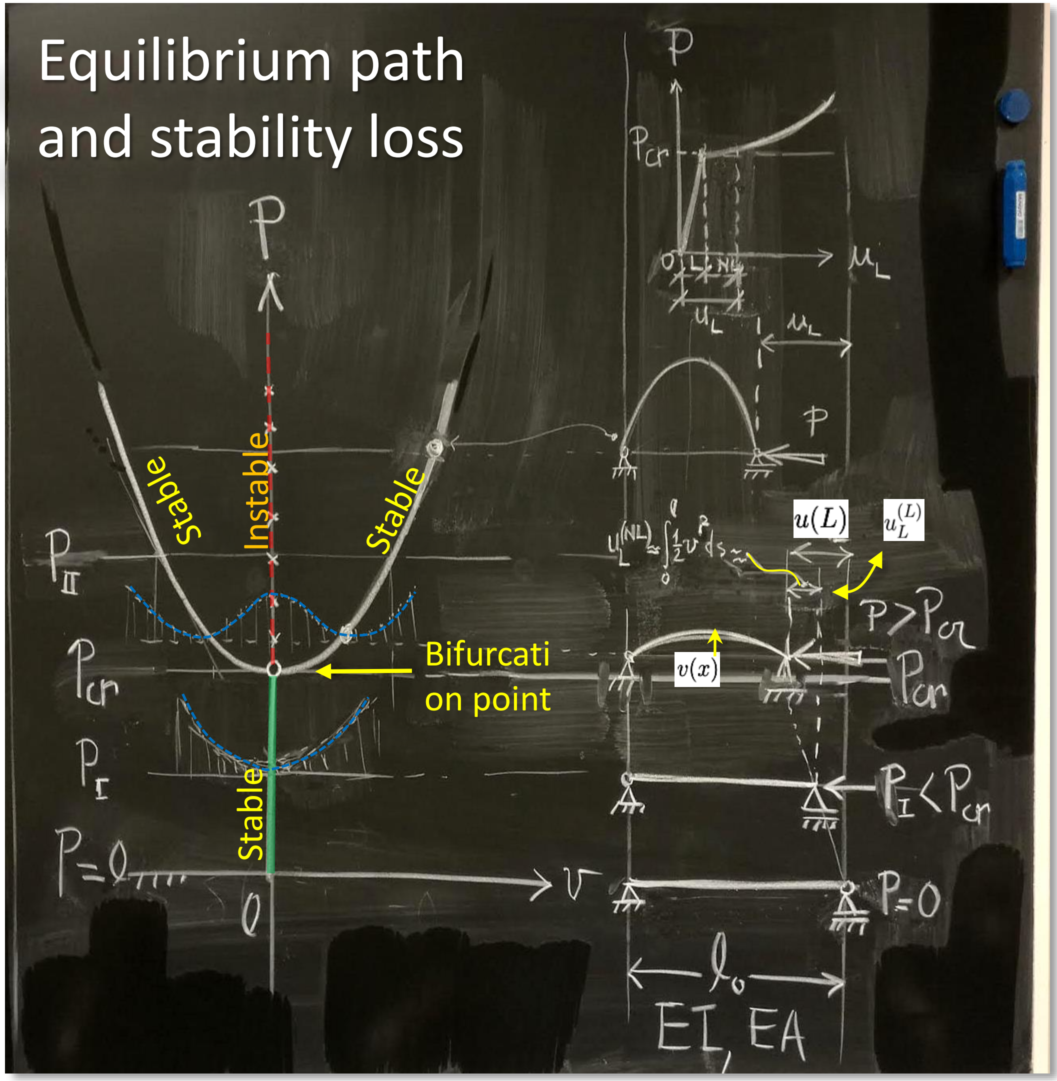
Stability (loss) energy criterion

$$\delta(\Delta\Pi[v]) = 0, \forall \delta u$$

$$\Delta\Pi[v] = \frac{1}{2}\int_0^\ell EIv''^2 \mathrm{d}x - P\int_0^\ell \frac{1}{2}v'^2 \mathrm{d}x$$

Euler-Lagrange equations stability of a column

$$(EIv'')'' + Pv'' = 0 \quad \& \quad 4 \quad \text{BCs.}$$



Skriiva liitutaululla …

Energy criteria for determination of instability of elastic structures

Change of total potential energy **between which** two **states**?

?

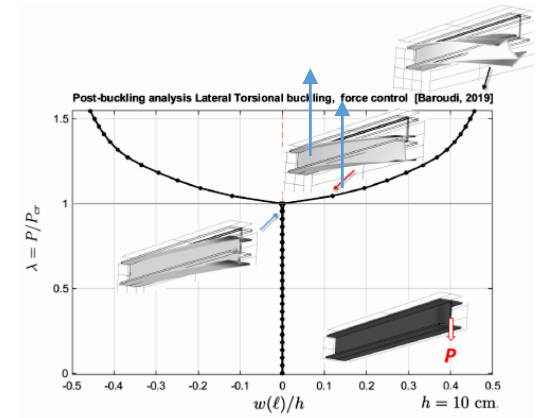
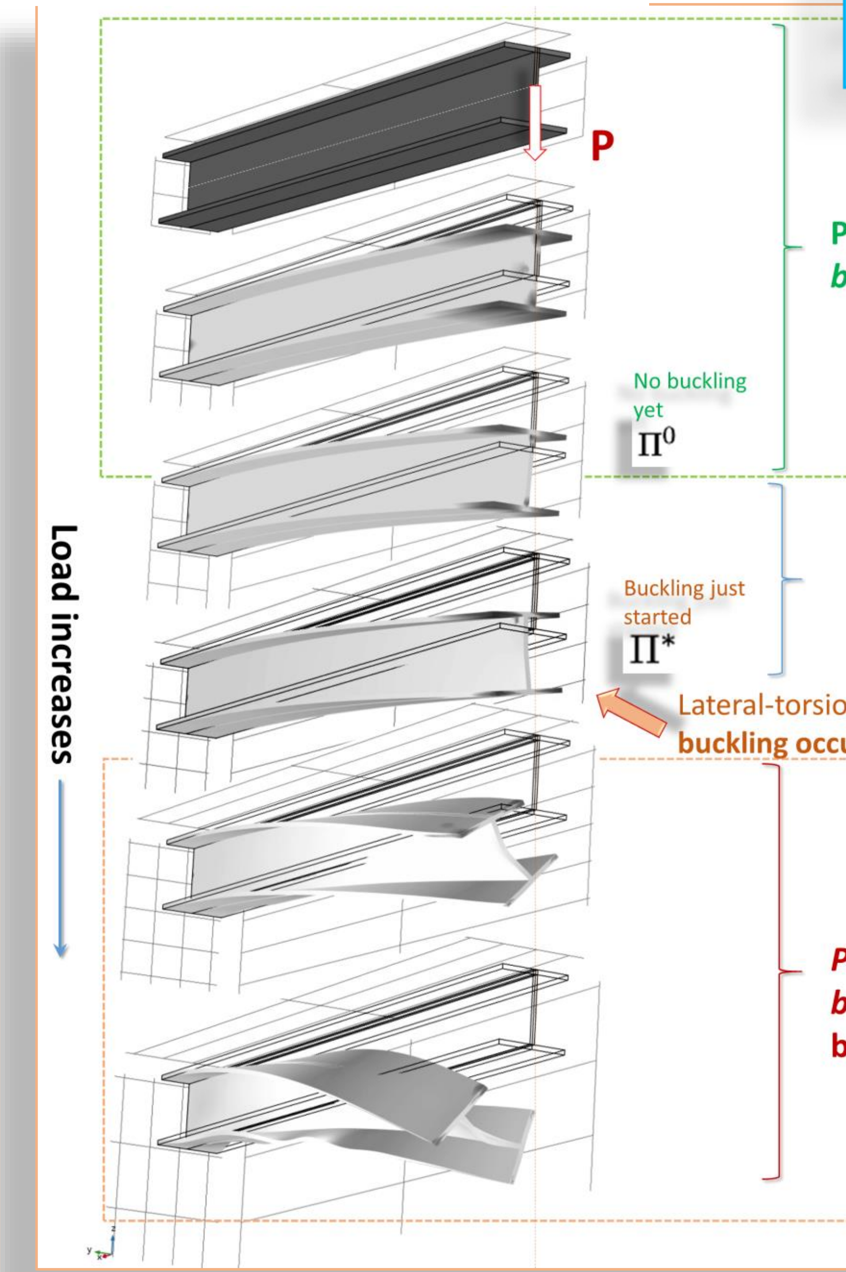


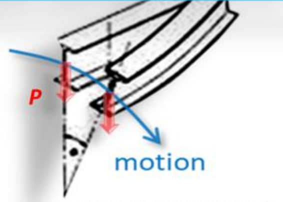
Figure 3.122: Equilibrium paths. FE-post-buckling analysis of an aluminium I-beam cantilever. The transversal tip-load is at the centroid.

N.B. The perturbed configuration [.]* can be thought achieved keeping the load constant and for instance, giving a tiny kinematical (virtual) perturbation to a an adjacent equilibrium configuration v*



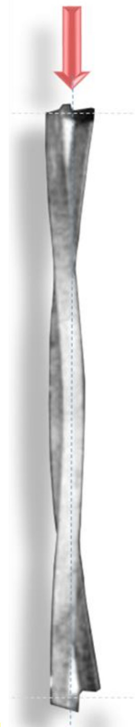
Primary *pre-buckling* state

***Post-buckling* behavior**



Lateral torsional buckling

Torsional buckling



$$\Delta\Pi = \Pi^* - \Pi^0$$

$$\Delta\Pi = \Pi(u^0 + \delta u) - \Pi(u^0) = \underbrace{\delta\Pi|_{u_0}}_{=0} + \frac{1}{2}\delta^2\Pi|_{u_0} + \dots$$

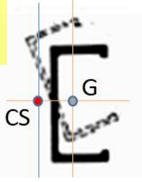
$$\delta(\Delta\Pi) = 0 \implies \delta\left(\frac{1}{2}\delta^2\Pi|_{u_0} + \dots\right) = 0$$

General *stability loss* criterion

Keeping up to quadratic terms

$$\implies \delta(\delta^2\Pi) = 0$$

Trefftz *stability loss* criterion



$$\Downarrow$$

This criticality condition for bifurcation provides the Buckling Equations

Example of use of stability criteria in the form $\delta(\Delta\Pi) = 0$

Stability (loss) energy criterion

$$\Delta\Pi[v] = \frac{1}{2}\int_0^\ell EIv''^2\mathrm{d}x - P\int_0^\ell \frac{1}{2}v'^2\mathrm{d}x =$$

$$\delta(\Delta\Pi[v]) = 0, \forall\delta u \implies \delta\left(\tfrac{1}{2}\int_0^\ell EIv''^2\mathrm{d}x - P\int_0^\ell \tfrac{1}{2}v'^2\mathrm{d}x\right) = 0, \forall\delta u$$

$$= \int_0^\ell EIv''\delta v''\mathrm{d}x - P\int_0^\ell v'\delta v'\mathrm{d}x = 0$$

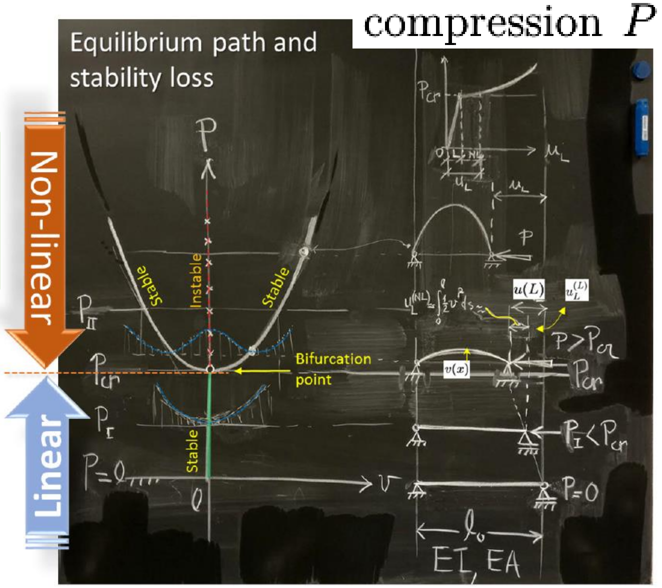
$$\Downarrow$$

Euler-Lagrange equations stability of a column

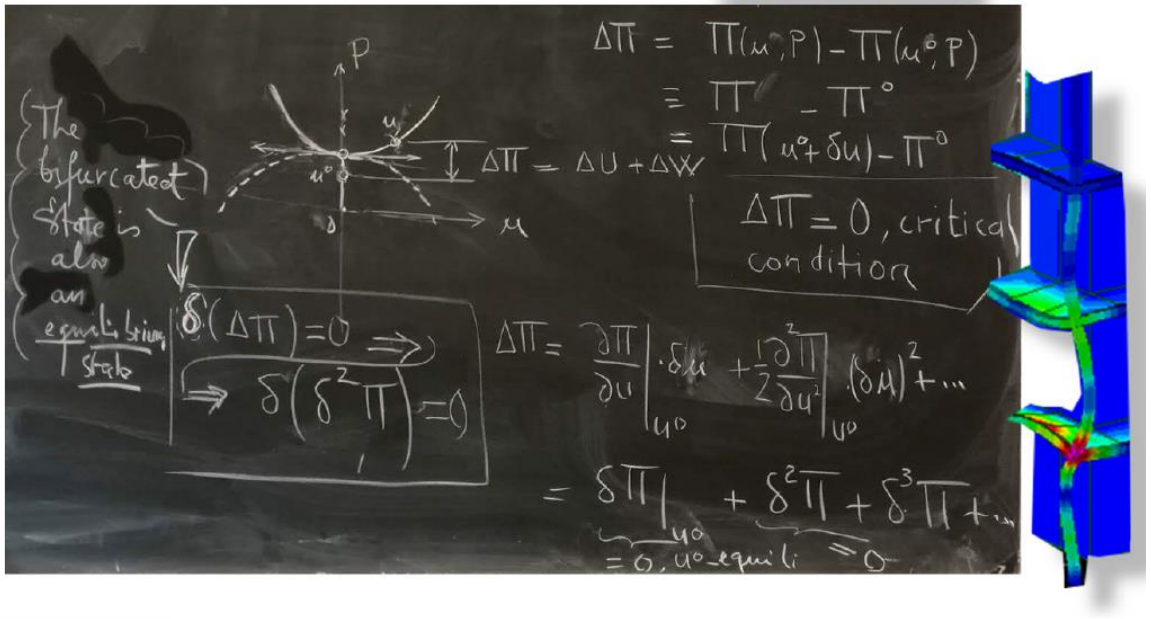
$$\boxed{(EIv'')'' + Pv'' = 0 \quad \& \quad 4 \quad \text{BCs.}}$$

The above homogeneous differential equation describes the stability problem and its solution provides us the critical buckling load together with the associated buckling-modes once the relevant four boundary conditions are specified.

compression $P > 0$

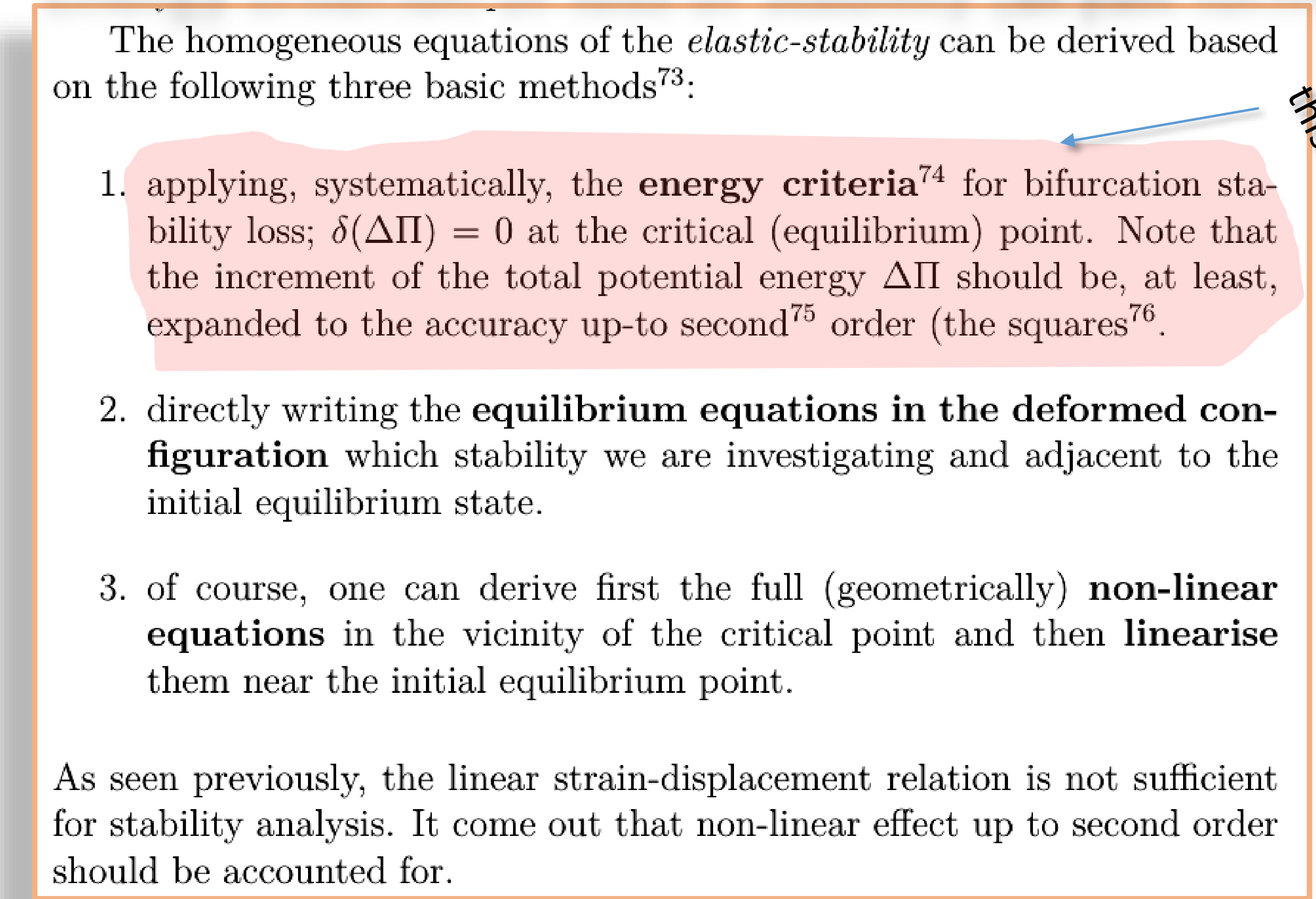


Critical condition for loss: $\delta(\Delta\pi) = 0$



Stability of an equilibrium.

Energy criterion of loss of stability (**Bryan** form)



The homogeneous equations of the *elastic-stability* can be derived based on the following three basic methods[73]:

1. applying, systematically, the **energy criteria**[74] for bifurcation stability loss; $\delta(\Delta\Pi) = 0$ at the critical (equilibrium) point. Note that the increment of the total potential energy $\Delta\Pi$ should be, at least, expanded to the accuracy up-to second[75] order (the squares[76].
2. directly writing the **equilibrium equations in the deformed configuration** which stability we are investigating and adjacent to the initial equilibrium state.
3. of course, one can derive first the full (geometrically) **non-linear equations** in the vicinity of the critical point and then **linearise** them near the initial equilibrium point.

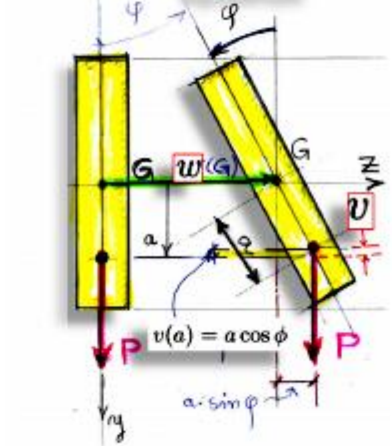
As seen previously, the linear strain-displacement relation is not sufficient for stability analysis. It come out that non-linear effect up to second order should be accounted for.

We use systematically this condition

$$\boxed{\Delta\Pi = \frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V + \int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V.}$$

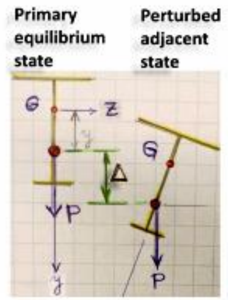
+ should also include increment of work of external work not already accounted in by the work of initial stresses

critical equilibrium → infinitesimal perturbation toward adjacent equilibrium

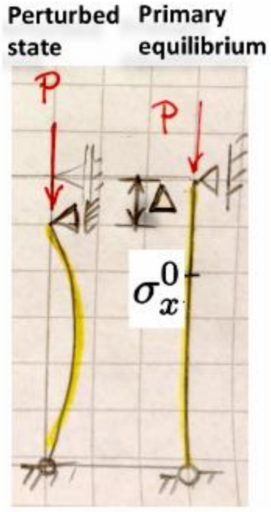


$\hat{\mathbf{u}} \equiv \delta\mathbf{u} \equiv \underbrace{\hat{\mathbf{u}} = (0, v, w)}_{} \equiv \mathbf{u}$

infinitesimal displacement increment between critical point and disturbed state



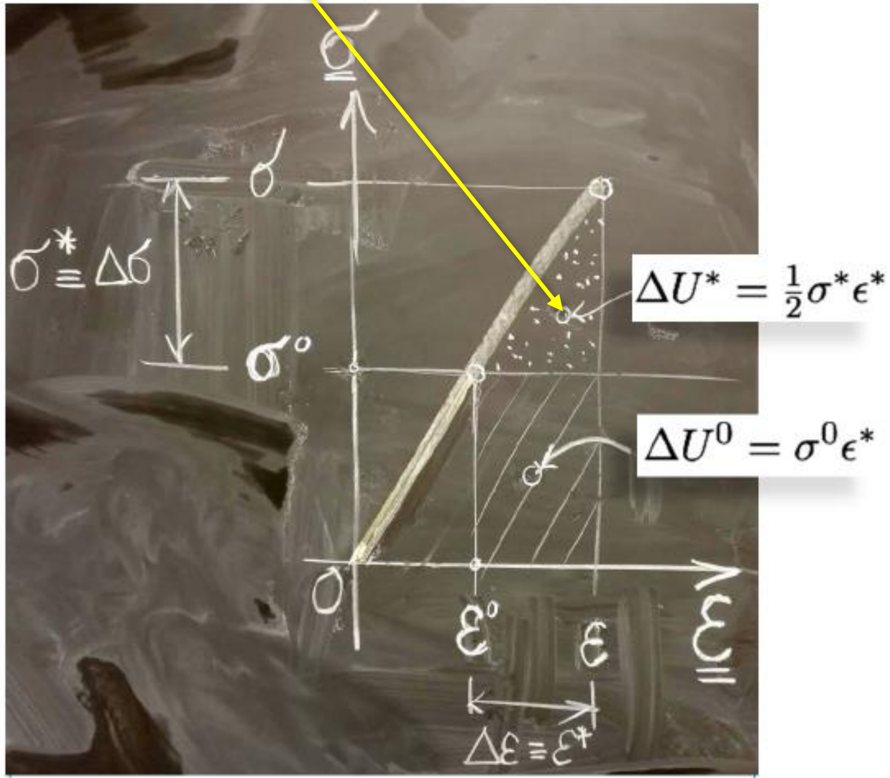
Additional work $\Delta W_{\mathrm{ext}} = P \cdot \Delta$ (*Lateral torsional buckling*)



Additional work $\Delta W_{\mathrm{ext}} = P \cdot \Delta$ (*Flexural buckling*)

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\text{linear part of strain increments in } \Delta U} + \underbrace{\int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part of strain increments in } \Delta W(\sigma^0)}$$

- Additional work of external force not included in the pre-stress



$\Delta U^* = \frac{1}{2}\sigma^*\epsilon^*$

$\Delta U^0 = \sigma^0\epsilon^*$

$$\Delta U = \Delta U^0 + \Delta U^* = \sigma^0\epsilon^* + \tfrac{1}{2}\sigma^*\epsilon^*$$

The strain energy change between reference equilibrium state $\mathbf{u}^0$ and a perturbed neighbouring (equilibrium) state $\mathbf{u}$. The change in strains being $\epsilon^* = \Delta\epsilon = \epsilon - \epsilon^0$ and in stresses $\sigma^* = \Delta\sigma = \sigma - \sigma^0$

Example: Buckling of a column

end-thrust $-P = N^0(x) < 0$

The total potential energy increment in Bryan form was

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2 \mathrm{d}x + \frac{1}{2}\int_0^\ell N^0(x)(v')^2 \mathrm{d}x,$$

$\epsilon_1 = -yv''(x)$

Linear part of the strain

$\sigma_x^0 A = N^0(x)$

Initial stress

$\epsilon_2 = \frac{1}{2}(v')^2$

Quadratic part of the strain

Finite deformation (strains)

$$\epsilon^*_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$$

After order of magnitude analysis for the strain increment, and keeping only up-to second order terms (the non-linear (quadratic) part can expressed in terms of rotations) one finally obtains

$$\begin{aligned}
\epsilon_x &= e_x + \frac{1}{2}\left(\omega_z^2 + \omega_y^2\right) \\
\epsilon_y &= e_y + \frac{1}{2}\left(\omega_x^2 + \omega_z^2\right) \\
\epsilon_z &= e_z + \frac{1}{2}\left(\omega_y^2 + \omega_x^2\right) \\
\gamma_{xy} &= 2e_{xy} - \omega_x\omega_y \\
\gamma_{yz} &= 2e_{yz} - \omega_y\omega_z, \\
\gamma_{zx} &= 2e_{zx} - \omega_z\omega_x.
\end{aligned}$$

The rotation component

$$\begin{aligned}
\omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \\
\omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \\
\omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),
\end{aligned}$$

quadratic part

the linear part

$$e_x = \frac{\partial u}{\partial x}, \qquad e_y = \frac{\partial v}{\partial y}, \qquad e_z = \frac{\partial v}{\partial z},$$

$$\begin{aligned}
e_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \\
e_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \\
e_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}.
\end{aligned}$$

What deformations are significant in buckling?

- In stability analysis while deriving the linear stability loss equations (the linear Eigen-value problem) the amplitude of the linear part e_i of the strains, during the infinitesimal perturbation of the initial equilibrium to the (bifurcated) adjacent one, remains small[a] as compared to changes in the rotation components of ω_i.
- Consequently, the quadratic terms in terms in strains e_i^2 and $\omega_i e_j$ are of second order increments as compared to changes in the rotation components, and for that reason will be dropped (ignored). In the above strain increments expressions, only terms shown in the above strains are retained for stability analysis.
- In addition to that, (*Cf.* Alfutov), terms containing the derivatives of initial primary displacements can be neglected (this, *their contribution to the increment of total potential energy* $\Delta\Pi$ *can be neglected*) too.

---

[a] As a consequence of the choice of the initial primary equilibrium and the close neighbouring adjacent (bifurcated) equilibrium. These two states are infinitesimally close.

Ex. Plate: **quadratic** part of strains

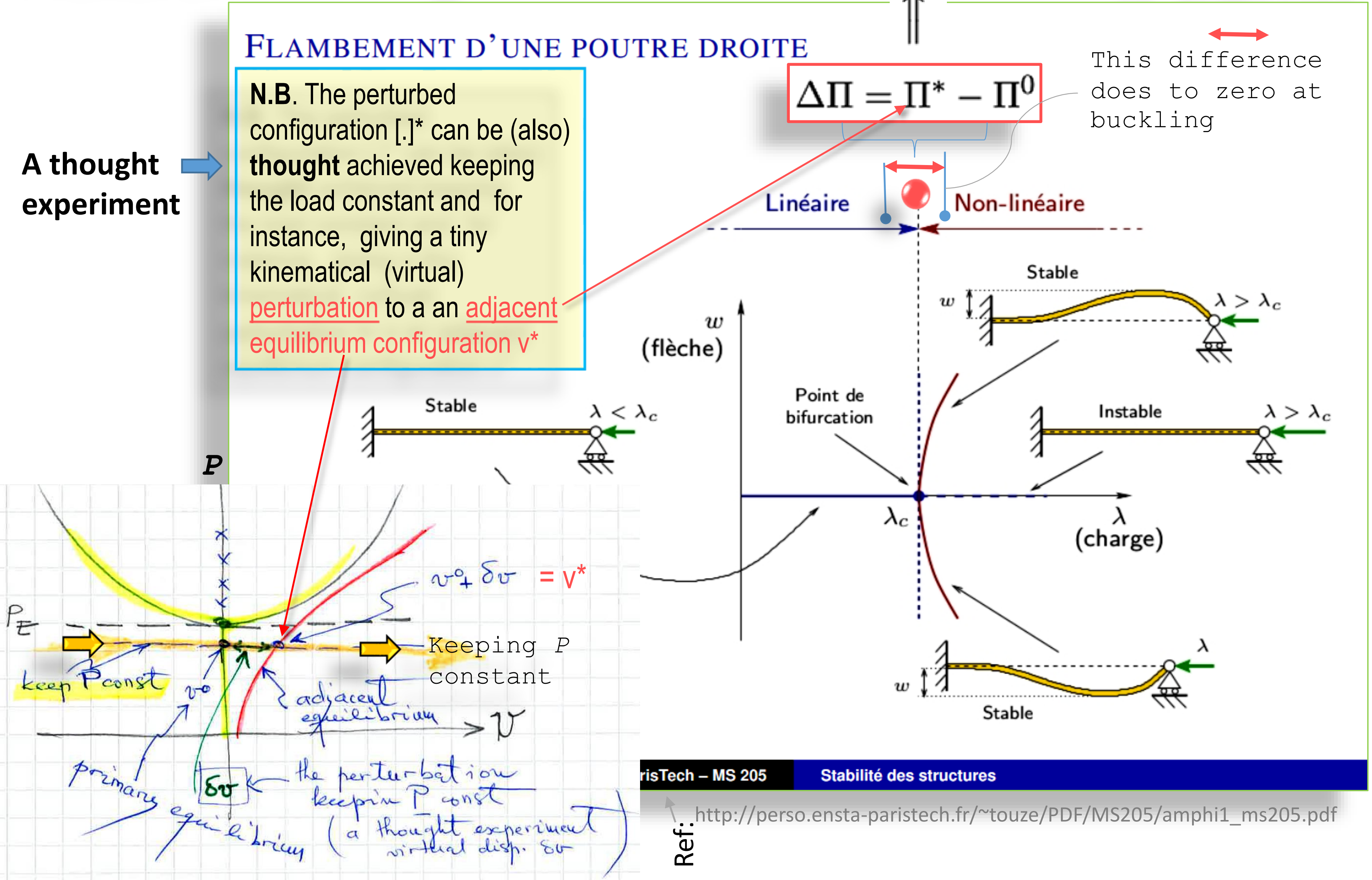
$$\begin{aligned}
\epsilon^*_{xx} &= \frac{1}{2}[\underbrace{u_{,x}^2 + v_{,x}^2}_{\approx 0 \ll w_{,x}^2} + w_{,x}^2] \approx \frac{1}{2}w_{,x}^2, \\
\epsilon^*_{yy} &= \frac{1}{2}[\underbrace{u_{,y}^2 + v_{,y}^2}_{\approx 0} + w_{,y}^2] \approx \frac{1}{2}w_{,y}^2, \\
\gamma^*_{xy} &= 2\epsilon^*_{xy} = \underbrace{u_{,x}u_{,y} + v_{,x}v_{,y}}_{\approx 0} + w_{,x}w_{,y} \approx w_{,x}w_{,y}.
\end{aligned}$$

$$\Delta\Pi = \underbrace{\frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V}_{\text{linear part of strain increments in } \Delta U} + \underbrace{\int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V}_{\text{quadratic part of strain increments in } \Delta W(\sigma^0)}$$

Flexural buckling

$$\delta(\Delta\Pi) = 0 \implies$$ Equations (of loss) of stability

A thought experiment



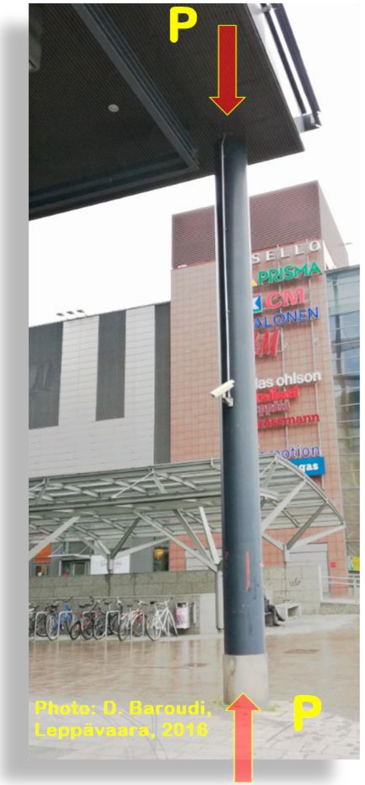
FLAMBEMENT D'UNE POUTRE DROITE

N.B. The perturbed configuration [.]* can be (also) **thought** achieved keeping the load constant and for instance, giving a tiny kinematical (virtual) perturbation to a an adjacent equilibrium configuration v*

$$\Delta\Pi = \Pi^* - \Pi^0$$

This difference does to zero at buckling

Ref: http://perso.ensta-paristech.fr/~touze/PDF/MS205/amphi1_ms205.pdf



Estimate the critical load!

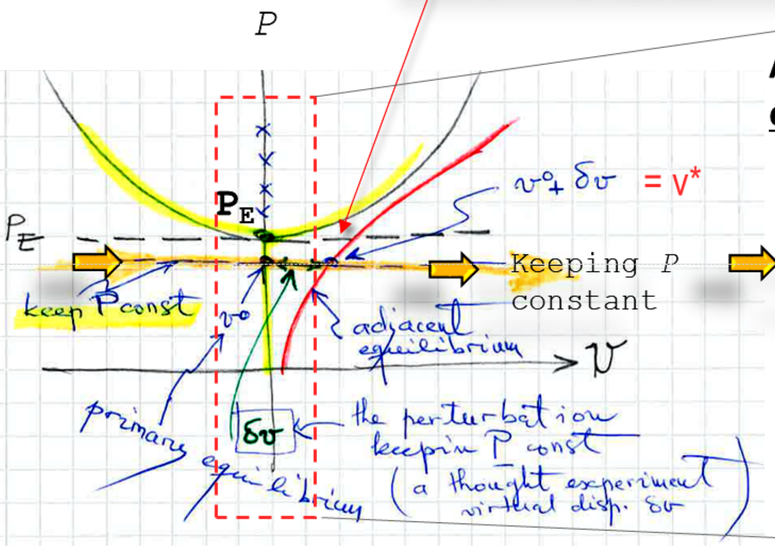
1) One way to think how form the increment of total potential energy is through a real loading sequence where the load increases quasi-statically and monotonically from zero to the buckling load $P_E^+ = P_E + \varepsilon$ where it buckles where ε being infinitesimally small > 0. The primary non-buckled configuration (primary equilibrium) corresponds to $P_E^- = P_E - \varepsilon$. Now one can form the increment of the total potential energy between these two real states and takes the limit when $\varepsilon \to 0$ to say that we are at the bifurcation or limit-point where now the critical load being P_E.

$$\boxed{\Delta\Pi = \Pi^* - \Pi^0} \Longrightarrow \delta(\Delta\Pi) = 0 \Longrightarrow$$

Equations (of loss) of stability

N.B. The perturbed configuration [.]* can be (also) **thought** achieved keeping the load constant and for instance, giving the primary equilibrium configuration v_0 a tiny kinematical (virtual) perturbation to a an adjacent equilibrium configuration v*

$$\boxed{\Delta\Pi = \Pi^* - \Pi^0}$$



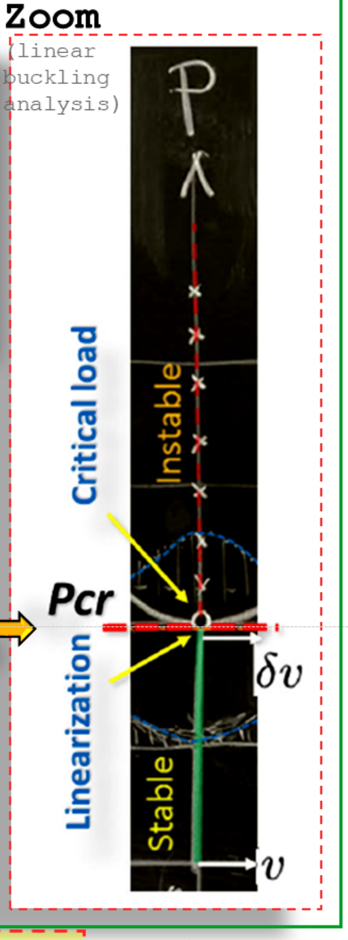
A thought experiment

Keeping P constant

Keeping P constant

Zoom

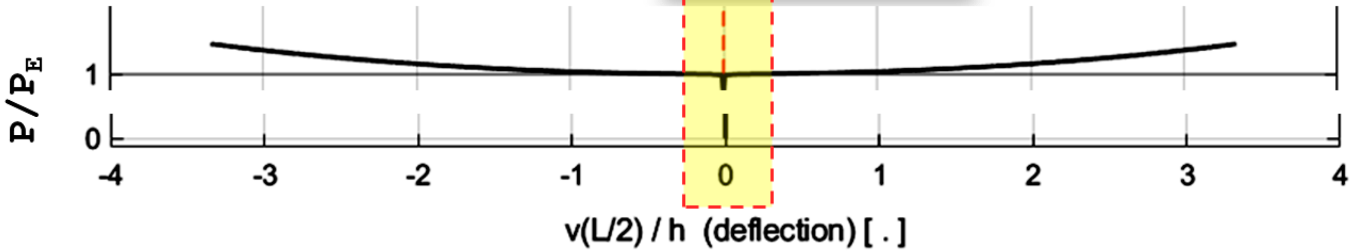
(linear buckling analysis)



2) the other more classical way how form the increment of total potential energy is by a *thought experiment where* we give an infinitesimal virtual perturbation to the primary equilibrium configuration to an adjacent neighbor equilibrium configuration while keeping all the loads unchanged. Then we write the increment od total potential energy between these to states of equilibrium.

A Finite element post-buckling analysis of a simply supported column under axial thrust.

This shows how 'sallow' is the critical point infinitesimal neighborhood



Buckling of a beam-column

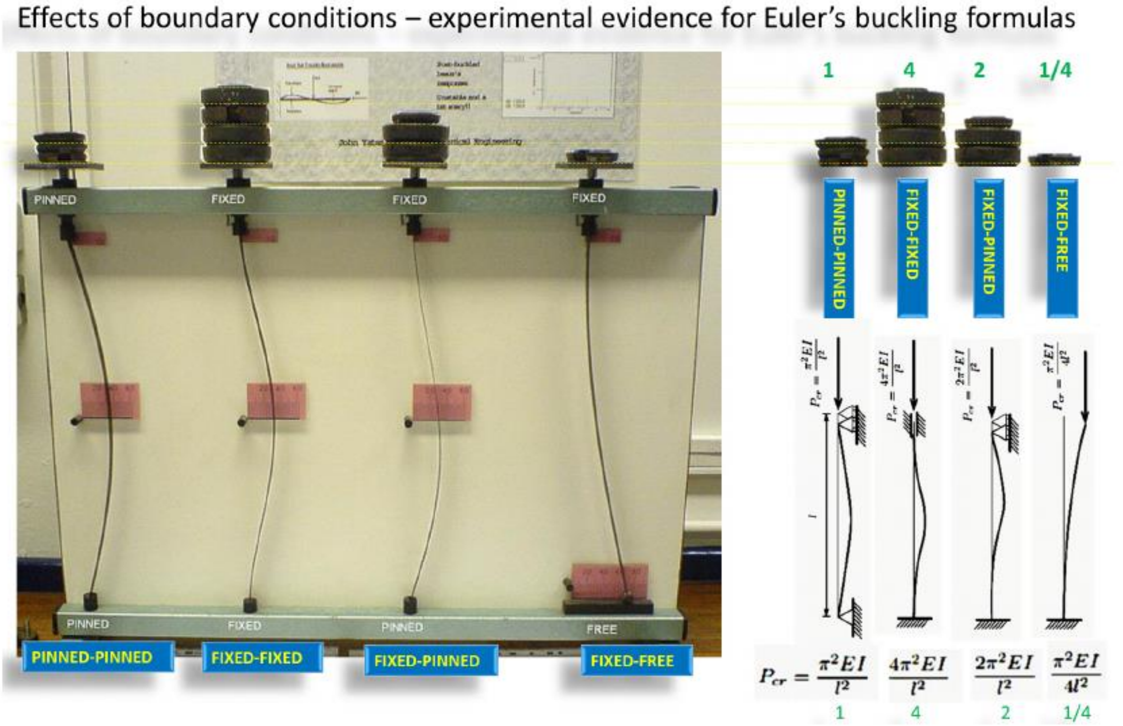
Solutions for some classical cases

$$P_{cr} = \mu\pi^2 \frac{EI}{\ell^2} \equiv P_E$$

$$\sigma_{cr} \equiv \sigma_E = \frac{P_E}{A} = \mu\pi^2 \frac{EI}{A\ell^2} = \mu\pi^2 E \left(\frac{r_{min}}{\ell}\right)^2 = \mu\pi^2 E/\lambda_{min}^2,$$

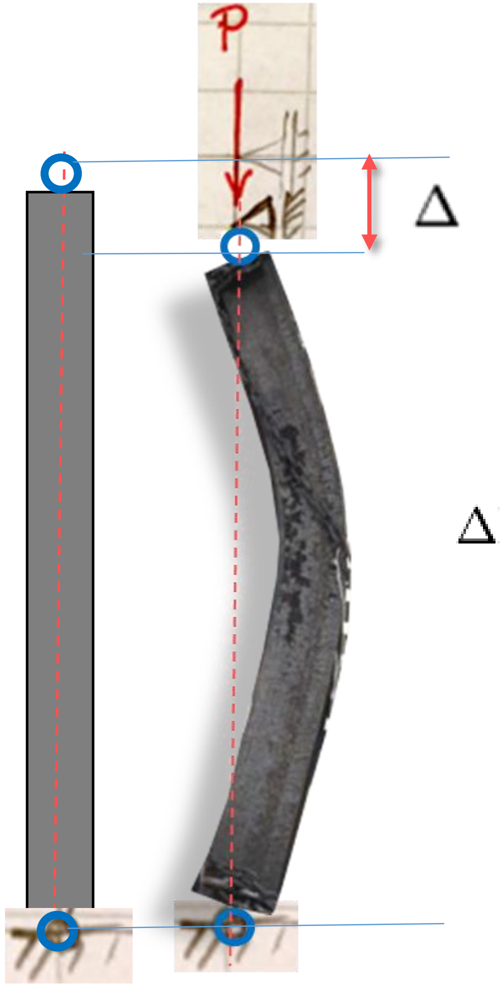
Critical strain

$$\epsilon_{cr}^0 \equiv \epsilon_E = \frac{\sigma_E}{E} = \mu\pi^2 \left(\frac{r_{min}}{\ell}\right)^2 = \mu\pi^2/\lambda_{min}^2$$



Rudimentary experimental evidence for Euler's basic buckling formulas and the effect of boundary conditions on the buckling load.

Buckling of a beam-column



Buckling just started

Π^*

$$\boxed{\Delta\Pi = \Pi^* - \Pi^0}$$

No buckling

$$\Downarrow$$

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2\mathrm{d}x - \frac{1}{2}P\int_0^\ell (v')^2\mathrm{d}x.$$

$$\delta(\Delta\Pi) = 0$$

$$\Downarrow$$

This criticality condition for bifurcation provides the Buckling Equations

$$(EIv'')'' + Pv'' = 0$$

& four boundary conditions.

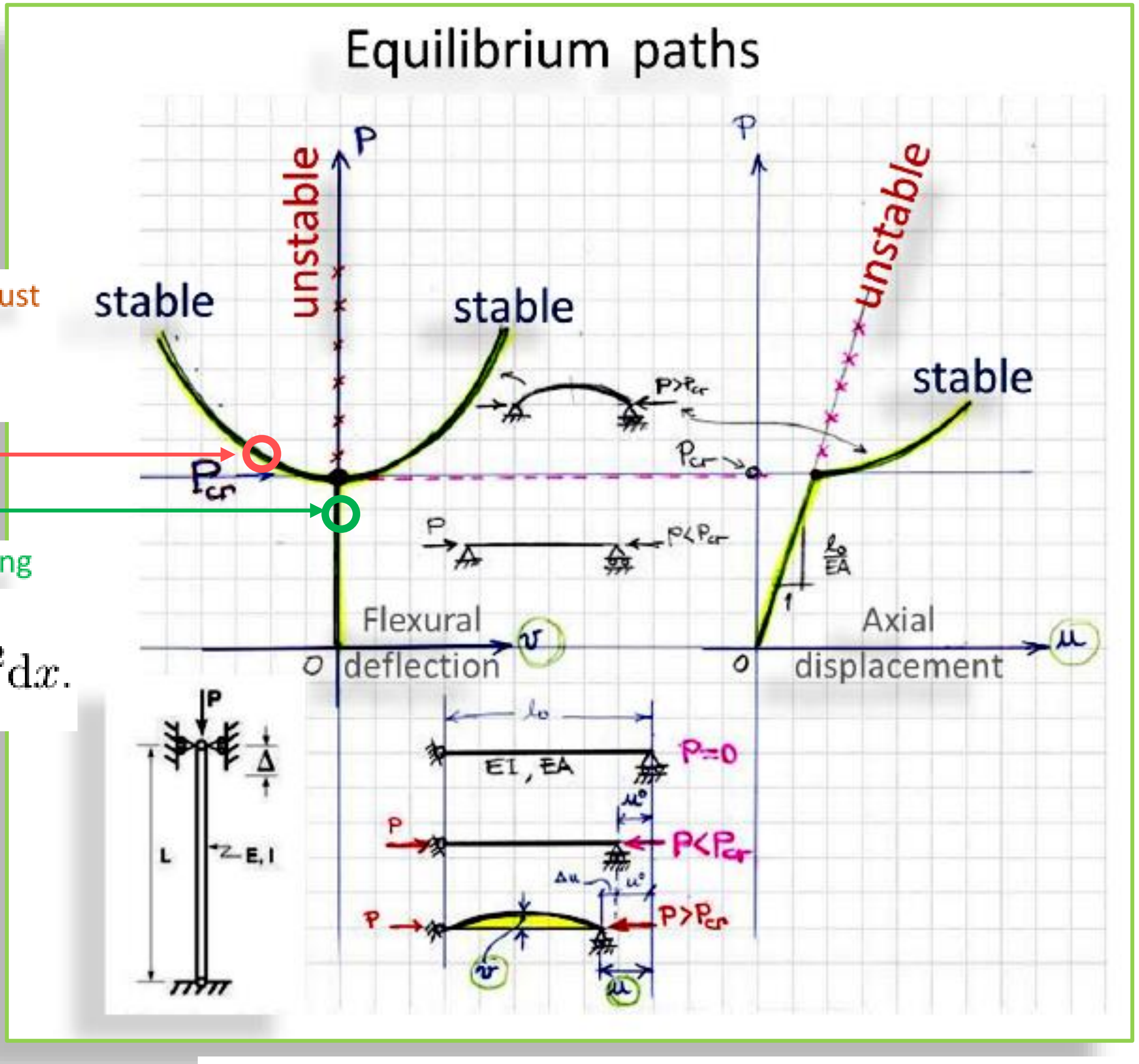


Illustration for equilibrium paths and bifurcation points for perfect structure (no imperfections).

Combined compression and bending

Linearised theory of buckling

The transition from the straight stretched beam-column equilibrium initial configuration to the neighbour adjacent buckled (flexural) equilibrium state occurs with no additional stretching for very small bifurcational deflection v. Therefore, it is assumed that the changes in length are of higher order. Consequently, the axial force does not changes $N \approx N_0$ from the axial force obtained in the straight state of equilibrium.

In the linearisation, we keep, in the Taylor's series, the first terms and higher terms are ignored. All the external loads are assumed constant in amplitude and direction.

All the external loads are assumed constant in amplitude and direction.

Linearisation:

$$\theta = v', \sin(\theta) \approx \theta, \sin(\theta + \mathrm{d}\theta) \approx \theta + \mathrm{d}\theta, \cos(\theta) \approx 1, \cos(\theta + \mathrm{d}\theta) \approx 1$$

Application examples of stability study using energy principles

Buckling of a beam-column

The total potential energy increment in Bryan form was

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2\mathrm{d}x + \frac{1}{2}\int_0^\ell N^0(x)(v')^2\mathrm{d}x,}$$

$\epsilon_1 = -yv''(x)$

Linear part of the strain

$\sigma_x^0 A = N^0(x)$

Initial stress

$\epsilon_2 = \frac{1}{2}(v')^2$

Quadratic part of the strain

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2\mathrm{d}x - \frac{1}{2}P\int_0^\ell (v')^2\mathrm{d}x.$$

end-thrust $-P = N^0(x) < 0$

Stability loss criteria

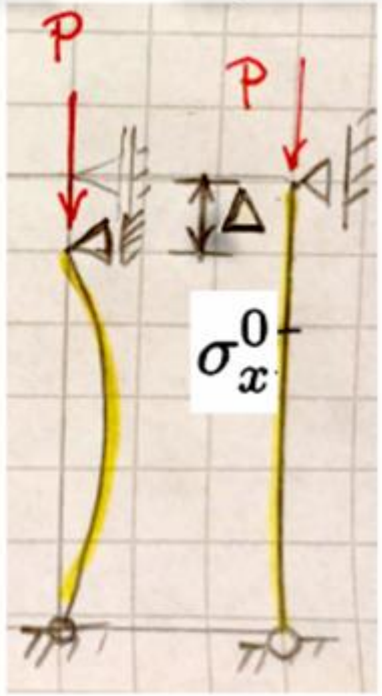
Taking the variation $\delta(\Delta\Pi) = 0$ $\Longrightarrow \int_0^\ell EIv''\delta v'' - P\int_0^\ell v'\delta v'\mathrm{d}x = 0,\ \forall\delta v$

which gives after twice integration by parts

$$\Longrightarrow \int_0^\ell \underbrace{[EIv^{(4)} + Pv'']}_{=0}\delta v\mathrm{d}x + [\underbrace{EIv''}_{-M}\delta v']_0^\ell - [(\underbrace{EIv'''}_{-Q} + Pv')\delta v]_0^\ell = 0,\ \forall\delta v$$

Field equation BCs BCs

Perturbed state Primary equilibrium



Additional work $\Delta W_{\mathrm{ext}} = P\cdot\Delta$ (*Flexural buckling*)

The linearised buckling equation

$$\Longrightarrow \boxed{\begin{array}{l}(EIv'')'' + Pv'' = 0\\ \&\ \text{four boundary conditions.}\end{array}}$$

Buckling of a beam-column

General solution

Stability equations

$$(EIv'')'' + Pv'' = 0$$

& four boundary conditions.

general solution $v(x)$ for the buckling of such column-beam

$$\begin{cases} v(x) = A\sin(kx) + B\cos(kx) + Cx + D + v_0(x), & P > 0 \text{ compression} \\ v(x) = A\sinh(kx) + B\cosh(kx) + Cx + D + v_0(x), & P < 0 \text{ tension} \end{cases}$$

where $k^2 = P/EI$

The few following slides are a recall form *Beams and Frames course (2018)*
Related to how the *stability equations* are derived by considering equilibrium of a deformed differential beam element

Combined compression and bending

Linearised theory of buckling

Writing the equilibrium equations (both vertical and horizontal resultant vanish - `FBD and equilibrium as during our` 1^{st} `lecture` for a differential material element ds one obtains the *basic equation of stability theory* for a straight beam-column as

$$\boxed{(EIv'')'' - (Nv')' = q} \tag{38}$$

Accounting for the linearisation around the initial equilibrium, we have $N \approx N_0$ and in our case only external compressive load $P > 0$ at the tip

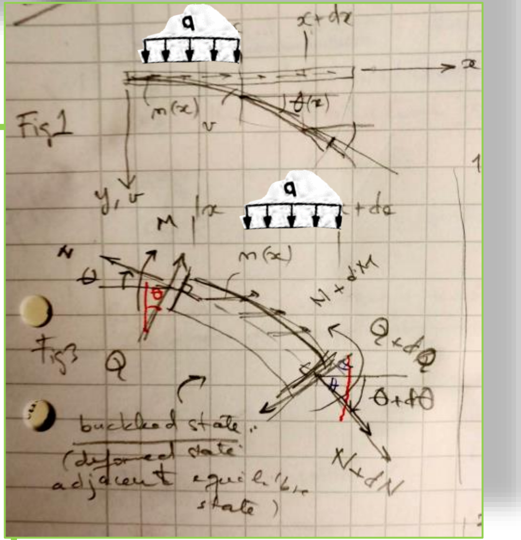
$$\boxed{(EIv'')'' - (N_0 v')' = q} \tag{39}$$

Assuming $N \approx N_0$ and for external compressive load $P > 0$, $N_0 = -P_0$ at one end of the column-beam is acting, and accounting for $M' = Q$ together with the constitutive relation $M = -EIv''$ we obtain

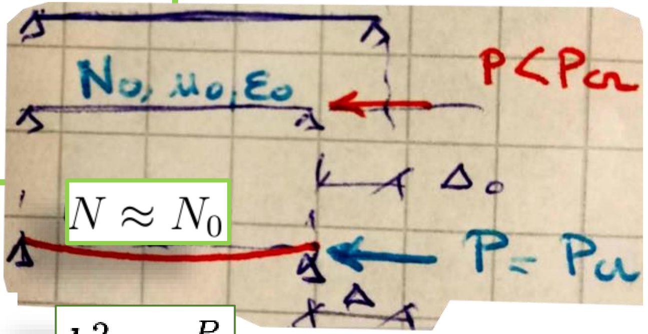
$$\boxed{(EIv'')'' + (Pv')' = q \quad \& \quad 4 \quad \text{Bcs}}$$

(compression $P > 0$) $\quad v(x) = A\sin(kx) + B\cos(kx) + Cx + D + \bar{v}(x)$

tension $P < 0$ $\quad v(x) = A\sinh(kx) + B\cosh(kx) + Cx + D + \bar{v}(x)$



$$\boxed{(EIv'')'' - (Nv')' = q}$$

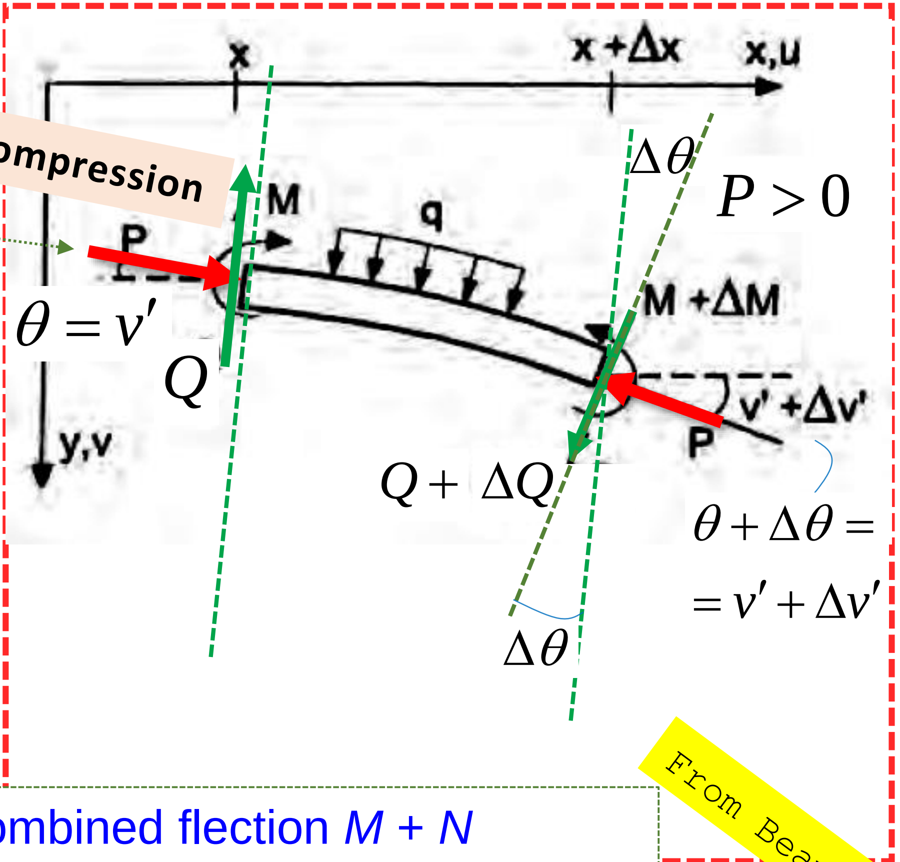


$\boxed{N \approx N_0}$

$\boxed{k^2 = \frac{P}{EI}}$

Combined compression and bending

To account for the second order effects, the idea is to write the **equilibrium equation in the deformed configuration** **/geometrical nonlinearity/** (account for the nonlinear part of the strain tensor)



Assumptions:

- **Large displacements**
- **Moderate rotations**
- **Linear elastic material (Hooke's law)**

Combined flection *M* + *N*
The superposition principle does not hold anymore

From Beams and Frames course

'Moderate' rotations

$$\tan\theta = v',\ |\theta| << 1 \Rightarrow \tan\theta \approx \theta,$$

$$\sin\theta \approx \theta,\ \cos\theta \approx 1$$

$$\Downarrow$$

$$\Delta Q\cos(\Delta\theta) \approx \Delta Q \qquad P\sin\theta \approx P\theta = Pv'$$

$$(Q + \Delta Q)\cos(\Delta\theta) \approx Q + \Delta Q$$

$$-Q + (Q + dQ) + Pv' - P(v' + dv') + q\,dx = 0$$

Equilibrium

0

Combined compression and bending

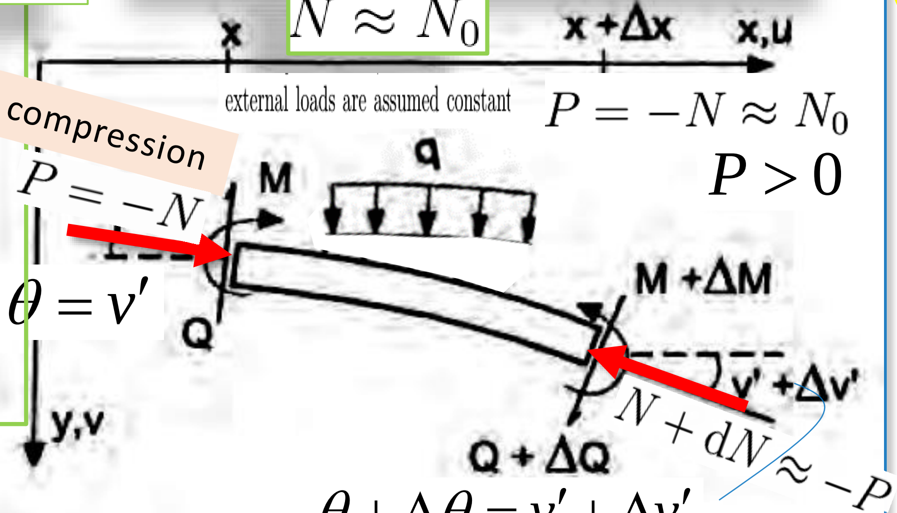
To account for the second order effects, the idea is to write the **equilibrium equation in the deformed configuration**

/geometrical nonlinearity/ (accounts for the nonlinear part of the strain tensor) and membrane forces $N \approx N_0$ from the undeformed

$$\epsilon^\circ = \frac{\mathrm{d}u}{\mathrm{d}x} + \boxed{\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2}$$

Linearised theory of buckling

$N \approx N_0$



external loads are assumed constant

$P = -N \approx N_0$

$P > 0$

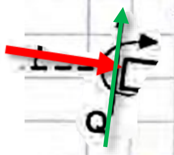
$P = -N$

$\theta = v'$

$N + \mathrm{d}N \approx -P$

$\theta + \Delta\theta = v' + \Delta v'$

From Beams and Frames course



$$-Q + Q + dQ + Pv' - P(v' + dv') + q\,dx = 0 ,$$

$$\Rightarrow \frac{dQ}{dx} - P\cdot\frac{dv'}{dx} + q = 0 . \Rightarrow \boxed{M'' - \boxed{Pv''} + q = 0}$$

$Q = M'$

$M = EI\kappa = -EIv''$

$\Rightarrow$ Saadaan vihdoin:

$$\boxed{\begin{cases}(EIv'')'' + \overset{\sim}{Pv''} = q \\ + \text{Reunaehdot}\end{cases}} \quad (1)$$

geometrinen epälineaarisuus tai nk. toisen asteen efektit

$\Rightarrow \boxed{P > 0}$

Tapauksessa $EI = $ vakio (1) $\Rightarrow$

$$\boxed{\begin{cases}v^{(4)} + k^2 v'' = \frac{q}{EI} , \quad \text{jossa} \quad k = \sqrt{P/EI} \\ 4\text{kpl. reunaehtoja}\end{cases}}$$

Linearisation:

$\theta = v', \sin(\theta) \approx \theta, \sin(\theta + \mathrm{d}\theta) \approx \theta + \mathrm{d}\theta,$
$\cos(\theta) \approx 1, \cos(\theta + \mathrm{d}\theta) \approx 1$

$$\boxed{\begin{cases}(EIv'')'' + \overset{\sim}{Pv''} = q \\ + \text{Reunaehdot}\end{cases}}$$

ovat 4.:nnen kertaluvun tavallisia diff yhtälöitä.

$(P > 0)$

ratkaisu on

$$\boxed{\begin{gathered}v(x) = \underbrace{A\sin(kx) + B\cos(kx) + Cx + D}_{\text{homogeenisen diff.yht. yleinen ratk.}} + \overset{\uparrow}{v_0(x)} \\ + 4\text{kpl reunaehtoja}\end{gathered}}$$

jokin yksityisratkaisu (q:sta riippuva)

pilareiden nurjahduskuorma tutkimalla (3) kun $q \equiv 0$ erityyppisellä reunaehdoilla.

The General solution (for compression case)

$$v(x) = A\sin(kx) + B\cos(kx) + Cx + D + \bar{v}(x)$$

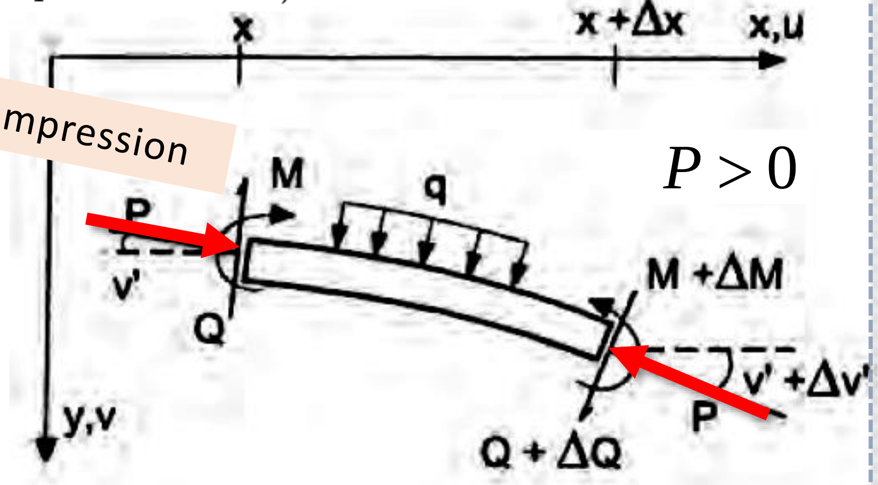
Combined compression/tension and bending

From Beams and Frames course

$$v(x) = A\sin(kx) + B\cos(kx) + Cx + D + \bar{v}(x)$$

$$v(x) = A\sinh(kx) + B\cosh(kx) + Cx + D + \bar{v}(x)$$

(compression $P > 0$)



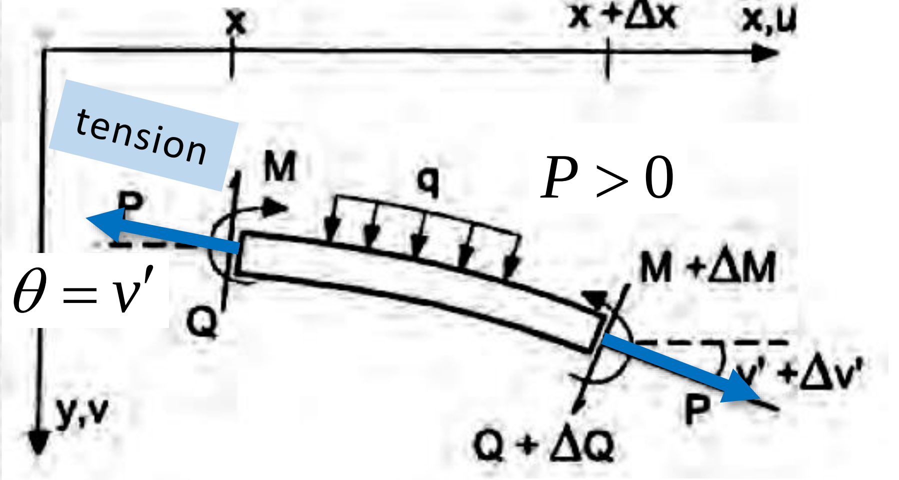
compression

$P > 0$

$\theta + \Delta\theta = v' + \Delta v'$

$\left\{ (EIv'')'' + \overbrace{Pv''}^{} = q \right.$
+ Reunaehdot

tension $P < 0$



tension

$P > 0$

$\theta = v'$

$\theta + \Delta\theta = v' + \Delta v'$

$\left\{ (EIv'')'' - \overbrace{Pv''}^{} = q \right.$
+ Reunaehdot

$$k^2 = \frac{P}{EI}$$

The General solution
(for compression case)

NB. The compression have a softening (of the $P > 0$ effective bending rigidity) effect on bending

$$v(x) = A\sin(kx) + B\cos(kx) + Cx + D + v_0(x)$$

The General solution
(for tension case)

NB. The tension have a stiffening effect on bending

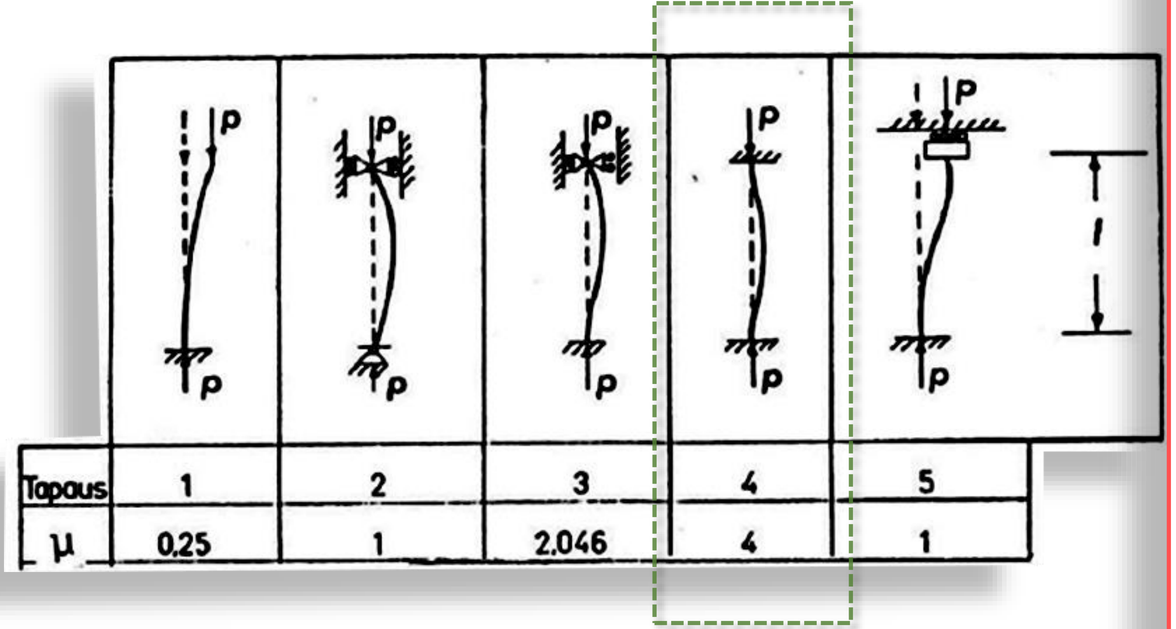
$$v(x) = A\sinh(kx) + B\cosh(kx) + Cx + D + v_0(x)$$

***N.B.* for** $P = 0 \longrightarrow \quad v(x) = A + Bx + Cx^2 + Dx^3 + v_0(x)$

Euler's basic buckling cases

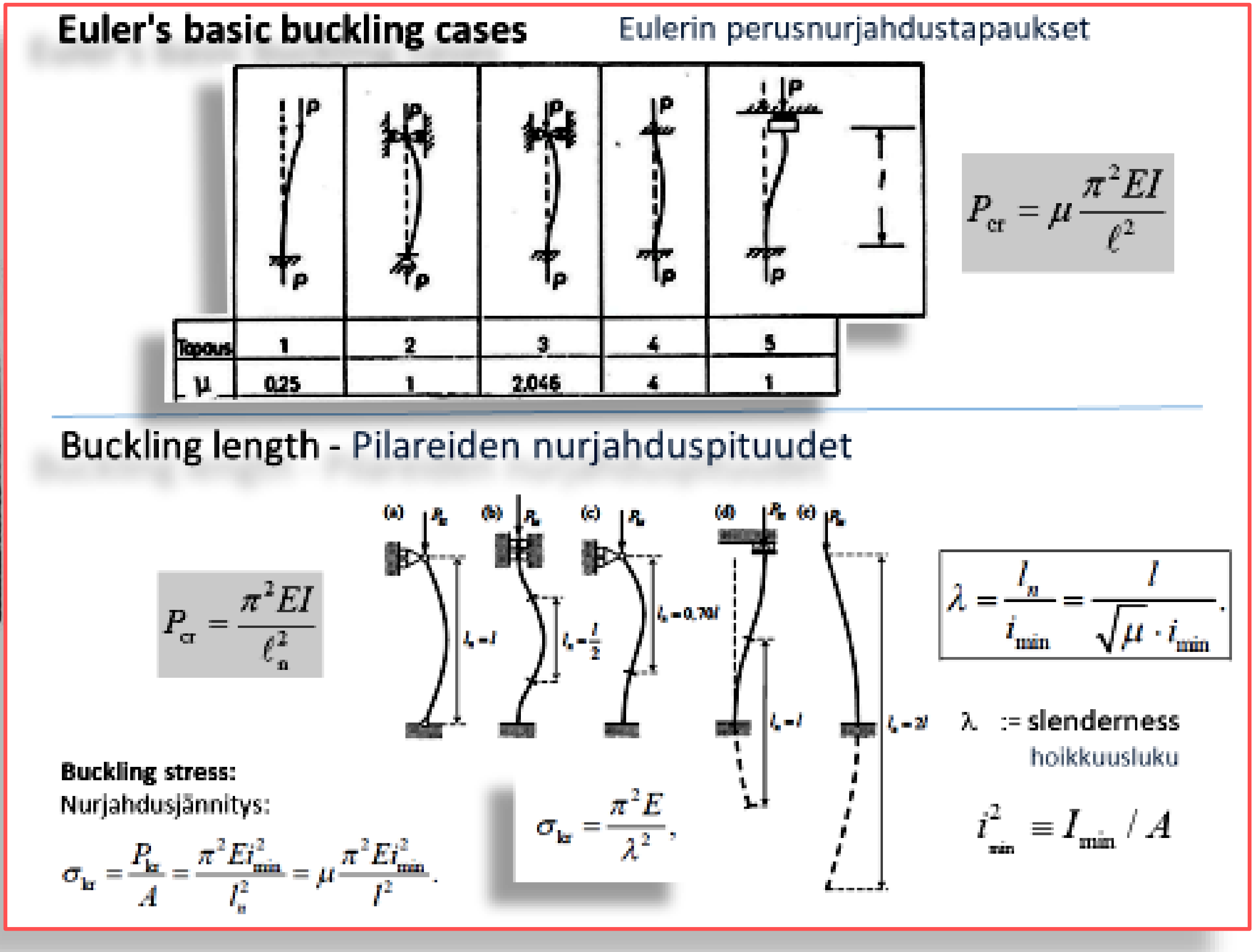
Eulerin perusnurjahdustapaukset

$$P_{\text{cr}} = \mu \frac{\pi^2 EI}{\ell^2}$$



Tapous	1	2	3	4	5
μ	0.25	1	2.046	4	1

$$P_{\text{cr}} = 4\frac{\pi^2 EI}{\ell^2}$$



Euler's basic buckling cases Eulerin perusnurjahdustapaukset

Tapous	1	2	3	4	5
μ	0.25	1	2.046	4	1

$$P_{\text{cr}} = \mu \frac{\pi^2 EI}{\ell^2}$$

Buckling length - Pilareiden nurjahduspituudet

$$P_{\text{cr}} = \frac{\pi^2 EI}{\ell_n^2}$$

$$\lambda = \frac{l_n}{i_{\min}} = \frac{l}{\sqrt{\mu} \cdot i_{\min}}.$$

λ := slenderness

hoikkuusluku

$$i_{\min}^2 = I_{\min} / A$$

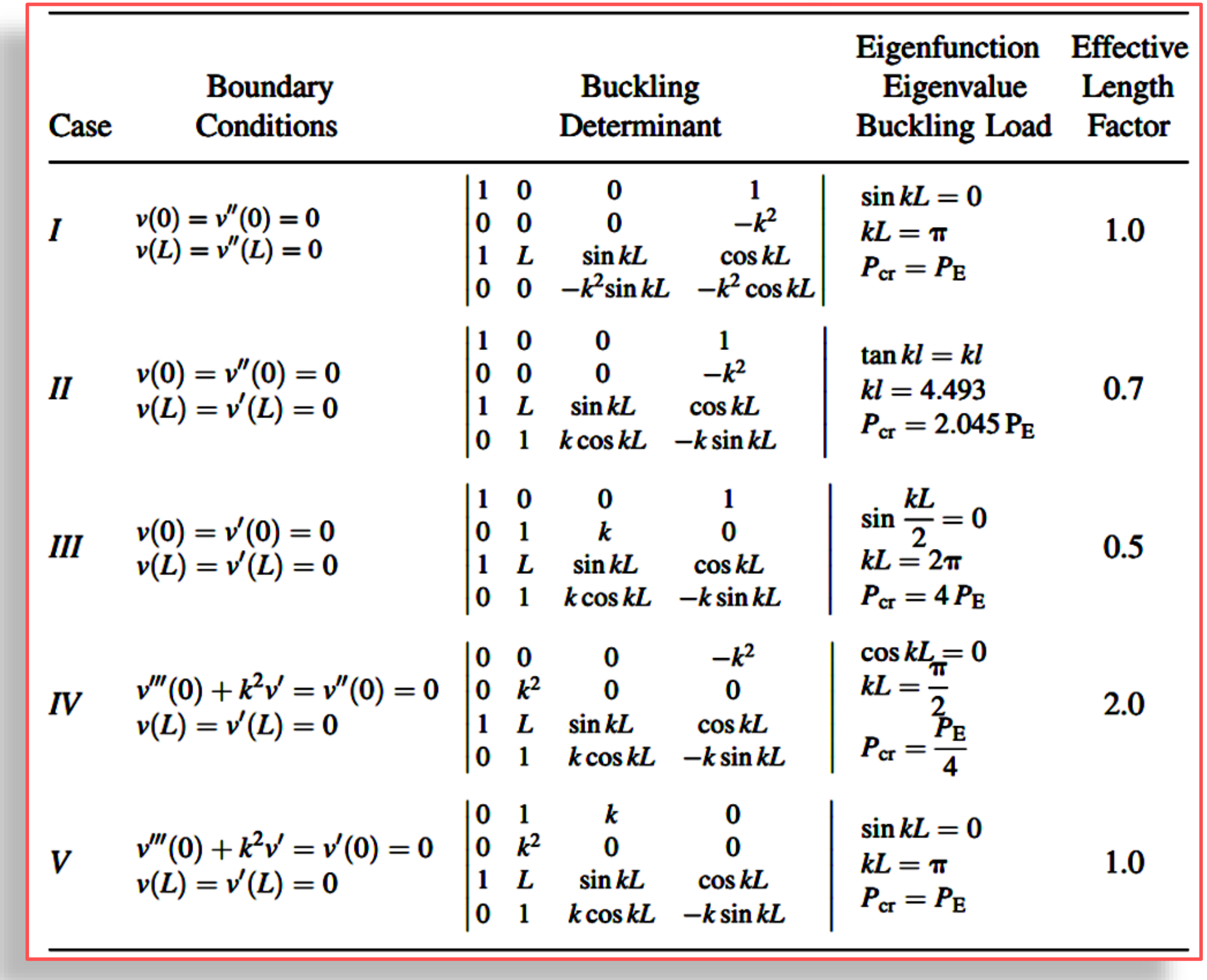
Buckling stress:

Nurjahdusjännitys:

$$\sigma_{kr} = \frac{P_{kr}}{A} = \frac{\pi^2 E i_{\min}^2}{l_n^2} = \mu \frac{\pi^2 E i_{\min}^2}{l^2}.$$

$$\sigma_{kr} = \frac{\pi^2 E}{\lambda^2},$$

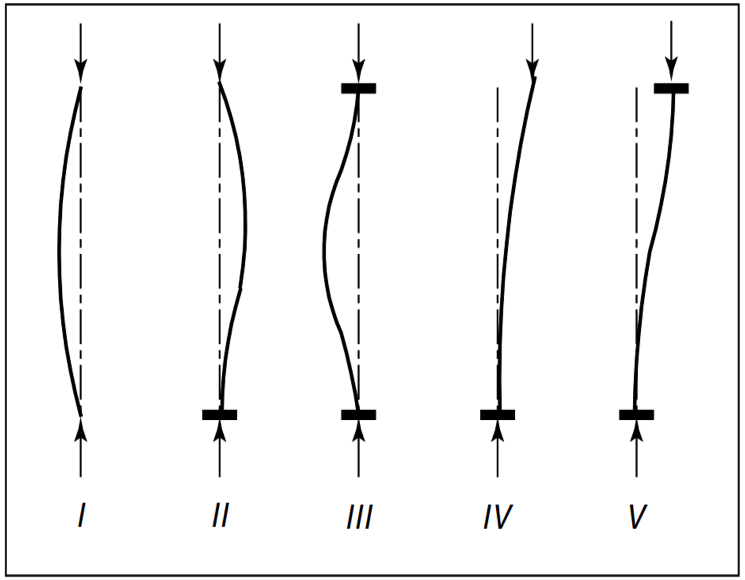
Five Fundamental Cases of Column Buckling



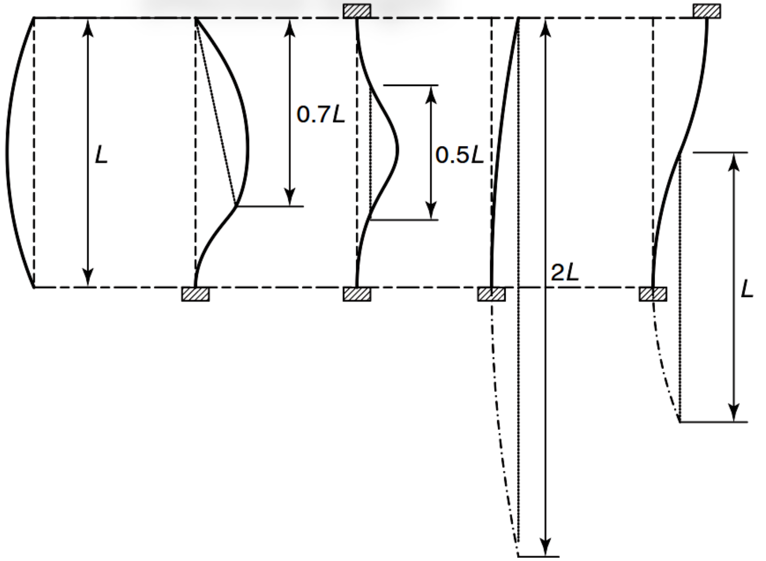
Case	Boundary Conditions	Buckling Determinant	Eigenfunction Eigenvalue Buckling Load	Effective Length Factor
I	$v(0) = v''(0) = 0$ $v(L) = v''(L) = 0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 0 & -k^2 \sin kL & -k^2 \cos kL \end{vmatrix}$	$\sin kL = 0$ $kL = \pi$ $P_{cr} = P_E$	1.0
II	$v(0) = v''(0) = 0$ $v(L) = v'(L) = 0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k \cos kL & -k \sin kL \end{vmatrix}$	$\tan kl = kl$ $kl = 4.493$ $P_{cr} = 2.045\,P_E$	0.7
III	$v(0) = v'(0) = 0$ $v(L) = v'(L) = 0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & k & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k \cos kL & -k \sin kL \end{vmatrix}$	$\sin \frac{kL}{2} = 0$ $kL = 2\pi$ $P_{cr} = 4\,P_E$	0.5
IV	$v'''(0) + k^2 v' = v''(0) = 0$ $v(L) = v'(L) = 0$	$\begin{vmatrix} 0 & 0 & 0 & -k^2 \\ 0 & k^2 & 0 & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k \cos kL & -k \sin kL \end{vmatrix}$	$\cos kL = 0$ $kL = \frac{\pi}{2}$ $P_{cr} = \frac{P_E}{4}$	2.0
V	$v'''(0) + k^2 v' = v'(0) = 0$ $v(L) = v'(L) = 0$	$\begin{vmatrix} 0 & 1 & k & 0 \\ 0 & k^2 & 0 & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k \cos kL & -k \sin kL \end{vmatrix}$	$\sin kL = 0$ $kL = \pi$ $P_{cr} = P_E$	1.0

Adapted from the reference:
STRUCTURAL STABILITY OF STEEL: CONCEPTS AND APPLICATIONS FOR STRUCTURAL ENGINEERS. THEODORE V. GALAMBOS ANDREA E. SUROVEK
JOHN WILEY & SONS, INC.

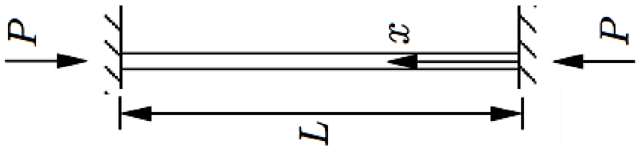
Elementary buckling cases



Geometric interpretation of the effective length



Example – rigidly fixed ends column



$$v(0) = v'(0) = v(L) = v'(L) = 0.$$

$$\boxed{v(x) = A\sin kx + B\cos kx + Cx + D}$$

$$v'(x) = Ak\cos kx - Bk\sin kx + C.$$

$$\begin{aligned} B + D &= 0, \\ kA + C &= 0, \\ A\sin kL + B\cos kL + CL + D &= 0, \\ kA\cos kL - kB\sin kL + C &= 0. \end{aligned}$$

$$\boxed{\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ \sin kL & \cos kL & L & 1 \\ k\cos kL & -k\sin kL & 1 & 0 \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

Cf.

$$\boxed{\Longrightarrow \mathbf{Hq} = \mathbf{0}\,,}$$

$$\boxed{\det\{\mathbf{H}\} = 0}$$

Non-trivial solution: the determinant vanishes: $\boxed{\det\{\mathbf{H}\} = 0}$

$$\boxed{4k\sin\frac{kL}{2}\left(\sin\frac{kL}{2} - \frac{kL}{2}\cos\frac{kL}{2}\right) = 0.}$$

$\Longrightarrow$ Criticality: $\sin\frac{kL}{2} = 0$ or $\tan\frac{kL}{2} = \frac{kL}{2}$,

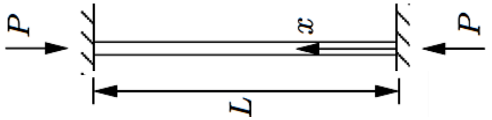
The zeros of the determinant: $\frac{kL}{2} = n\pi\,, \quad n = 1, 2, \ldots\,,$ $\qquad \frac{kL}{2} \approx 4.493.$

The critical load is the smallest: $k_1 = \frac{2\pi}{L}, \quad (n = 1),$ $\boxed{P_1 \equiv P_{kr} = \frac{4\pi^2 EI}{L^2}.}$

The critical load from the Euler's 'Table' : $P_{\text{cr}} = 4\frac{\pi^2 EI}{\ell^2}$

Adapted from ref: prof. Tuomala M.

Examples – rigidly fixed ends column



Four BCs: $v(0) = v'(0) = v(L) = v'(L) = 0.$

$$v'(x) = Ak\cos kx - Bk\sin kx + C.$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ \sin kL & \cos kL & L & 1 \\ k\cos kL & -k\sin kL & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Non-trivial solution: the determinant vanishes:

$$4k\sin\frac{kL}{2}\left(\sin\frac{kL}{2} - \frac{kL}{2}\cos\frac{kL}{2}\right) = 0.$$

(Stability loss criterion) Criticality: $\sin\frac{kL}{2} = 0$ or $\tan\frac{kL}{2} = \frac{kL}{2},$

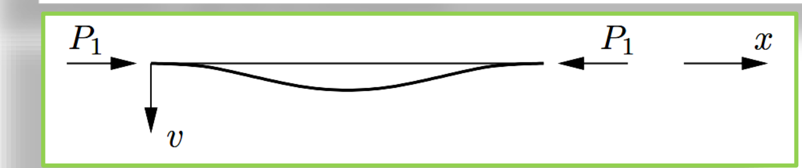
The zeros of the determinant: $\frac{kL}{2} = n\pi\ , \quad n = 1, 2, \ldots\ ,$ $\quad \frac{kL}{2} \approx 4.493.$

The critical load is the smallest: $k_1 = \frac{2\pi}{L}, \quad (n = 1),$

$$P_1 \equiv P_{kr} = \frac{4\pi^2 EI}{L^2}.$$

The corresponding Eigen- (buckling) mode:

(insert the solution back and solve the integration constants…up to a constant)



$$\begin{aligned} B + D &= 0, \\ k_1 A + C &= 0, \\ A\sin k_1 L + B\cos k_1 L + CL + D &= 0, \\ k_1 A\cos k_1 L - k_1 B\sin k_1 L + C &= 0, \end{aligned}$$

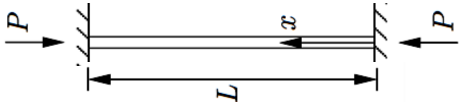
$$A = C = 0, \quad D = -B,$$

$$v(x) = B(\cos\frac{2\pi x}{L} - 1).$$

$$v(x) = B(\cos\frac{2\pi x}{L} - 1).$$

Examples – what is the buckling length?

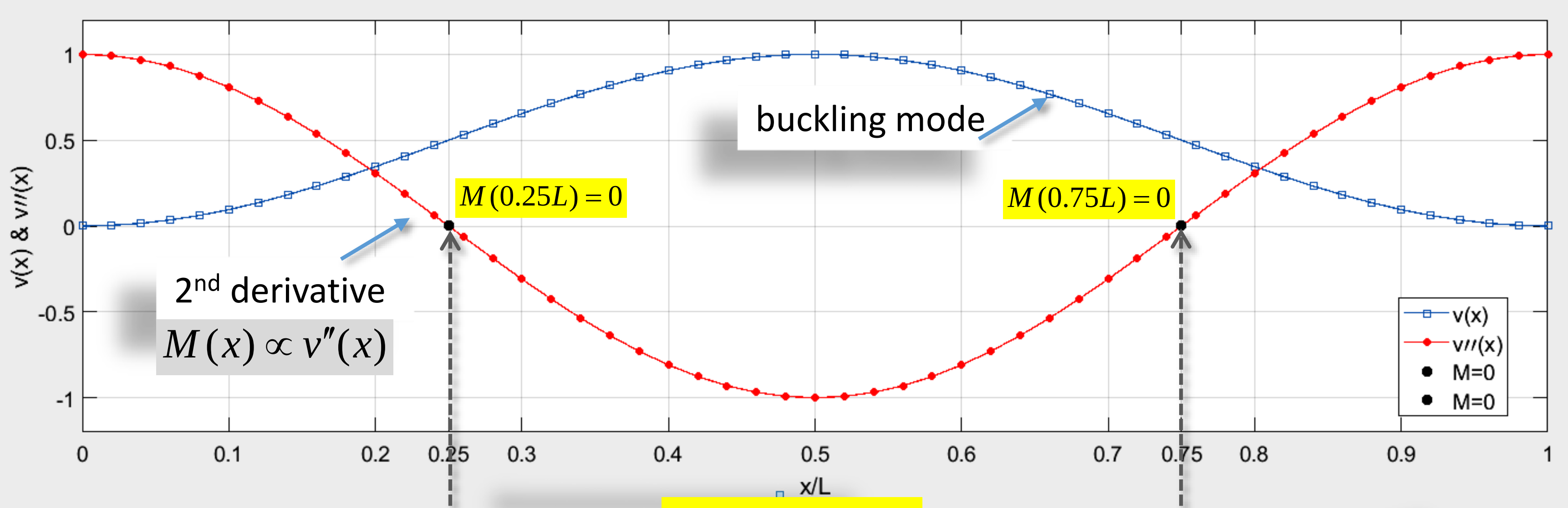
corresponding buckling mode:



critical load: $P_{\mathrm{cr}} = 4\frac{\pi^2 EI}{L^2}$

$$P_1 \equiv P_{kr} = \frac{4\pi^2 EI}{L^2}.$$

$$v(x) = B(\cos\frac{2\pi x}{L} - 1).$$



buckling mode

$M(0.25L) = 0$

$M(0.75L) = 0$

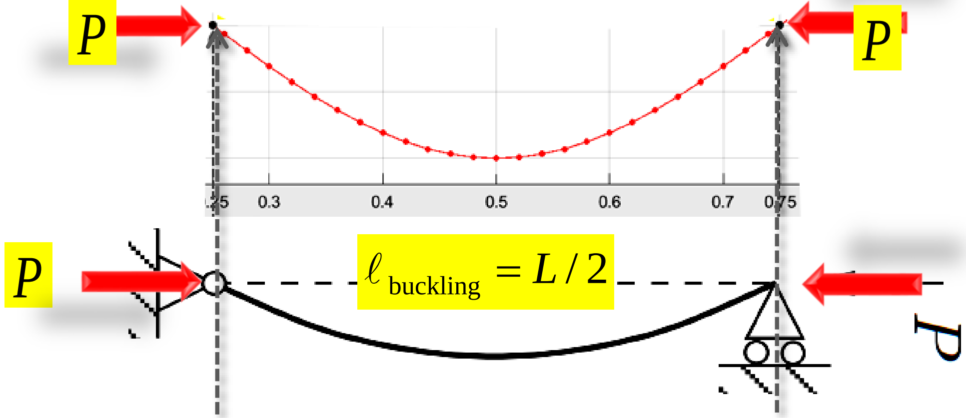
2nd derivative

$M(x) \propto v''(x)$

$\ell_{\text{buckling}} = L/2$

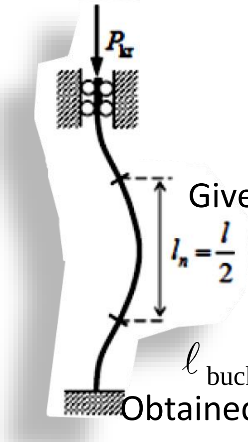
buckling length

$$P_{\mathrm{cr}} = 4\pi^2 EI / L^2 = \pi^2 EI / \ell^2_{\text{buckling}} \Rightarrow \ell_{\text{buckling}} = L/2$$



$\ell_{\text{buckling}} = L/2$

$$\ell_{\text{buckling}} = \frac{1}{\sqrt{\mu}} L$$



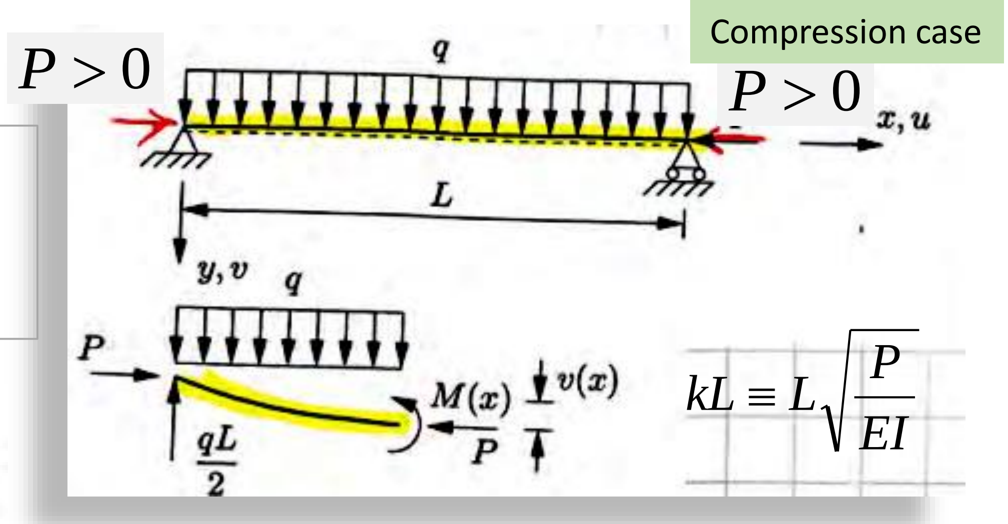
Given by tables

$\ell_{\text{buckling}} = L/2$

Obtained from theory

Second-order effects

The stress-problem:
Solve the deflection *v*(*x*) as function of the axial load *P* (the loading parameter)



$$EIv''(x) + Pv(x) = -\frac{qL^2}{2}\frac{x}{L}\left(1-\frac{x}{L}\right).$$

$$v(x) = \frac{qL^2}{P}\left[\frac{\sin k(L-x)+\sin kx}{(kL)^2 \sin kL} - \frac{1}{(kL)^2} - \frac{1}{2}\frac{x}{L} - \frac{1}{2}\left(\frac{x}{L}\right)^2\right],$$

$$\Downarrow$$

Homework: Show* that max. bending moment reduces to:

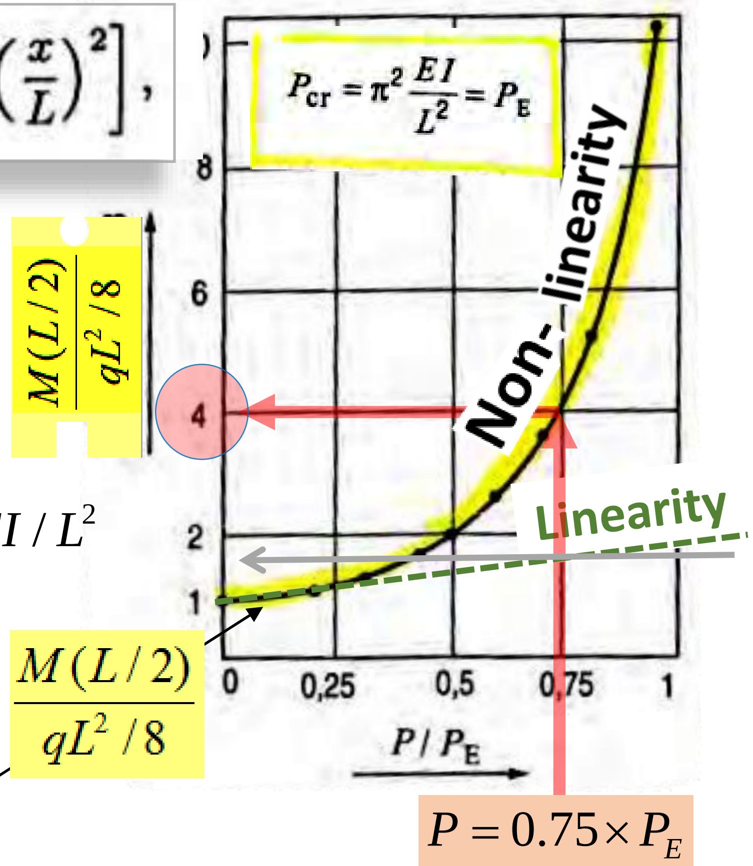
$$\Rightarrow \frac{M(L/2)}{qL^2/8} = \frac{8}{\pi^2 P_E}\left(1/\cos\left(\frac{\pi}{2}\sqrt{P/P_E}\right)-1\right)$$

$$q = 0 \rightarrow P_E = \pi^2 EI/L^2$$

and that maximum deflection is:

$$\Rightarrow v(L/2) = \frac{q}{\pi k^2}\left[1/\cos(kL/2)-1\right] - \frac{q(L/2)^2}{2P}$$

Second-order effects = non-linear effects



Frames – recall from 'beams and frames course for the **slope-deflection method** with Berry's stability functions

Slope-deflection method – Stiffness-equation

The stiffness equations of the slope-deflection method with axial load

$$M_{ij} = A_{ij}(P)\varphi_{ij} + B_{ij}(P)\varphi_{ij} - C_{ij}(P)\psi_{ij} + MK_{ij}(P)$$

$$ij = \{\ 12,\ 21\}$$

Stiffness-coefficients and loading terms depend on the member axial force

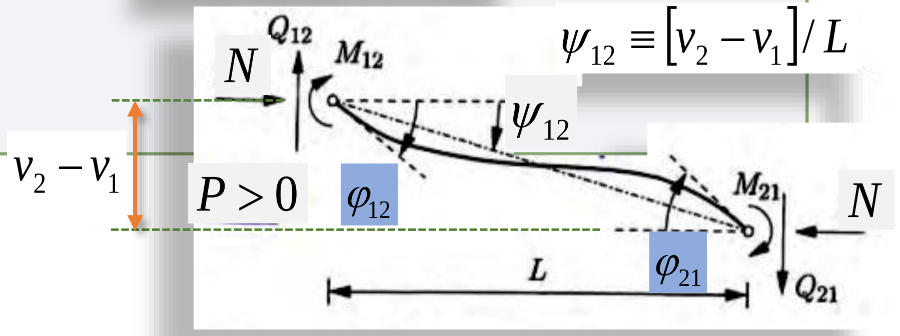
$$\lambda \equiv kL \equiv L\sqrt{\frac{P}{EI}}$$

$$M_{12} = A^0_{12}(kL)\varphi_{12} - C^0_{12}(kL)\psi_{21} + MK^0_{12}(kL)$$

Stiffness-coefficients are symmetric with respect to *i* and *j*

Member axial force can be compressive or tensile. The stiffness-coefficients are different in compression and in tension.

Compression : P > 0



$N \equiv -N_{12} = P > 0$ Case of compression : P > 0

Frames – recall from 'beams and frames course for the slope-deflection method with Berry's stability functions

The stiffness coefficients – axial compression and bending

Compression : P > 0 $\psi_{12} \equiv [v_2 - v_1]/\ell$

NB. Notation: $y \equiv v$ $\theta \equiv \varphi$

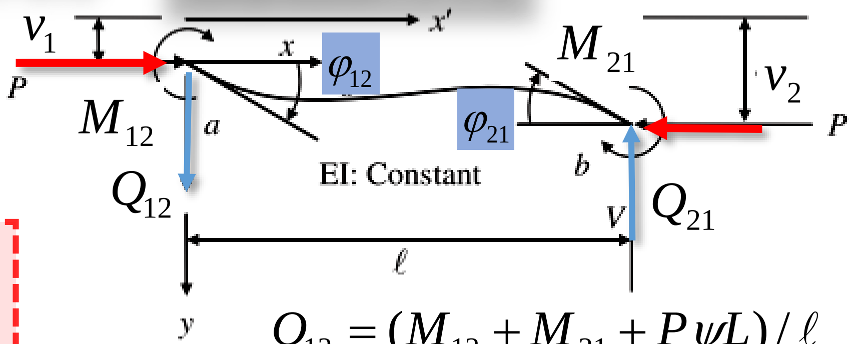
$$v^{(4)}(x) + k^2 v''(x) = 0$$

$$v(x) = A\sin(kx) + B\cos(kx) + Cx + D$$

Boundary conditions:

$v(0) = v_1 = 0$ $v(\ell) = v_2 \equiv \psi_{12}\ell = v_2 - v_1 \equiv \Delta$

$v'(0) = \varphi_{12}$ and $v'(\ell) = \varphi_{21}$



$$Q_{12} = (M_{12} + M_{21} + P\psi L)/\ell$$

$$\Delta = \psi_{12}\ell = v_2 - v_1 = v_2 - 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin\beta & \cos\beta & \ell & 1 \\ k & 0 & 1 & 0 \\ k\cos\beta & -k\sin\beta & 1 & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ \Delta \\ \varphi_{12} \\ \varphi_{21} \end{Bmatrix}$$

$\Downarrow$

However, it is more practical to express the stiffness coefficients in terms of Berry's functions as we did till now.

$$M_{12} = M(0) = -EIv''(0) = EIBk^2$$

$$\beta \equiv k\ell \equiv \lambda$$

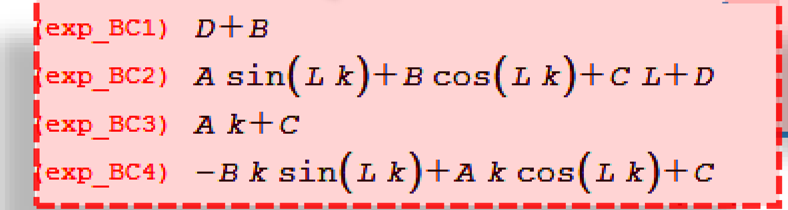
$$= \left[\frac{EIk^2}{k(2\cos\beta + \beta\sin\beta - 2)}\right][(\beta\cos\beta - \sin\beta)\varphi_{12} + (\sin\beta - \beta)\varphi_{21} + (k - k\cos\beta)\Delta]$$

$$= \left[\frac{EI\beta}{\ell(2\cos\beta + \beta\sin\beta - 2)}\right]\left[(\beta\cos\beta - \sin\beta)\varphi_{12} + (\sin\beta - \beta)\varphi_{21} + (\beta - \beta\cos\beta)\frac{\Delta}{\ell}\right]$$

$$A_{12}(k\ell) = M(0, k\ell) \downarrow$$

$$A_{12}(\lambda) = \frac{\lambda(\lambda\cos\lambda - \sin\lambda)}{2\cos\lambda + \lambda\sin\lambda - 2}$$

$$\sin\beta = 2\sin(\beta/2)\cos(\beta/2)$$



```
(exp_BC1) D+B
(exp_BC2) A sin(L k)+B cos(L k)+C L+D
(exp_BC3) A k+C
(exp_BC4) −B k sin(L k)+A k cos(L k)+C
```

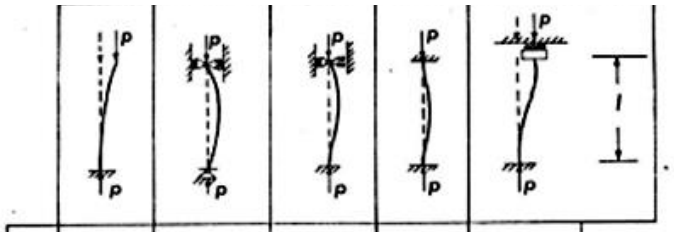
We have earlier established these eqs previously when using Maxima

$$\beta \equiv k\ell \equiv \lambda$$

Frames – recall from ‘beams and frames course

Formulary

Eulerin peruskaavat nurjahdukselle: $P_{cr} = \mu \cdot \dfrac{\pi^2 EI}{l^2}$



Tapaus	1	2	3	4	5
μ	0.25	1	2.046	4	1

The stiffness coefficients (are symmetric)

Puristettu ja taivutettu sauva:

Kulmanmuutosmenetelmä

$M_{ij} = A_{ij}\varphi_{ij} + B_{ij}\varphi_{ji} - C_{ij}\psi_{ij} + \overline{MK}_{ij}$

$M_{ij} = A^0_{ij}\varphi_{ij} - C^0_{ij}\psi_{ij} + \overline{MK}^0_{ij}$ (sauvan päässä j on nivel)

Tasajäykkä sauva :

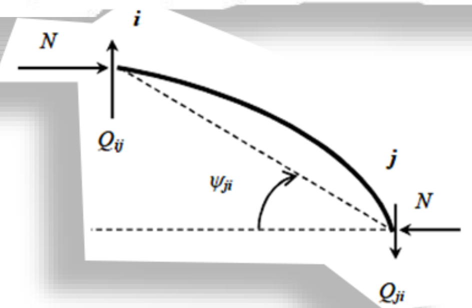
$A_{ij} = A_{ji} = \dfrac{2\psi(kL)}{4\psi^2(kL) - \phi^2(kL)}\dfrac{6EI}{L}$, $B_{ij} = B_{ji} = \dfrac{\phi(kL)}{4\psi^2(kL) - \phi^2(kL)}\dfrac{6EI}{L}$ ja $C_{ij} = A_{ij} + B_{ij}$,

$\overline{MK}_{ij} = -A_{ij}\overline{\alpha}^0_{ij} - B_{ji}\overline{\alpha}^0_{ji}$, $\qquad \overline{MK}_{ji} = -A_{ji}\overline{\alpha}^0_{ji} - B_{ij}\overline{\alpha}^0_{ij}$,

$A^0_{ij} = C^0_{ij} = \dfrac{1}{\psi(kL)}\dfrac{3EI}{L}$, $\qquad \overline{MK}^0_{ij} = -A_{ij}\overline{\alpha}^0_{ij}$

Leikkausvoima:

$Q_{ij} = Q^0_{ij} - (M_{ij} + M_{ji})/L - N\psi_{ij}$ (N positiivinen, kun sauva puristettu)



Berry’s functions (stability function)

Berryn funktiot:

Olkoon $\lambda \equiv kL$,

Puristettu sauva:

Compression

$\phi(\lambda) = \dfrac{6}{\lambda}\left(\dfrac{1}{\sin\lambda} - \dfrac{1}{\lambda}\right)$, $\psi(\lambda) = \dfrac{3}{\lambda}\left(\dfrac{1}{\lambda} - \dfrac{1}{\tan\lambda}\right)$, ja $\chi(\lambda) = \dfrac{24}{\lambda^3}\left(\tan\dfrac{\lambda}{2} - \dfrac{\lambda}{2}\right)$,

Vedetty sauva:

$\phi(\lambda) = \dfrac{6}{\lambda}\left(-\dfrac{1}{\sinh\lambda} + \dfrac{1}{\lambda}\right)$, $\psi(\lambda) = \dfrac{3}{\lambda}\left(-\dfrac{1}{\lambda} + \dfrac{1}{\tanh\lambda}\right)$, ja $\chi(\lambda) = \dfrac{24}{\lambda^3}\left(-\tanh\dfrac{\lambda}{2} + \dfrac{\lambda}{2}\right)$,

Extension

$$M_{ij} = A_{ij}\varphi_{ij} + B_{ij}\varphi_{ij} - C_{ij}\psi_{ij} + \overline{M}_{ij}$$

EI constant

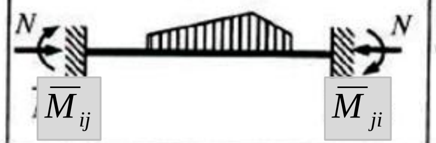
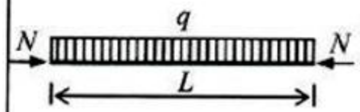
$$A_{12} = \frac{2\psi(kL)}{4\psi^2(kL) - \phi^2(kL)}\frac{6EI}{L} = A_{21},$$

$$B_{12} = \frac{\phi(kL)}{4\psi^2(kL) - \phi^2(kL)}\frac{6EI}{L} = B_{21},$$

$$C_{12} = A_{12} + B_{12}, \quad C_{21} = A_{21} + B_{21}.$$

Loading terms

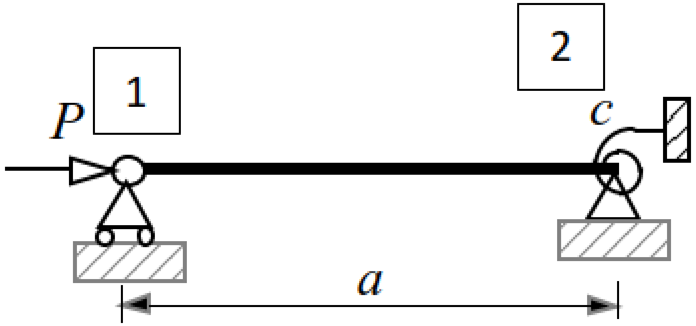
$\overline{M}_{ij} = -\overline{M}_{ji}$

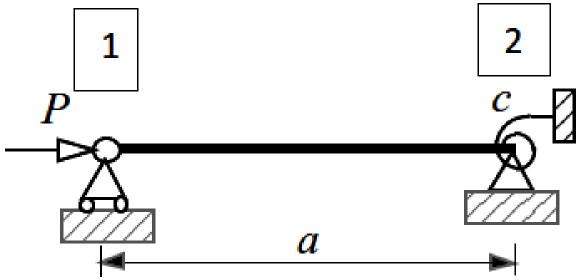
N:o	Kuormitus	Kiinnitysmomentit:  N N $\overline{M}_{ij}$ $\overline{M}_{ji}$
1	 q N N L	$\overline{MK}_1 = -\overline{MK}_2$ $= -\dfrac{qL^2}{12}\dfrac{\chi(kL)}{\tan(\frac{kL}{2})/(\frac{kL}{2})}$

The stiffness coefficients – axial compression and bending

Example from exam 2018

A straight beam is simply supported at one end, and supported by a rotational spring, with spring constant $c=\alpha EI/a$, at the other. Its length is *a*, and bending stiffness *EI*. Determine the critical compressive load of the beam, when $\alpha=1$. Show further that the result is covering the cases where the right hand end of the beam is simply supported and clamped by varying the coefficient α.





1. Easiest way is to apply the slope-deflection method. Thus the equilibrium equation is $M_{21}+M_{2s}=0 \Rightarrow (A_{21}^{o}+c)\varphi_2=0$.

$A_{21}^{o}+c=-\frac{1}{\Psi(ka)}\frac{3EI}{a}+\alpha\frac{EI}{a}=0 \Rightarrow \Psi(ka)=\frac{3}{\alpha}$. Jos $\Psi(ka)=\frac{3}{ka}\left(\frac{1}{ka}-\frac{1}{\tan ka}\right)$

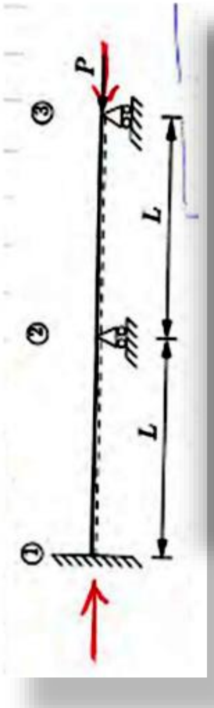
$\Rightarrow \tan ka=\frac{\alpha ka}{\alpha+(ka)^2}$ If $\alpha=1 \Rightarrow \tan ka=\frac{ka}{1+(ka)^2} \Rightarrow ka=3.405 \Rightarrow P_{cr}=1.175\frac{\pi^2 EI}{a^2}$

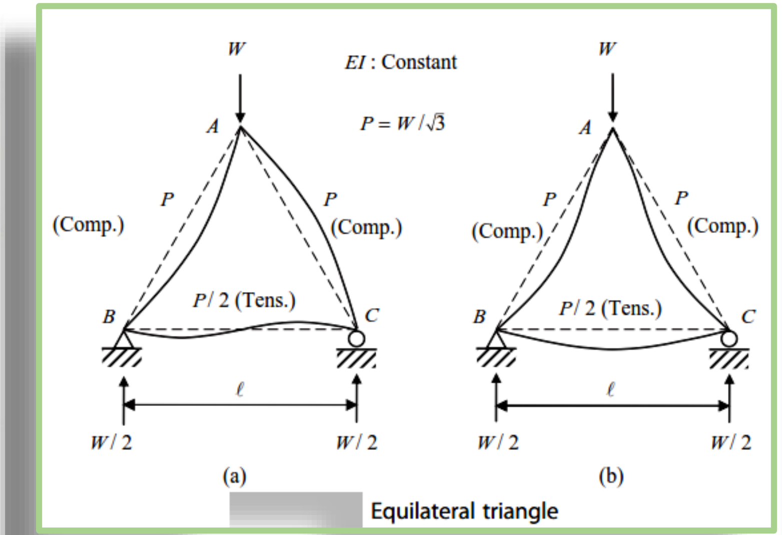
If $\alpha=0 \Rightarrow \tan ka=0 \Rightarrow ka=n\pi \Rightarrow P_{cr}=\frac{\pi^2 EI}{a^2}$. If $\alpha=\infty \Rightarrow \tan ka=ka \Rightarrow P_{cr}=2.046\frac{\pi^2 EI}{a^2}$.

From differential equation, the solution is $v(x)=C_1\sin kx+C_2\cos kx+C_3x+C_4$ where $k^2=P/EI$ and the boundary conditions $v(0)=v''(0)=v(a)=0,\ cv'(a)=-EIv''(a)$ yielding $C_2=C_4=0$, $C_3=-C_1\sin ka/a$ and the condition $c(k\cos ka-\sin ka/a)=P\sin ka$, yielding the same result.

Buckling of Continuous Beam-Columns and Frames

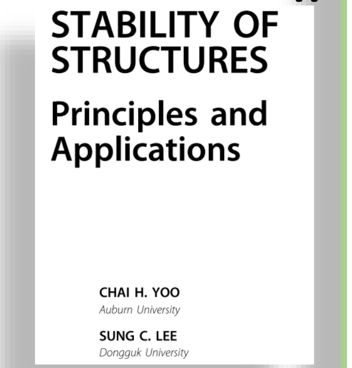
Frames – recall from 'beams and frames course





Equilateral triangle

Examples from textbook:



STABILITY OF STRUCTURES

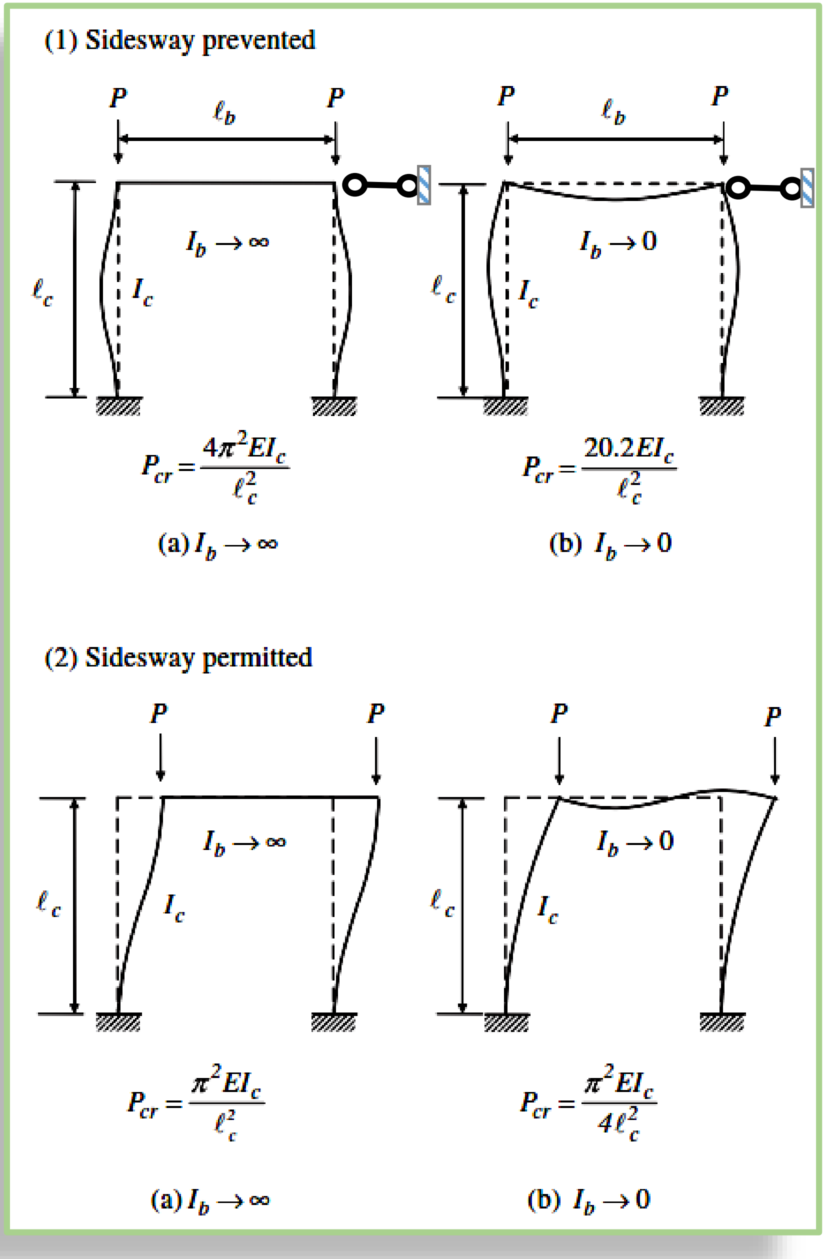
Principles and Applications

CHAI H. YOO

Auburn University

SUNG C. LEE

Dongguk University



Frames – recall from 'beams and frames course

Buckling of frames - no side sway

only beam 1-2 is axially compressed

SOLUTION

Solution: $\varphi_{21}=\varphi_{23}\equiv\varphi_2,$

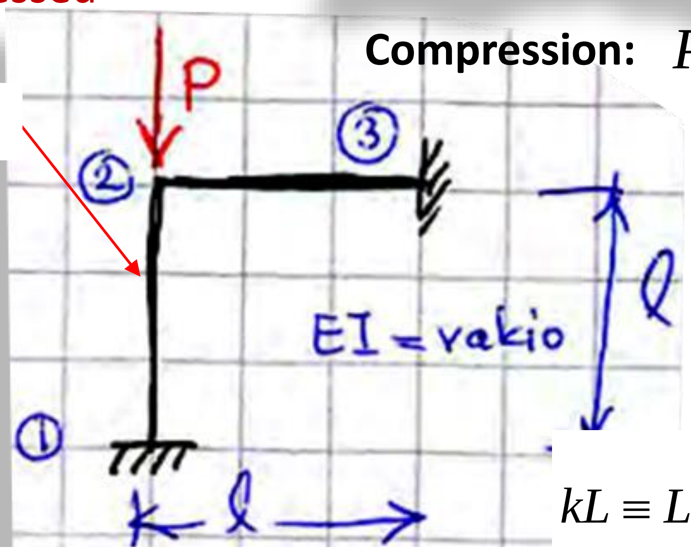
$$M_{21}+M_{23}=0\Rightarrow\left(A_{21}+a_{23}\right)\varphi_2=0,$$

compression $P>0$

normal force = 0

$$\frac{2\psi(kL)}{4\psi^2(kL)-\phi^2(kL)}\frac{6EI}{L},\qquad\frac{4EI}{L}$$

no side sway: $\psi_{ij}=0$



Compression: $P>0$

$$kL\equiv L\sqrt{\frac{P}{EI}}$$

Critical condition = non-trivial solution exists:

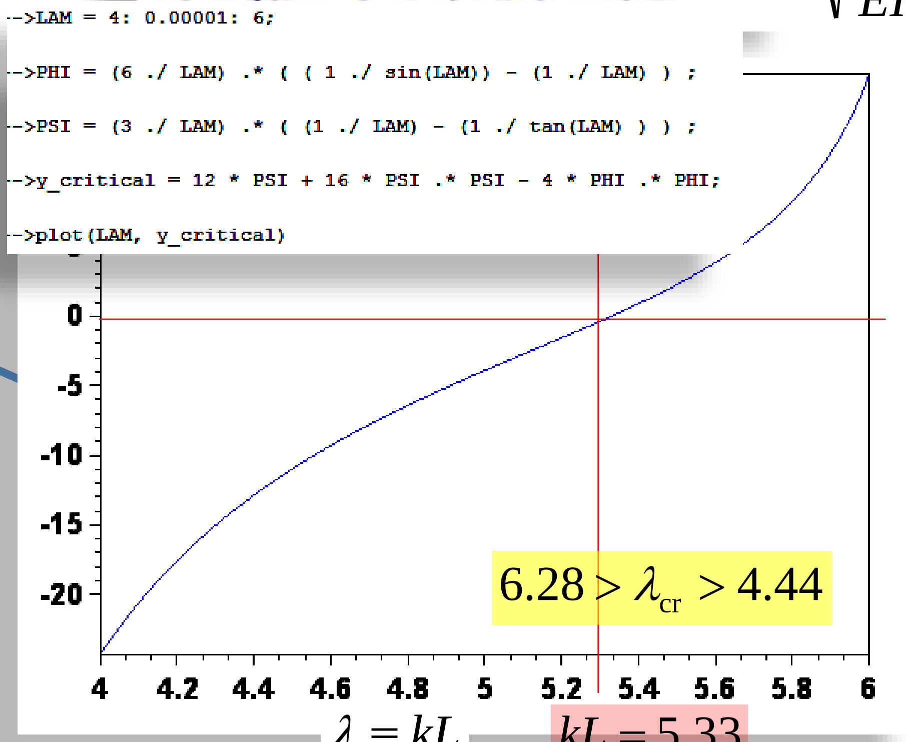
$$\phi_2\neq0\Rightarrow A_{21}(kL)+a_{23}=0\Rightarrow kL=?$$

$$\Downarrow$$

$$12\psi(kL)+16\psi^2(kL)-4\phi^2(kL)=0$$

$$\Downarrow$$

$$kL=5.33\Rightarrow P_{cr}=2.88\pi^2\frac{EI}{L^2}$$



```
-->LAM = 4: 0.00001: 6;

-->PHI = (6 ./ LAM) .* ( ( 1 ./ sin(LAM)) - (1 ./ LAM) ) ;

-->PSI = (3 ./ LAM) .* ( (1 ./ LAM) - (1 ./ tan(LAM) ) ) ;

-->y_critical = 12 * PSI + 16 * PSI .* PSI - 4 * PHI .* PHI;

-->plot(LAM, y_critical)
```

$6.28>\lambda_{cr}>4.44$

$\lambda\equiv kL$ $\quad kL=5.33$

Berry's stability functions: $\lambda\equiv kL$

$$\psi(\lambda)=\frac{3}{\lambda}\left(\frac{1}{\lambda}-\frac{1}{\tan\lambda}\right),\qquad\phi(\lambda)=\frac{6}{\lambda}\left(\frac{1}{\sin\lambda}-\frac{1}{\lambda}\right)$$

$$M_{ij}=A_{ij}(P)\varphi_{ij}+B_{ij}(P)\varphi_{ij}-C_{ij}(P)\psi_{ij}+MK_{ij}(P),$$

Geometrically non-linear analysis of frames by the Slope-deflection method

Moderate rotations and loads close to critical load but not over

Recall from previous course (beams and frames)

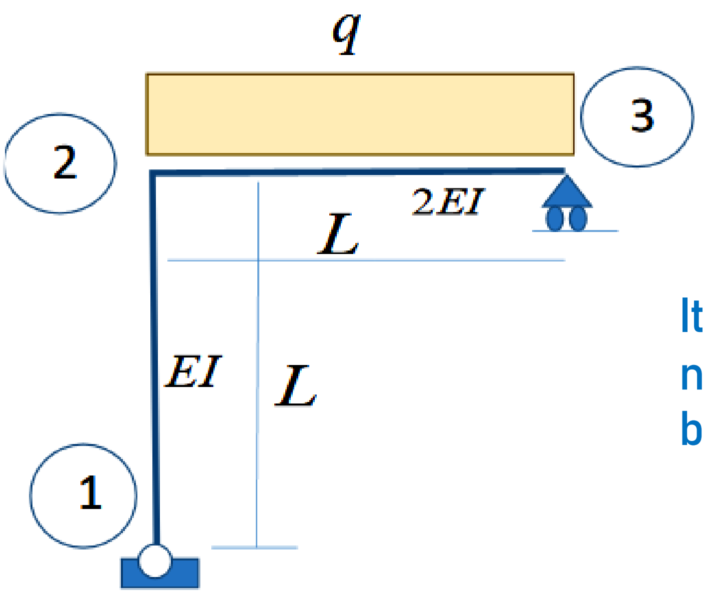
$$\varphi_{21}=\varphi_{23}\Rightarrow\frac{L}{3EI}\Psi(kL)M_{21}+\psi_{21}=\frac{L}{6EI}M_{23}+\frac{qL^3}{48EI}$$

$$(1+2\Psi(kL))M_2+\frac{6EI}{L}\psi_{21}=\frac{qL^2}{8}$$

$$Q_{21}=0\quad\Rightarrow\quad-\frac{M_2}{L}-P\psi_{21}=0\Rightarrow\psi_{21}=-\frac{M_2}{PL}$$

$$(1+2\Psi(kL))M_2-\frac{3M_2}{k^2L^2}=\frac{qL^2}{8}$$

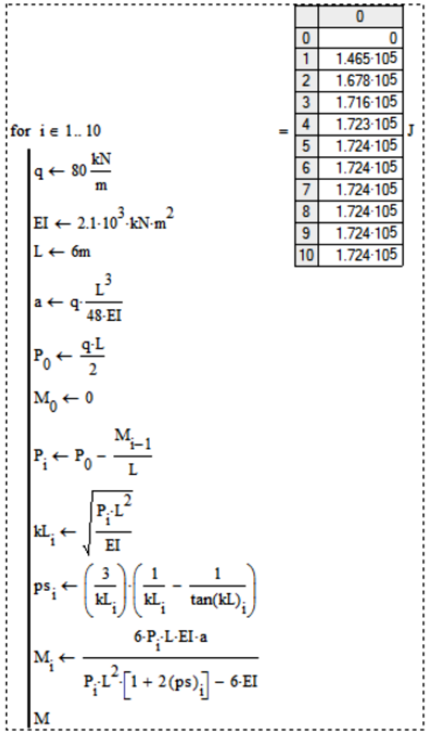
Q: DETERMINE THE BENDING MOMENT AT RIGID JOINT #2



Iterations are needed to solve the bending moment:

$$\Rightarrow\begin{cases}M_2=\dfrac{qL^2}{8}\dfrac{PL^2}{PL^2(1+2\Psi(kL))-6EI}\\[2ex] N_{21}+Q_{32}=0\end{cases}$$

Express Q_{32} in terms of end-moments

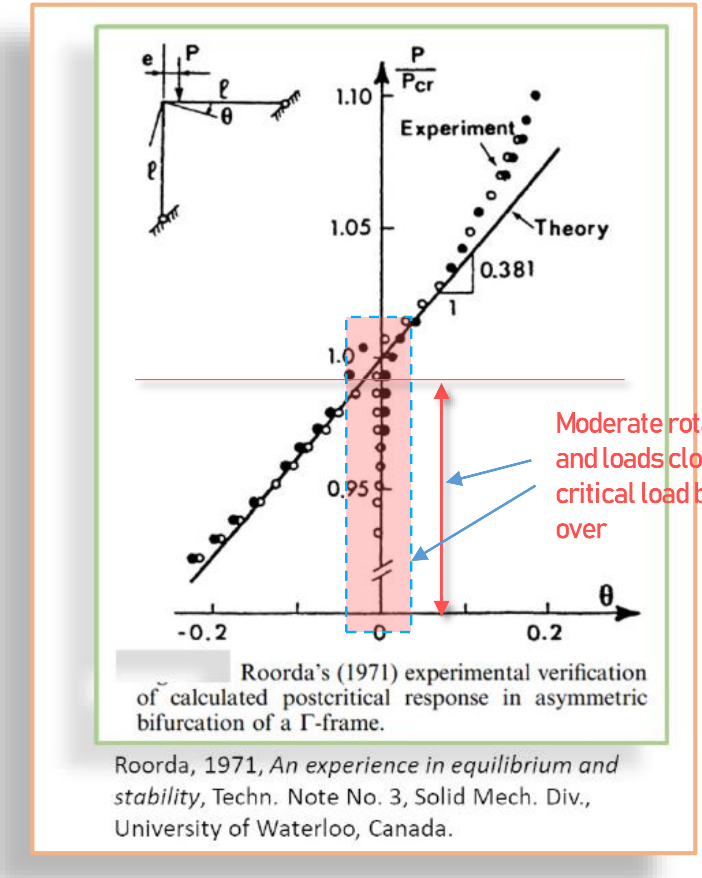


```
for i ∈ 1..10
    $q \leftarrow 80\,\frac{kN}{m}$
    $EI \leftarrow 2.1\cdot10^3\cdot kN\cdot m^2$
    $L \leftarrow 6m$
    $a \leftarrow q\cdot\frac{L^3}{48\cdot EI}$
    $P_0 \leftarrow \frac{q\cdot L}{2}$
    $M_0 \leftarrow 0$
    $P_i \leftarrow P_0 - \frac{M_{i-1}}{L}$
    $kL_i \leftarrow \sqrt{\frac{P_i\cdot L^2}{EI}}$
    $ps_i \leftarrow \left(\frac{3}{kL_i}\right)\cdot\left(\frac{1}{kL_i}-\frac{1}{\tan(kL)_i}\right)$
    $M_i \leftarrow \frac{6\cdot P_i\cdot L\cdot EI\cdot a}{P_i\cdot L^2\cdot\left[1+2(ps)_i\right]-6\cdot EI}$
    M
= (table below) J
```

	0
0	0
1	1.465·105
2	1.678·105
3	1.716·105
4	1.723·105
5	1.724·105
6	1.724·105
7	1.724·105
8	1.724·105
9	1.724·105
10	1.724·105

Geometrically non-linear analysis of frames by the Slope-deflection method

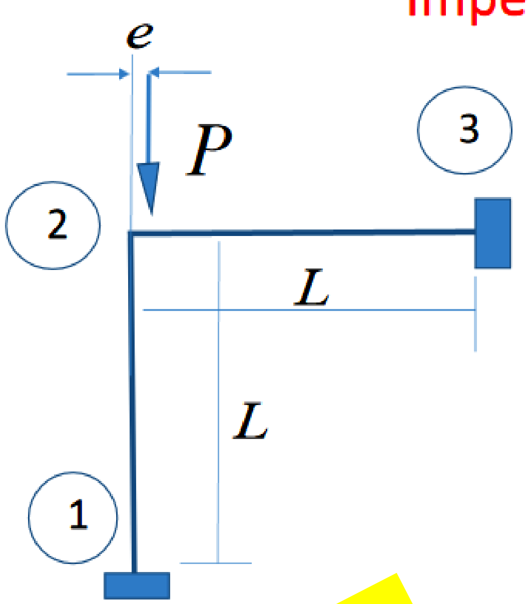
Moderate rotations and loads close to critical load but not over



Roorda's (1971) experimental verification of calculated postcritical response in asymmetric bifurcation of a Γ-frame.

Moderate rotations and loads close to critical load but not over

Roorda, 1971, *An experience in equilibrium and stability*, Techn. Note No. 3, Solid Mech. Div., University of Waterloo, Canada.



Imperfection, eccentricity

$$M_{21}+M_{23}-Pe=0 \Rightarrow \varphi_2=\frac{Pe}{(A_{21}+a_{23})}$$

$$\begin{cases} M_{21}=A_{21}\varphi_2=Pe\frac{A_{21}}{(A_{21}+a_{23})} \\ M_{23}=a_{23}\varphi_2=Pe\frac{a_{23}}{(A_{21}+a_{23})} \\ M_{32}=b_{32}\varphi_2=\frac{M_{23}}{2} \end{cases}$$

$$Q_{23}=-\frac{M_{23}+M_{32}}{L}=-\frac{3}{2}\frac{M_{23}}{L}=-Pe\frac{3a_{23}}{2(A_{21}+a_{23})}$$

Recall from previous course (beams and frames)

$$P=P+Q_{23}=P(1-e\frac{3a_{23}}{2(A_{21}+a_{23})})=P(1-e\frac{1}{\frac{\Psi(kL)}{4\Psi^2(kL)-\Phi^2(kL)}+\frac{2}{3}})$$

Should be N_{21}

(the normal stress resultant in column 12)

If now the eccentritcity *e* is negative, the value of the compressive load *P* is increasing all the time, and no convergence will be reached. If positive, the convergence is reached.

Linear and non-linear buckling analysis

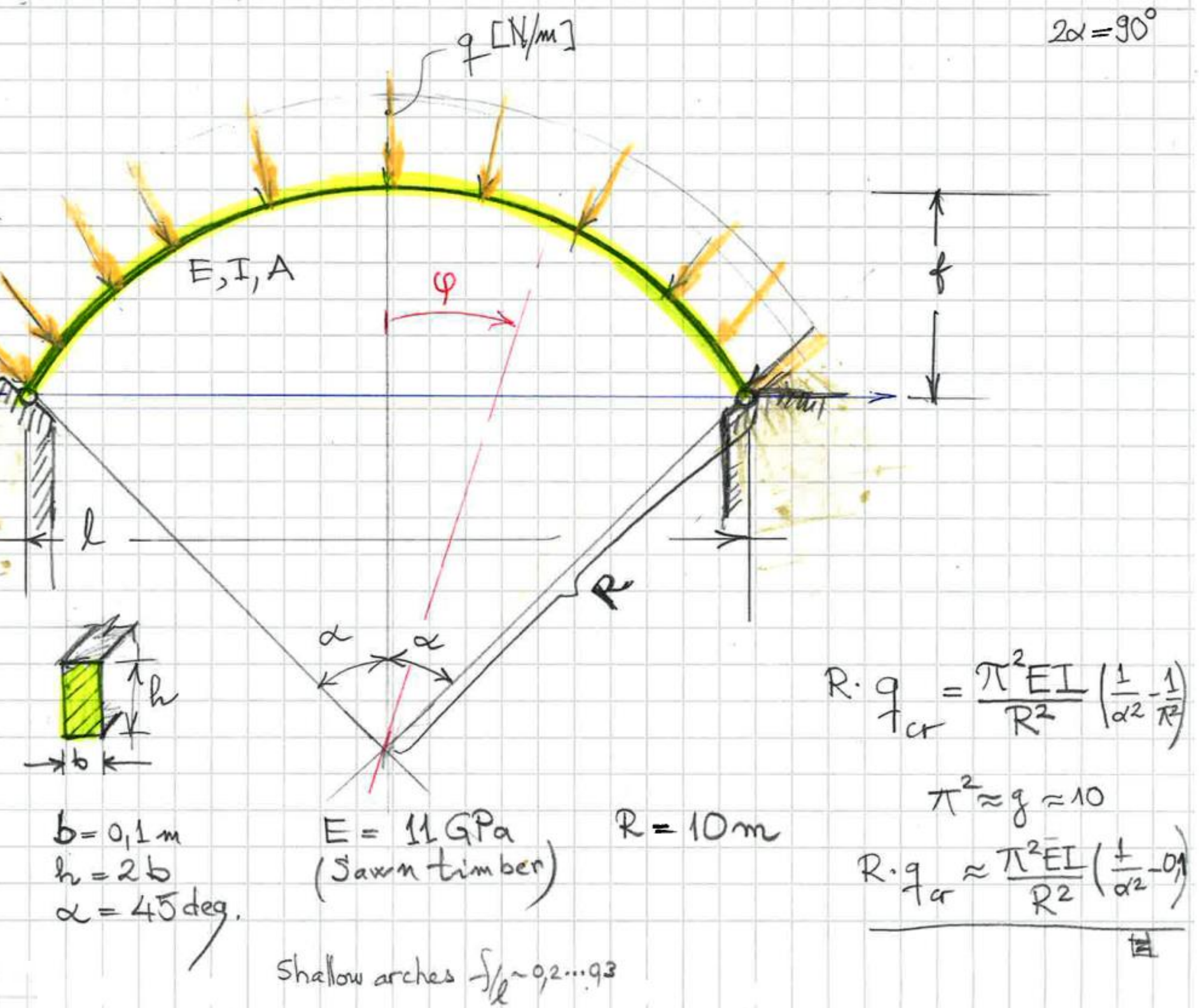
Two-three weeks time to return it

Free Exercise - 20 extra-points for HW

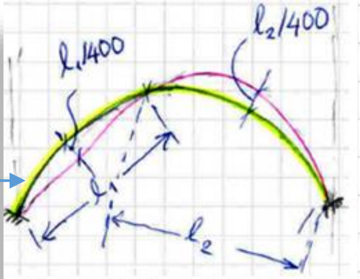
1. Perform linear buckling analysis for the perfect geometry and find the critical load and the respective buckling mode
2. Find the second buckling load and the buckling mode
3. Analysis the shape imperfection effect on the buckling load (GNA)

For that do:

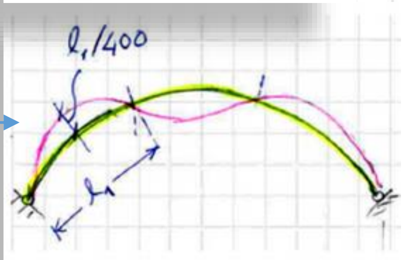
- Take the first buckling mode and then the second one (or their combination) multiplied by L/400 (L distance between mode nodes, as in Figs. on right) as a shape imperfection to add for the perfect geometry.
- Determine the load-displacement curve at some characteristic points
- What is the limit load? How much the buckling load of the perfect arch is reduced?



Assume that stresses remains in the elastic range.



b) loading is antisymmetric



a) loading is symmetric

Example of initial shape imperfections in an arch (Standards: design of wood structures - EN 1995-1-1)

Timoshenko column

Engesser (1891)

Timoshenko (1921)

There is cases when the effect of shear deformation should be considered.

$$\gamma = -\theta + v'.$$

$$\boxed{\Delta\Pi = \frac{1}{2}\int_\ell EI\kappa^2 \mathrm{d}x + \frac{1}{2}\int_\ell k_s GA\gamma^2 \mathrm{d}x - \frac{1}{2}P\int_\ell (v')^2 \mathrm{d}x}$$

the curvature

$\kappa = -v''(1-\alpha P)$

$\gamma = \alpha P v',$

$$\delta(\Delta\Pi) = 0$$

$$\Downarrow$$

linearised buckling equation

$$\boxed{(1-\alpha P)[EIv'']'' + Pv'' = 0}$$

mean shear stress $\bar{\tau} = Q_y(x)/A$

ξ being the *shear correction coefficient*

$$Q_y(x) = k_s GA\gamma = \frac{GA}{\xi}\gamma$$

$$\left\{\begin{aligned} &\gamma \equiv \gamma_{xy} = \frac{\tau_{xy}}{G} = \xi\frac{Q_y}{GA} \equiv \alpha Q_y \\ &\gamma(x) \equiv \gamma_{xy} = u_y + v_x = -\theta(x) + v'(x), \\ &\gamma = \alpha P v', \end{aligned}\right.$$

$$M = EI\theta' = EI\kappa = EI(\gamma' - v'')$$

$$Q = GA\gamma/\xi = \gamma/\alpha$$

$$\begin{cases} Q - Pv' & = 0 \\ M'' - Pv'' & = 0, \end{cases}$$

Timoshenko column

Engesser (1891)

Timoshenko (1921)

There is cases when the effect of shear deformation should be considered.

$$\boxed{\Delta\Pi = \overbrace{\frac{1}{2}\int_\ell EI\kappa^2 \mathrm{d}x + \frac{1}{2}\int_\ell k_s GA\gamma^2 \mathrm{d}x}^{\text{Change of strain energy during buckling}} \underbrace{- \frac{1}{2}P\int_\ell (v')^2 \mathrm{d}x}_{\text{Increment of work of external force during buckling}}}$$

the curvature

$\kappa = -v''(1-\alpha P)$

$\gamma = \alpha P v',$

$\gamma = -\theta + v'.$

$$\delta(\Delta\Pi) = 0$$

$$\Downarrow$$

linearised buckling equation

$$\boxed{(1-\alpha P)[EIv'']'' + Pv'' = 0}.$$

linearised buckling equation

$$\begin{cases} v^{(4)} + k^2 v'' = 0 \\ k^2 = \dfrac{P}{EI}\dfrac{1}{1-\alpha P} \\ \alpha = \dfrac{\xi}{GA} \end{cases}$$

buckling of a cantilever column

$$\Downarrow$$

$$\boxed{P^{\mathrm{T}} = P^{\mathrm{E}}\frac{1}{1+\alpha P^{\mathrm{E}}},}$$

Timoshenko buckling load

Euler buckling load

Timoshenko column

Reduction coefficient of the Euler buckling load

Engesser (1891)

Timoshenko (1921)

Analysis of the results

all end-conditions excepts for fixed-pinned

$$P^{\mathrm{T}}=P^{\mathrm{E}}\frac{1}{1+\boxed{\frac{P^{\mathrm{E}}}{k_sGA}}}=P^{\mathrm{E}}\frac{1}{1+\boxed{\alpha P^{\mathrm{E}}}},$$

fixed-pinned ends

$$P^{\mathrm{T}}=P^{\mathrm{E}}\frac{1}{1+1.1\frac{P^{\mathrm{E}}}{k_sGA}}=P^{\mathrm{E}}\underbrace{\frac{1}{1+1.1\alpha P^{\mathrm{E}}}}$$

Reduction coefficient for the Euler buckling load

$$\alpha=\frac{\xi}{GA}$$

$$1+\boxed{\frac{P^{\mathrm{E}}}{k_sGA}}\ldots 1+1.1\frac{P^{\mathrm{E}}}{k_sGA}$$

buckling of a cantilever column

$$\Downarrow$$

$$P^{\mathrm{T}}=P^{\mathrm{E}}\frac{1}{1+\alpha P^{\mathrm{E}}},$$

Timoshenko buckling load

Euler buckling load

Reduction coefficient

$$\boxed{\alpha P^{\mathrm{E}}}=\frac{\xi}{GA}\cdot\mu\frac{\pi^2EI}{\ell^2}$$

$$=\xi\,\mu\,\pi^2\frac{E}{G}\left[\frac{I/A}{\ell}\right]^2$$

Boundary conditions effects

Material effects
Linear effect

Cross-section geometry effects
Quadratic effect

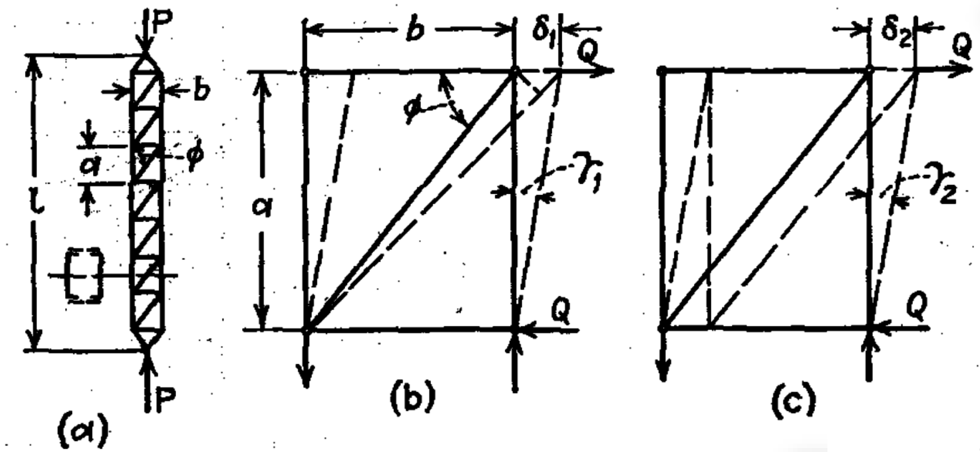
Usually the decrease of the buckling load due to transverse shear effects is negligible for bars with solid cross-section. On the contrary, for some open-cross sections, the reduction may be of 50 % even.

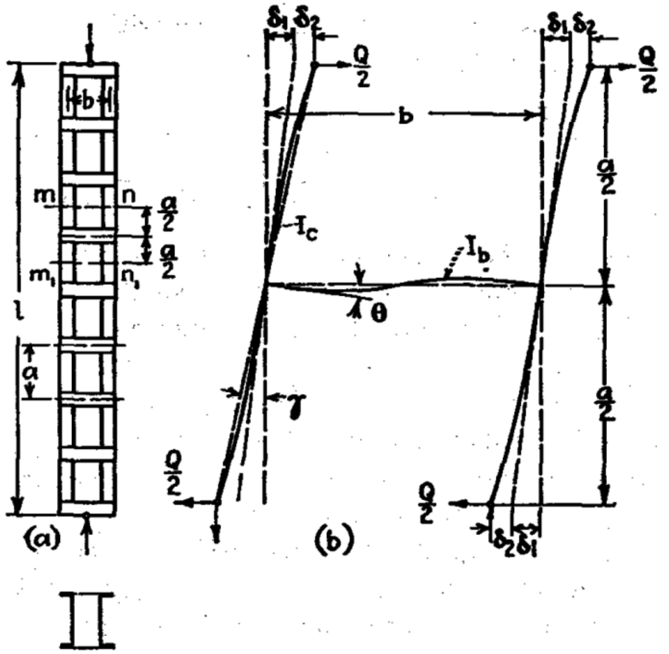
Timoshenko column

There is cases when the effect of shear deformation should be considered.

Built-in columns – 'ristikkopilari'

- Examples displayed for curiosity
- Ourdays, stability of such structures is analyzed computationally, especially because *torsional stability loss* is involved which is quite complex when not impossible to analyze theoretically

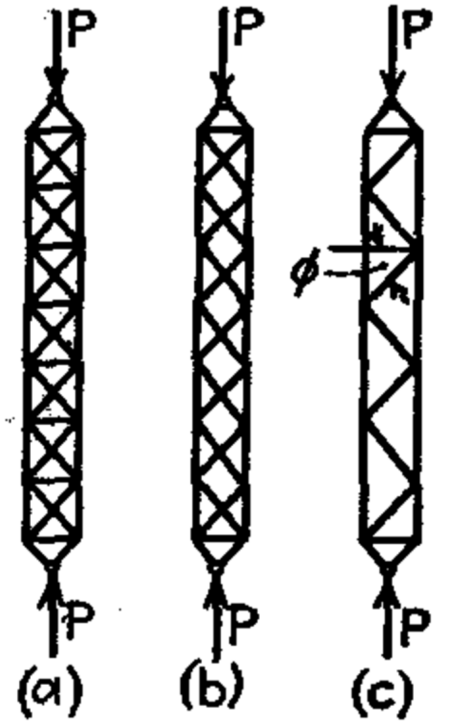




$$P_{cr} = \frac{\pi^2 EI}{l^2} \frac{1}{1 + \frac{\pi^2 EI}{l^2}\left(\frac{1}{A_d E \sin\phi \cos^2\phi} + \frac{b}{a A_b E}\right)}$$

$$P_{cr} = \frac{\pi^2 EI}{l^2} \frac{1}{1 + \frac{\pi^2 EI}{l^2}\left(\frac{ab}{12EI_b} + \frac{a^2}{24EI_c}\right)}$$

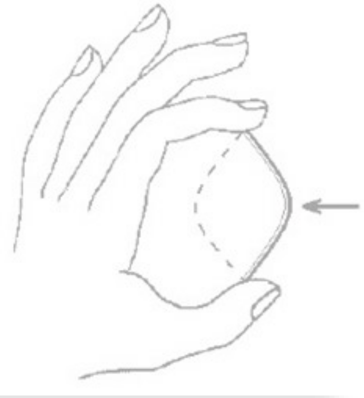
$$P_{cr} = \frac{\pi^2 EI}{l^2} \frac{1}{1 + \frac{\pi^2 EI}{l^2}\left(\frac{ab}{12EI_b} + \frac{a^2}{24EI_c} + \frac{na}{bA_bG}\right)}$$

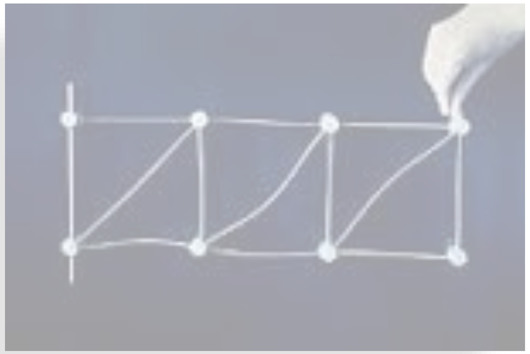


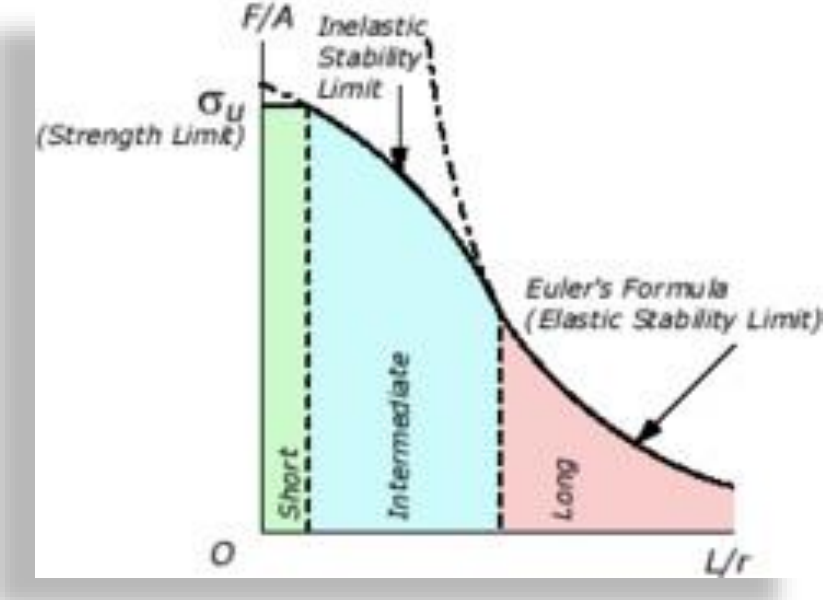
Ref. Timoshenko

Effects of imperfections

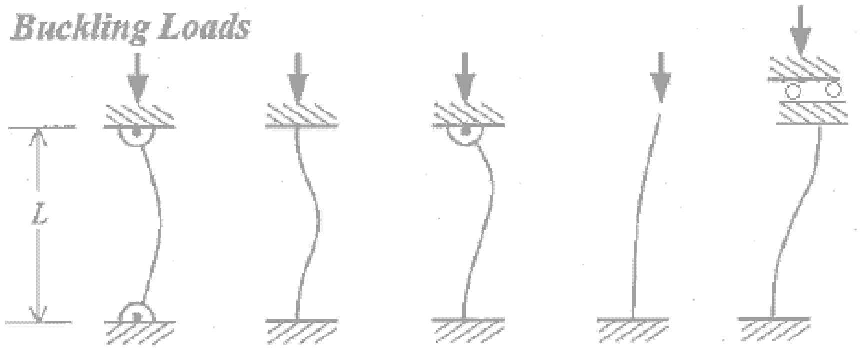
The well-known *Ayreton-Perry* design formula (Eurocode 3)







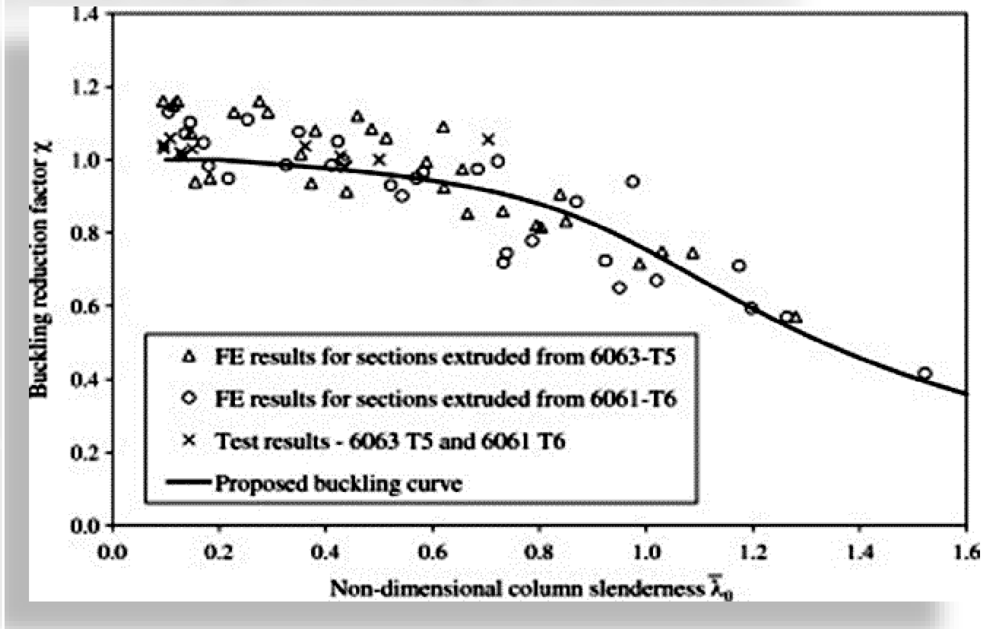
Buckling Loads



Buckling Load	$\frac{\pi^2 EI}{L^2}$	$\frac{4\pi^2 EI}{L^2}$	$\frac{2.045\pi^2 EI}{L^2}$	$\frac{\pi^2 EI}{4L^2}$	$\frac{\pi^2 EI}{L^2}$
Effective Length	L	$0.5L$	$0.699L$	$2L$	L

Slide from ” Beams and Frames – course’

Mistä nurjahduskäyrät *tulevat*?



Buckling curves

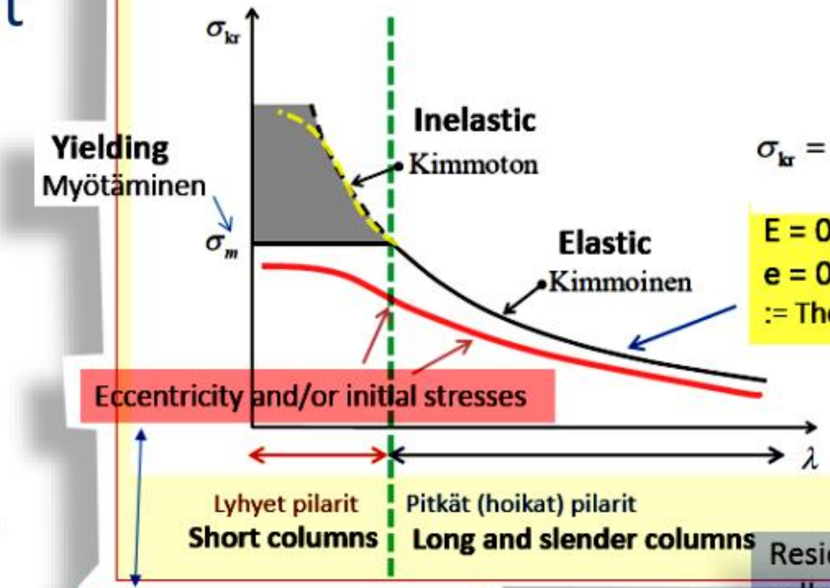
Nurjahduskäyrät

$$\sigma_{kr} = \frac{\pi^2 E}{\lambda^2},$$

Slide from ”Beams and Frames – course’

$$\sigma_{kr} = \frac{\pi^2 E}{\lambda^2},$$

Relation between buckling stress relation and slenderness
Nurjahdusjännityksen riippuvuus hoikkuusluvusta



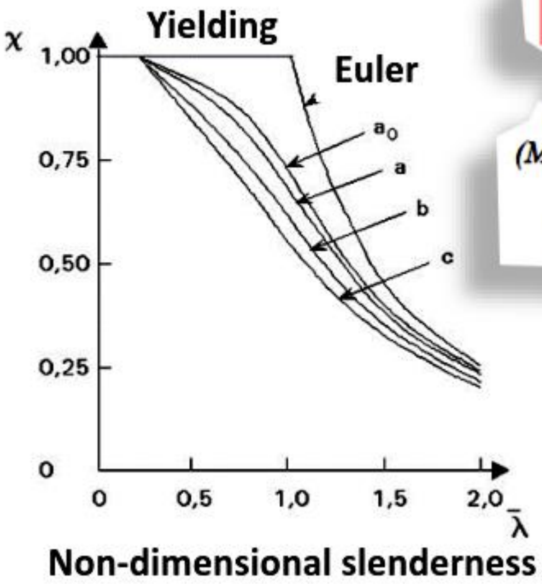
$$\sigma_{kr} = \frac{P_{kr}}{A} = \frac{\pi^2 E i_{min}^2}{l_n^2} = \mu \frac{\pi^2 E i_{min}^2}{l^2}.$$

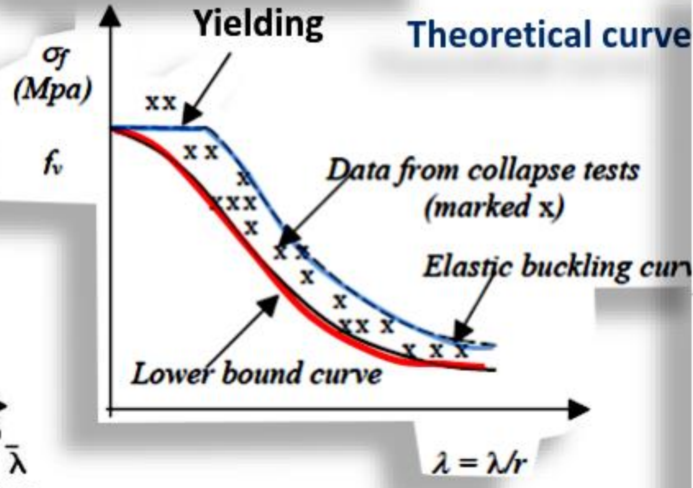
E = 0, no initial stresses
e = 0, *ei* esijännityksiä
:= The Euler’s elastic solution

$$\lambda = \frac{l_n}{i_{min}} = \frac{l}{\sqrt{\mu} \cdot i_{min}}.$$

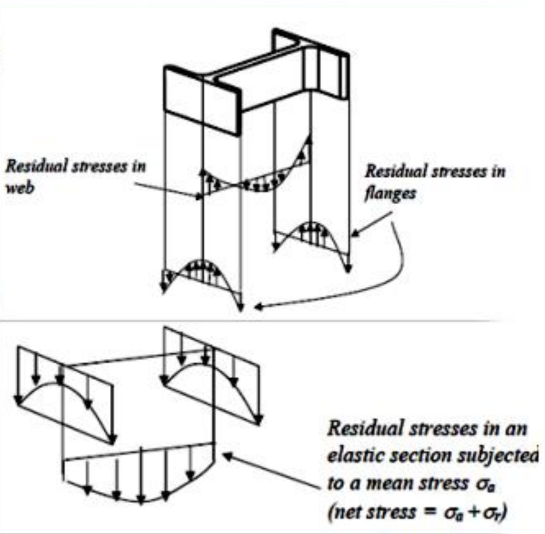
e = δ/L ja/tai esijännitys

Eurocode 3 multiple column curves (CEN, 2005)





Residual-stress distribution in rolled wide-flange shapes



Effects of imperfections

Ayreton-Perry design formula

(Eurocode 3)

6.3.1.2 Nurjahduskäyrät Buckling curves

(1) Aksiaalisesti puristetuille sauvoille muunnettua hoikkuutta $\bar{\lambda}$ vastaava pienennystekijä χ lask seuraavasta kaavasta käyttäen kyseeseentulevaa nurjahduskäyrää:

$$\chi = \frac{1}{\Phi + \sqrt{\Phi^2 - \bar{\lambda}^2}} \quad \text{mutta } \chi \leq 1{,}0$$

missä $\Phi = 0{,}5\left[1 + \alpha\left(\bar{\lambda} - 0{,}2\right) + \bar{\lambda}^2\right]$

$$\bar{\lambda} = \sqrt{\frac{Af_y}{N_{cr}}}$$ poikkileikkausluokille 1, 2 ja 3;

$$\bar{\lambda} = \sqrt{\frac{A_{eff} f_y}{N_{cr}}}$$ poikkileikkausluokalle 4;

α on epätarkkuustekijä;

N_{cr} on kimmoteorian mukainen bruttopoikkileikkauksen mukaan laskettu kriittinen voima kyseeseen tulevassa nurjahdusmuodossa.

6.3.1.1 Nurjahduskestävyys

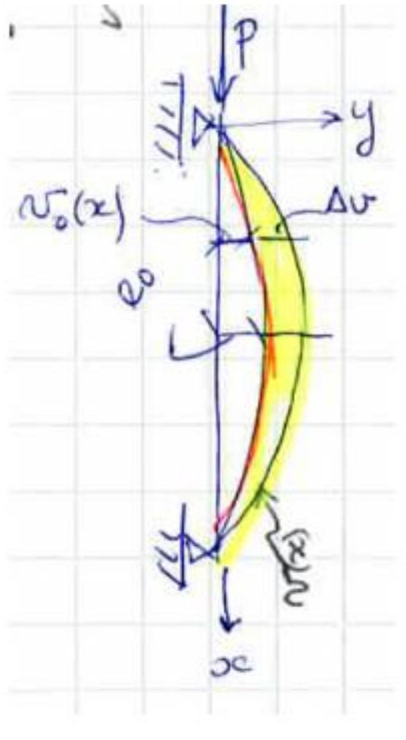
(1) Puristetut sauvat mitoitet

$$\frac{N_{Ed}}{N_{b,Rd}} \leq 1{,}0$$

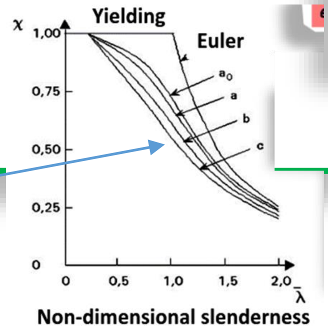
External axial load
Action

Resistance

$$N_{b,Rd} = \frac{\boxed{\chi} A f_y}{\gamma_{M1}}$$



Initial shape imperfection $w_0(x) = e_0 \sin(\pi x/\ell)$.



Nurjahduskäyrä	a_0	a	b	c	d
Epätarkkuustekijä α	0,13	0,21	0,34	0,49	0,76

Effects of imperfections

The well-known *Ayreton-Perry* design formula (Eurocode 3)

initial shape imperfection $w_0(x) = e_0 \sin(\pi x/\ell)$

$$(EIv'')'' + Pv'' = 0$$
& four boundary conditions.

$\Longrightarrow$

Eccentricity

$$w(x) = \frac{e_0}{1-(\lambda/\pi)^2}\sin(\pi x/\ell), \qquad \lambda^2 = \frac{P\ell^2}{EI},$$

$$\sigma_x^{max} = \frac{N_{max}}{A} + \frac{M_{max}}{W} \leq \sigma_y$$
$$\Downarrow \qquad = \frac{P}{A} + \frac{M_{max}}{I}\frac{h}{2} \leq \sigma_y.$$

$\Longrightarrow$

Ayrton-Perry formula (of Eurocode 3)

$$\bar{\lambda} = \sqrt{A\sigma_y/N_E}.$$

$$\boxed{\chi = \frac{1}{\phi + \sqrt{\phi^2 - \bar{\lambda}^2}}, \quad \text{where } \phi = \frac{1}{2}\left[1 + a\bar{\lambda} + \bar{\lambda}^2\right]}$$

Depends on *eccentricity*

$$M_{max} = M(\ell/2) = -EI(v''(\ell/2) - v_0''(\ell/2)),$$
$$= P_{cr}e_0\frac{(\lambda/\pi)^2}{1-(\lambda/\pi)^2},$$
$$\Downarrow \qquad = P_{cr}e_0\frac{P/P_{cr}}{1-P/P_{cr}}.$$

$$a\bar{\lambda} = [e_0 h/2]/i^2$$
$$a = \pi\sqrt{E/\sigma_y}\frac{e_0}{\ell}\frac{h/2}{i}$$
$$\chi = P/P_y.$$

... and obtains:

$$\sigma_y A = P + P_{cr}e_0\frac{h/2}{I/A}\frac{P/P_{cr}}{1-P/P_{cr}} \Longrightarrow$$
$$\frac{P_y}{P_{cr}} = \frac{P}{P_{cr}} + \frac{e_0 h/2}{i^2}\frac{P/P_{cr}}{1-P/P_{cr}}.$$

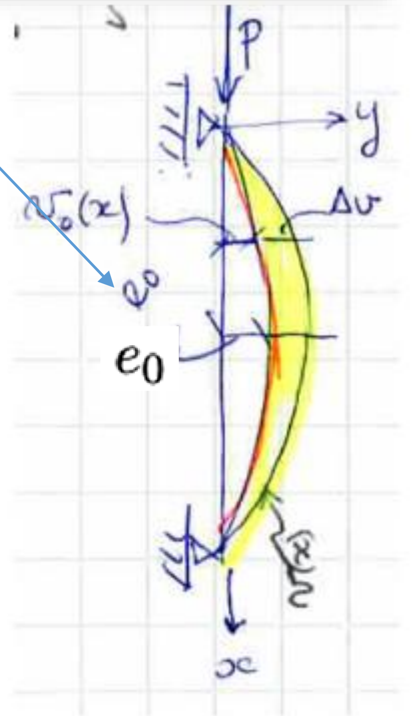
$$\bar{\lambda}^2 = a\bar{\lambda}\frac{\chi\bar{\lambda}^2}{1-\chi\bar{\lambda}^2} + \chi\bar{\lambda}^2 \Longrightarrow$$
$$\frac{1}{\chi} = a\bar{\lambda}\frac{1/\chi}{1/\chi - \bar{\lambda}^2} + 1$$
$$\Rightarrow \frac{1}{2}\frac{1}{\chi^2} - \underbrace{\frac{1}{2}\left[1 + a\bar{\lambda} + \bar{\lambda}^2\right]}_{\equiv\phi}\frac{1}{\chi} + \frac{1}{2}\bar{\lambda}^2 = 0,$$

Solve ϕ from this:

Design formula:

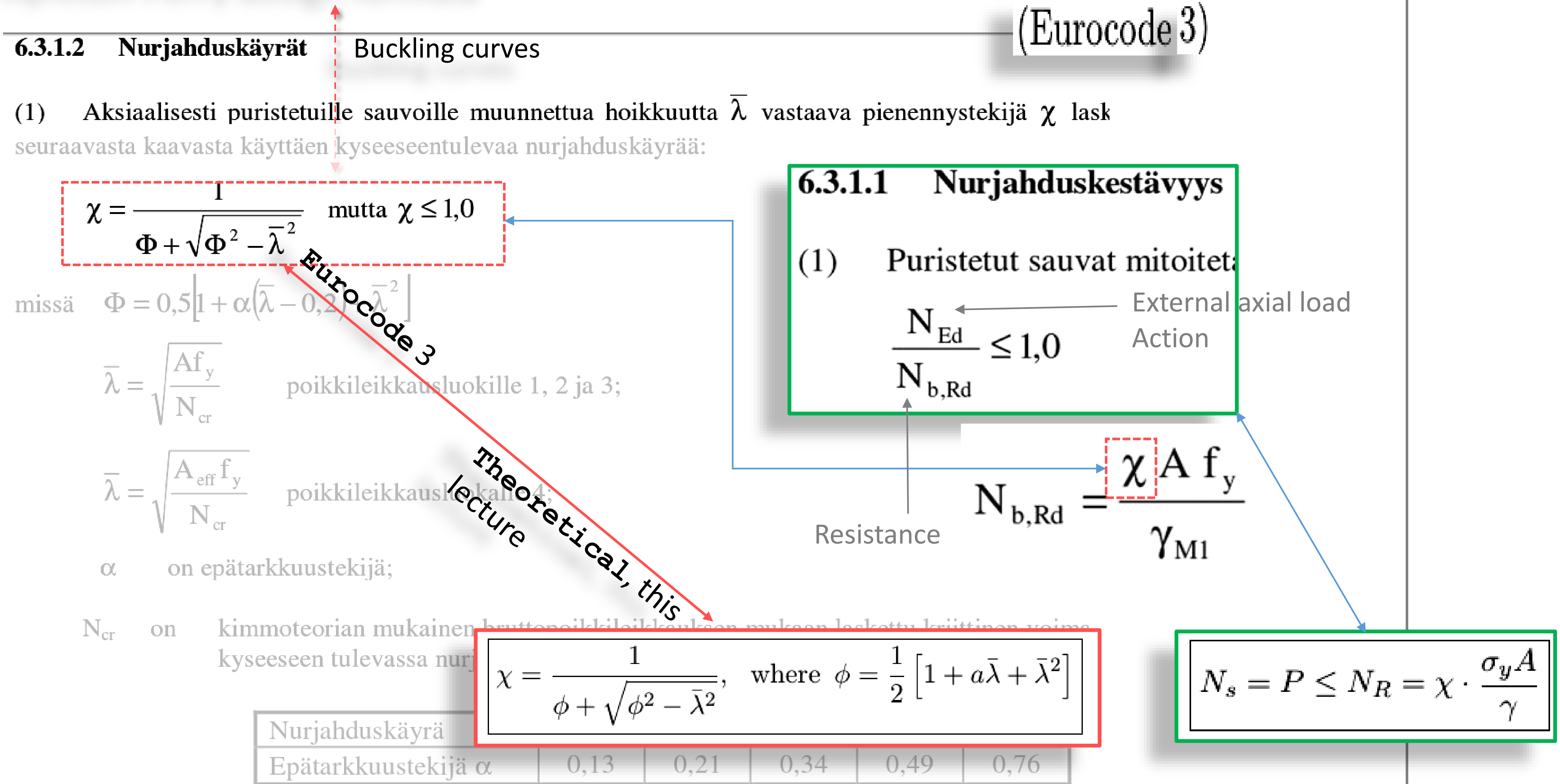
$$\boxed{N_s = P \leq N_R = \chi\sigma_y A.}$$

where $\bar{\lambda}$ is the column relative slenderness. How this formula is used in design of a steel column? Consider a steel section in compression with P. All the possible imperfections are now accounted through this buckling resistance reduction factor χ such that the inequality



Initial shape imperfection $w_0(x) = e_0 \sin(\pi x/\ell)$.

Ayreton-Perry design formula



(Eurocode 3)

6.3.1.2 Nurjahduskäyrät Buckling curves

(1) Aksiaalisesti puristetuille sauvoille muunnettua hoikkuutta $\bar{\lambda}$ vastaava pienennystekijä χ lask

seuraavasta kaavasta käyttäen kyseeseentulevaa nurjahduskäyrää:

$$\chi = \frac{1}{\Phi + \sqrt{\Phi^2 - \bar{\lambda}^2}} \quad \text{mutta } \chi \le 1{,}0$$

Eurocode 3

missä $\Phi = 0{,}5\left[1 + \alpha\left(\bar{\lambda} - 0{,}2\right) + \bar{\lambda}^2\right]$

$\bar{\lambda} = \sqrt{\frac{Af_y}{N_{cr}}}$ poikkileikkausluokille 1, 2 ja 3;

$\bar{\lambda} = \sqrt{\frac{A_{eff} f_y}{N_{cr}}}$ poikkileikkausluokalle 4;

α on epätarkkuustekijä;

N_{cr} on kimmoteorian mukainen bruttopoikkileikkauksen mukaan laskettu kriittinen voima kyseeseen tulevassa nurj

Theoretical, this lecture

$$\chi = \frac{1}{\phi + \sqrt{\phi^2 - \bar{\lambda}^2}}, \quad \text{where } \phi = \frac{1}{2}\left[1 + a\bar{\lambda} + \bar{\lambda}^2\right]$$

Nurjahduskäyrä					
Epätarkkuustekijä α	0,13	0,21	0,34	0,49	0,76

6.3.1.1 Nurjahduskestävyys

(1) Puristetut sauvat mitoiteta

$$\frac{N_{Ed}}{N_{b,Rd}} \le 1{,}0$$

External axial load

Action

Resistance

$$N_{b,Rd} = \frac{\boxed{\chi} A\, f_y}{\gamma_{M1}}$$

$$N_s = P \le N_R = \chi \cdot \frac{\sigma_y A}{\gamma}$$

Ayreton-Perry design formula

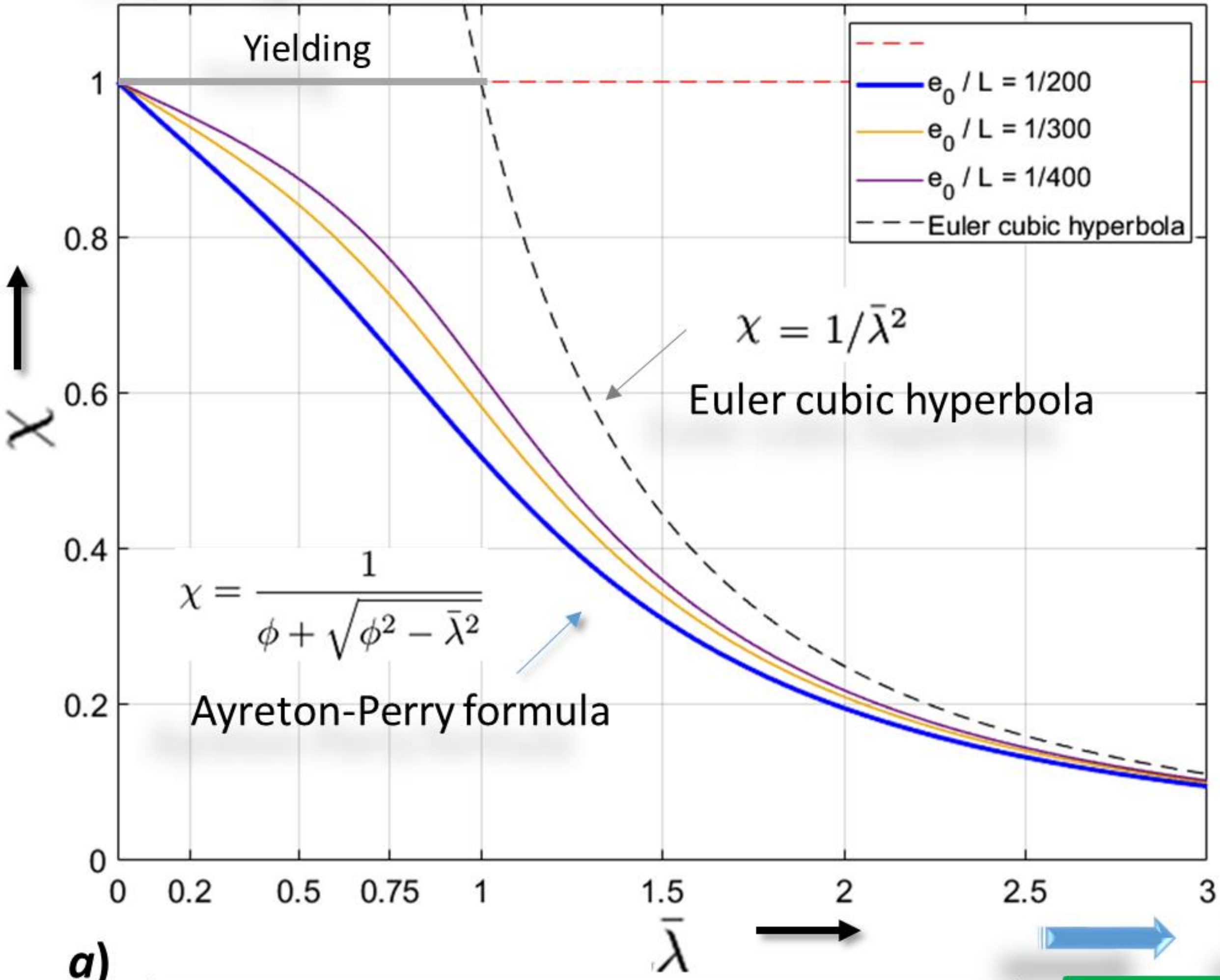
Steel $\ell = 2\text{m},$

$a = [0.446,\ 0.297,\ 0.223],$

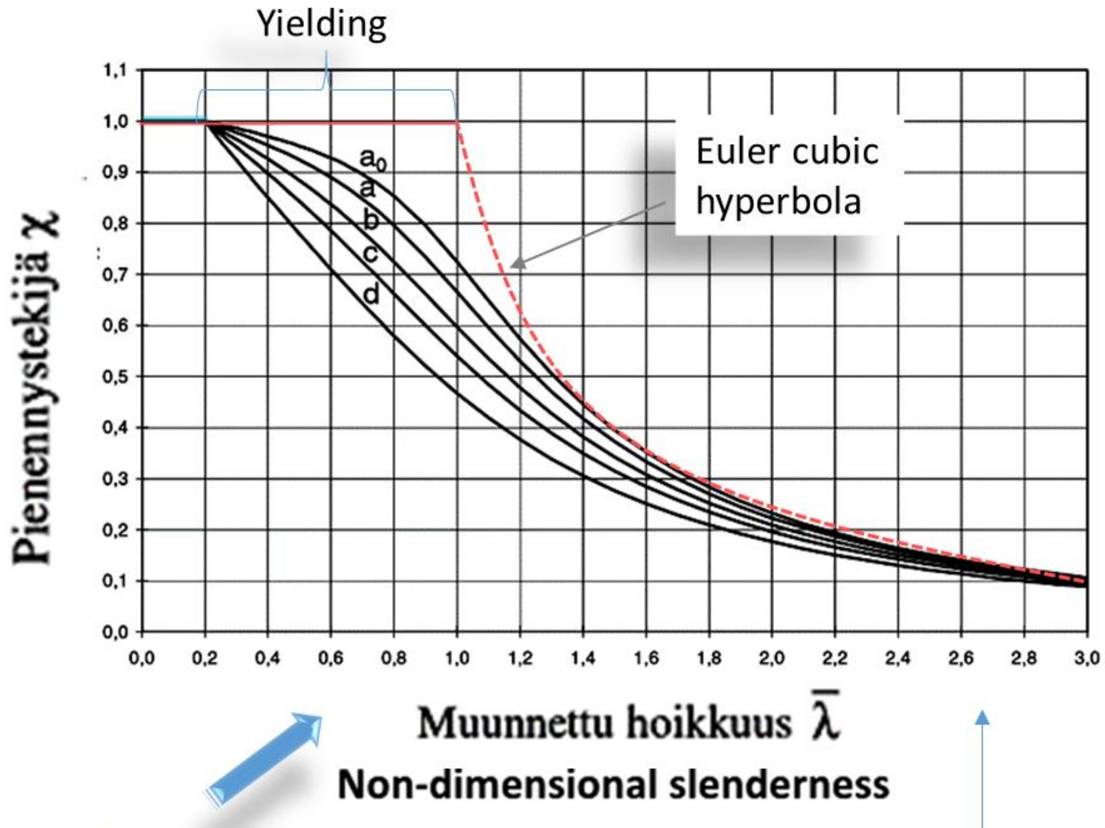
$i = 0.1714\ \text{m},\ \ h = 0.2\ \text{m},$

$e_0/\ell = [1/400,\ 1/300,\ 1/400]$

Buckling curves



Eurocode buckling curves



a)

$$\chi = \frac{1}{\phi + \sqrt{\phi^2 - \bar{\lambda}^2}}, \quad \text{where } \phi = \frac{1}{2}\left[1 + a\bar{\lambda} + \bar{\lambda}^2\right]$$

$$\boxed{N_s = P \leq N_R = \chi \cdot \frac{\sigma_y A}{\gamma}}$$

b)

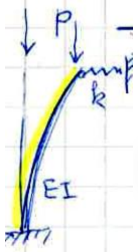
$$a\bar{\lambda} = [e_0 h/2]/i^2$$

$$a = \pi\sqrt{E/\sigma_y}\frac{e_0}{\ell}\frac{h/2}{i}$$

$$\chi = P/P_y.$$

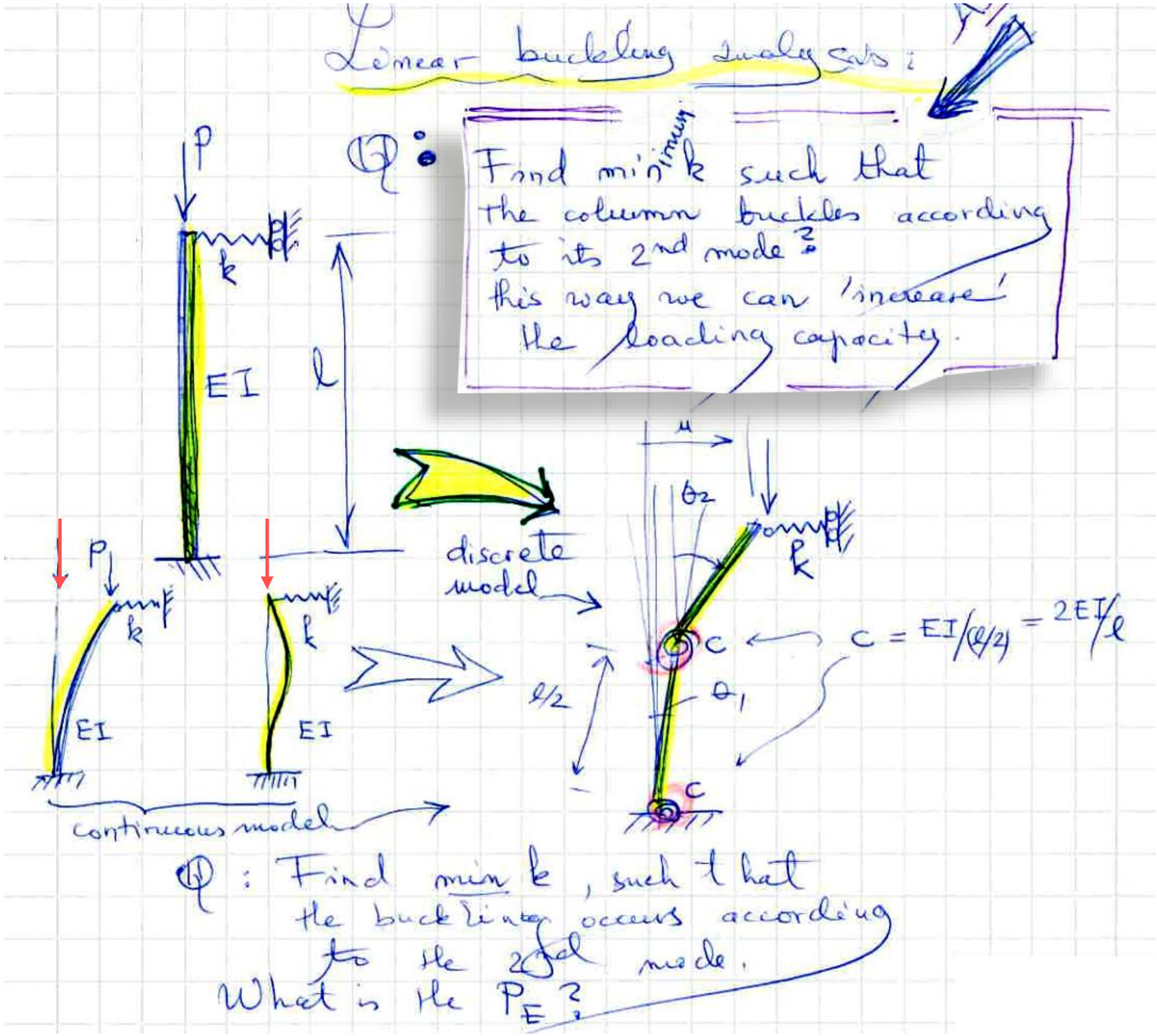
Adapted from Eurocode 3

Example of a design problem



1st mode

2nd mode



Linear buckling analysis:

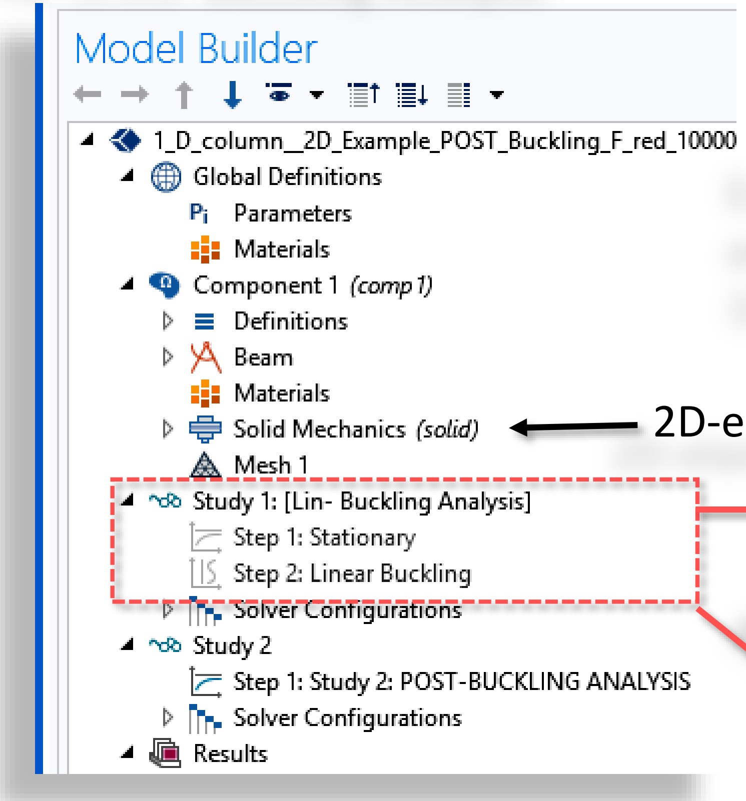
Q: Find min(imum) k such that the column buckles according to its 2nd mode?
this way we can 'increase' the loading capacity.

Q: Find min k, such that the buckling occurs according to the 2nd mode.
What is the P_E?

Ref: example adapted from RK.

Linear buckling analysis of simply supported column

FE- Linear **Buckling Analysis**



Model Builder

- 1_D_column__2D_Example_POST_Buckling_F_red_10000
  - Global Definitions
    - Parameters
    - Materials
  - Component 1 *(comp1)*
    - Definitions
    - Beam
    - Materials
    - Solid Mechanics *(solid)*
    - Mesh 1
  - Study 1: [Lin- Buckling Analysis]
    - Step 1: Stationary
    - Step 2: Linear Buckling
    - Solver Configurations
  - Study 2
    - Step 1: Study 2: POST-BUCKLING ANALYSIS
    - Solver Configurations
  - Results

2D-elasticity

Euler analytical 1D

Buckling load

- In this FE-analysis, the column was treated as a two-dimensional elastic domain

rectangular cross-section

$$P_{E,1D} = \pi^2 EI/\ell^2.$$

$$\text{height } h = 50 \text{ mm},$$

$$\text{width } b = \ell/10,$$

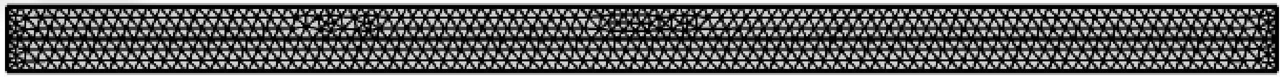
$$EI = 72.917 \text{ kN.m}^2$$

$$E = 70 \text{ GPa}, \nu = 0.33.$$

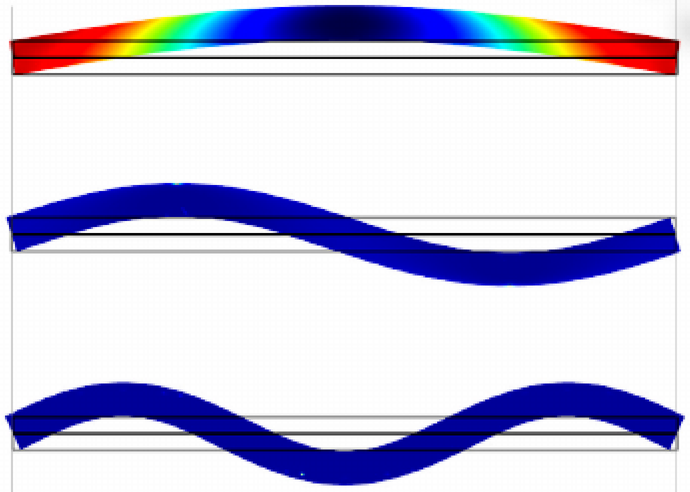
$$\boxed{P_{cr,2D} = 713 \text{ kN}}$$

$$P_{E,1D}/P_{cr,2D} = 1.01,$$

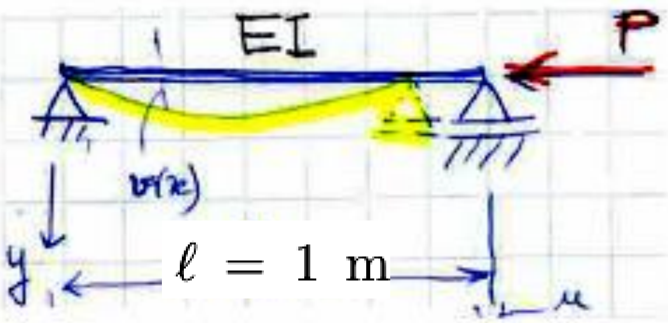
FE-Mesh



FE- Linear **Buckling Analysis**



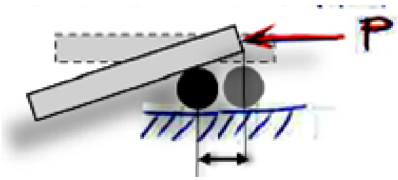
[FE-buckling analysis]
First three critical loads and respective buckling modes



P_E = 719.66 kN (analytical 1D)

$$\boxed{P_E = 720 \text{ kN (1-D)},}$$

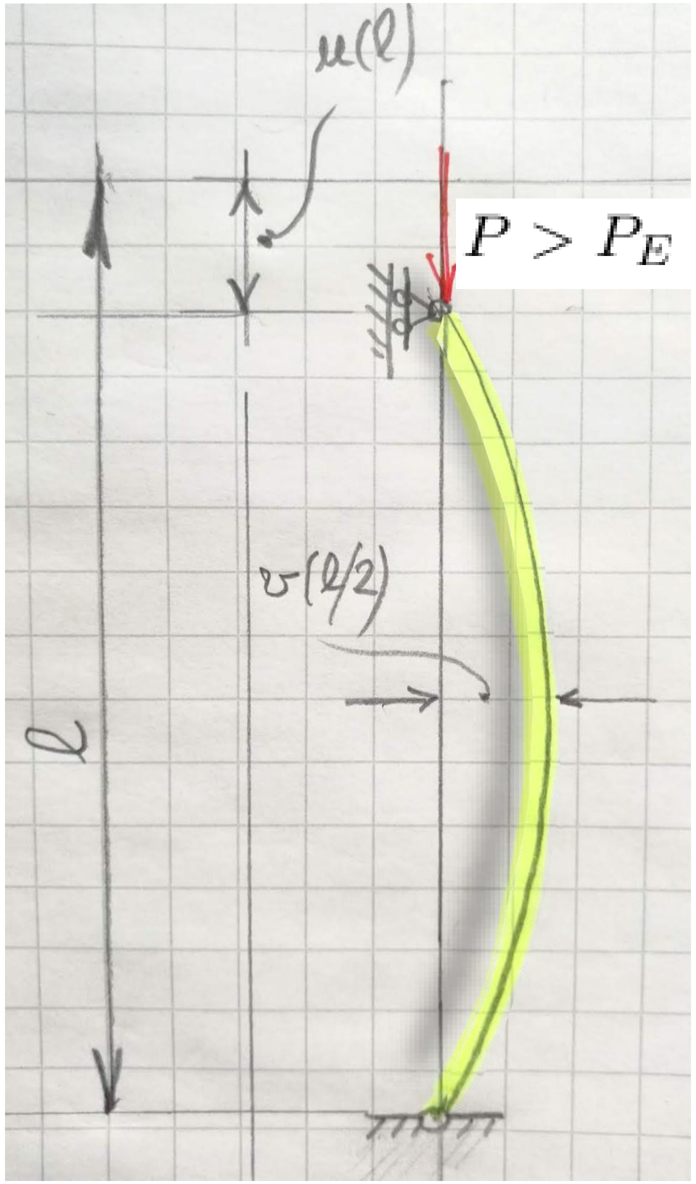
Asymptotic post-buckling analysis of simply supported column



Roller 'buckling' displacement.

$$\kappa = -\frac{v''}{\sqrt{1-v'^2}}$$

$$\mathrm{d}u = [1-\sqrt{1-v'^2}]\mathrm{d}x$$



Asymptotic post-buckling analysis of simply supported column

What to do: at buckling & for moderate increments

- ✓ estimate the **displacements**/rotation
- ✓ Study **stability** of post-buckling branch

- analytical approach is used

load increase $P = P_E + \Delta P$

few percent

How to do it?

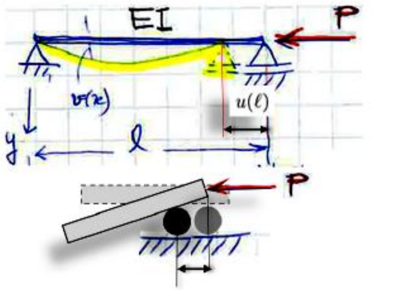
- we use the Lagrangian formulation
- assume a (bifurcational) flexural deflection mode

$$v(x) = v_0 \sin(\pi x/\ell).$$

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^\ell EI\kappa^2 \mathrm{d}x - P\int_0^\ell \left[1-\sqrt{1-(v')^2}\right]\mathrm{d}x,}$$

Lagrangian curvature $\kappa = -\frac{v''}{\sqrt{1-v'^2}}$

Shortening due to flexion $\mathrm{d}u/\mathrm{d}x = 1-\sqrt{1-(v')^2}$



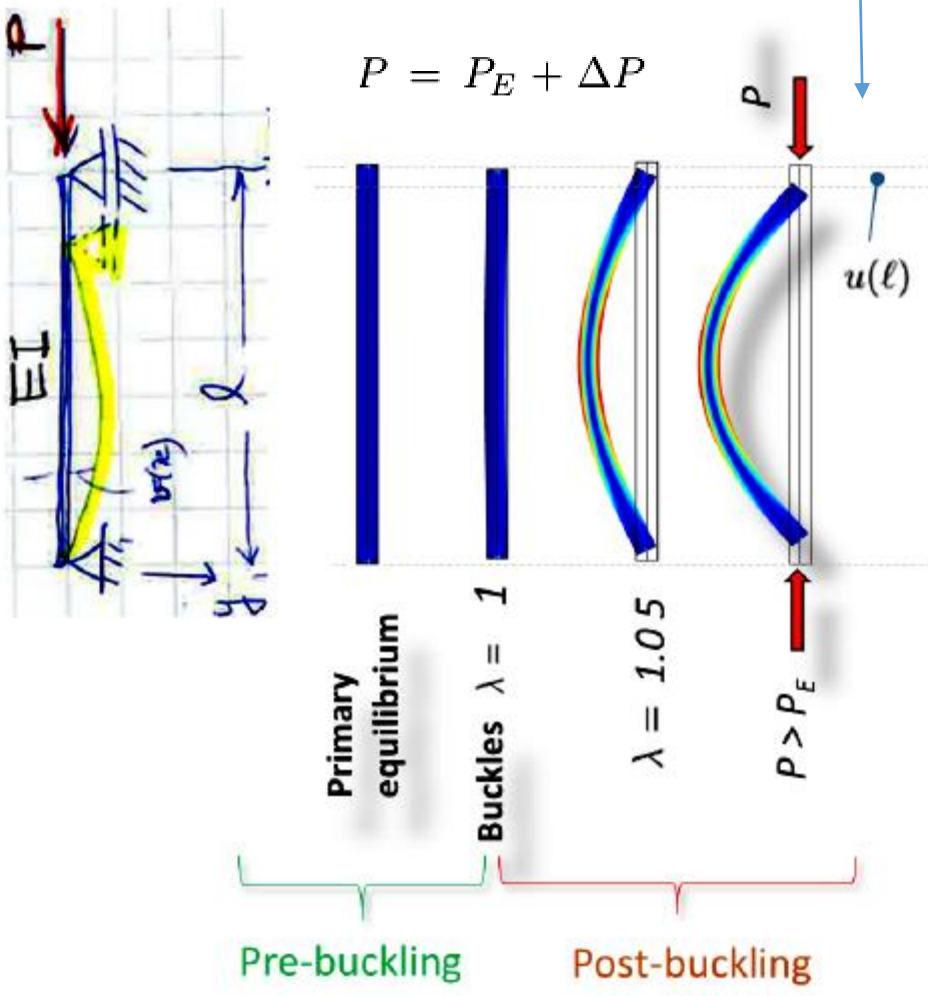
$$u(\ell) \approx \frac{\pi^2\ell}{4}\left(\frac{v_0}{\ell}\right)^2 + \frac{P_E\ell}{EA}$$

$$u(\ell) \approx \frac{\ell}{2}\left(\frac{P}{P_E}-1\right)\cdot(P \geq P_E) + \frac{P_E\ell}{EA}$$

Roller 'buckling' displacement.

Derive the force-displacement relation

FE- Post-Buckling Analysis



$\lambda = P/P_E$ **loading increases**

Post-buckling of simply supported column.

Asymptotic post-buckling analysis of simply supported column

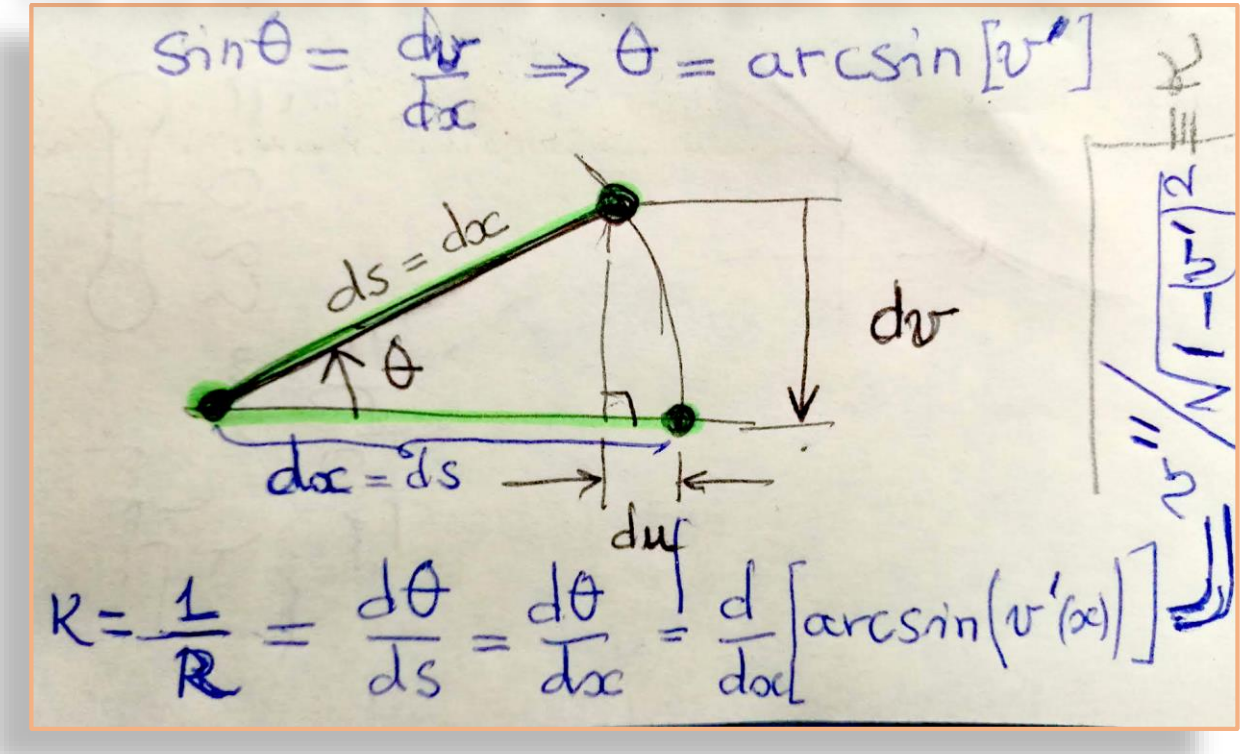
- we use the Lagrangian formulation
- assume a (bifurcational) flexural deflection mode

$$\Delta\Pi = \frac{1}{2}\int_0^{\ell} EI\kappa^2 \mathrm{d}x - P\underbrace{\int_0^{\ell}\left[1-\sqrt{1-(v')^2}\right]\mathrm{d}x,}_{\text{Shortening due to bending}}$$

Lagrangian curvature

$$\kappa = -\frac{v''}{\sqrt{1-v'^2}}$$

The curvature in the Lagrangian formulation:



The minus sign is because of sign convention for positive curvature

$$\kappa = -\frac{v''}{\sqrt{1-v'^2}}$$

From the right-angle triangle (1.74) and using Pythagoras one obtains the shortening

$$\mathrm{d}u = [1-\sqrt{1-v'^2}]\mathrm{d}x$$

Asymptotic post-buckling analysis of simply supported column

What to do: at buckling & for moderate increments

- ✓ estimate the **displacements**/rotation
- ✓ Study **stability** of post-buckling branch

$\Longrightarrow$ Derive the force-displacement relation

How to do it?

- we use the **Lagrangian** formulation
- assume a (bifurcational) flexural deflection mode

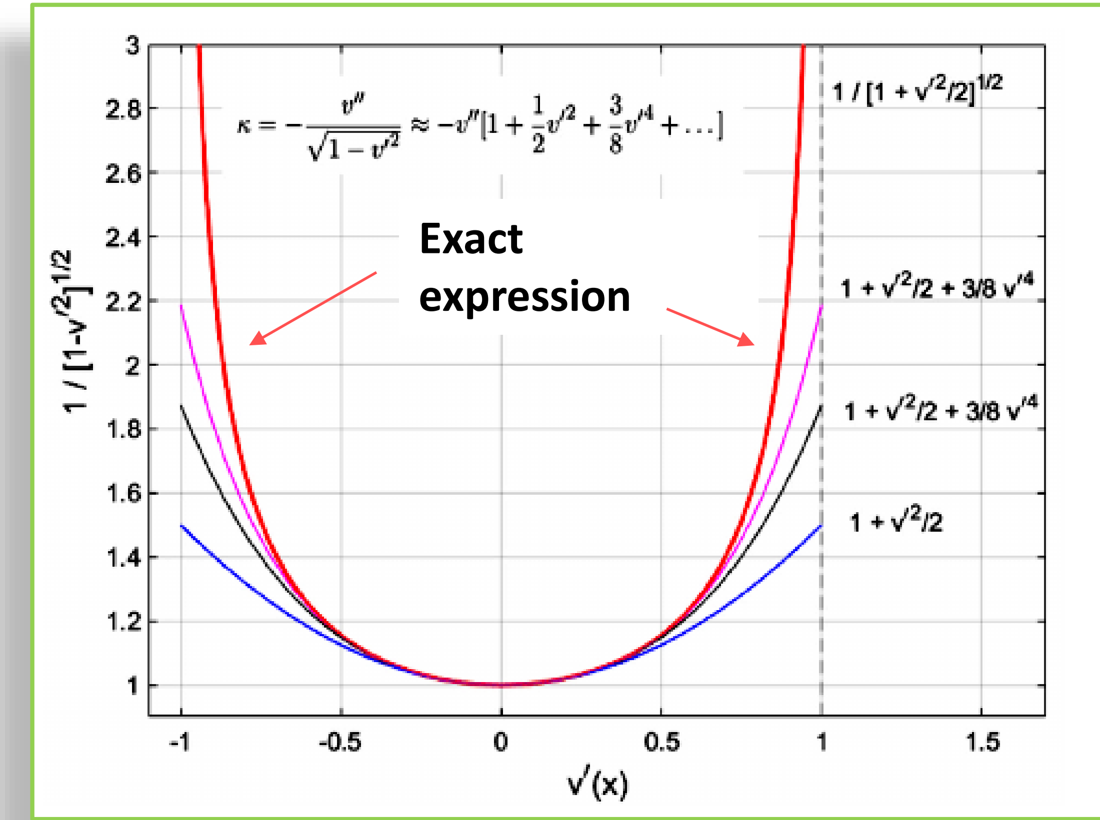
$$v(x) = v_0 \sin(\pi/x\ell)$$

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^\ell EI\kappa^2 \mathrm{d}x - P\underbrace{\int_0^\ell \left[1-\sqrt{1-(v')^2}\right]\mathrm{d}x}_{\text{Shortening due to flexion}},}$$

Lagrangian curvature

$$\kappa = -\frac{v''}{\sqrt{1-v'^2}} \approx -v''[1+\frac{1}{2}v'^2+\frac{3}{8}v'^4+\ldots]$$

$$\mathrm{d}u/\mathrm{d}x = 1-\sqrt{1-(v')^2} \approx 1-[1-\frac{1}{2}v'^2] = \frac{1}{2}v'^2$$



Taylor expansions

Taylor expansions with only two terms

$$\Longrightarrow \quad \boxed{\Delta\Pi \approx \frac{1}{2}\int_0^\ell EIv''^2[1+\frac{1}{2}v'^2]^2\mathrm{d}x - \frac{1}{2}P\int_0^\ell (v')^2\mathrm{d}x,}$$

Asymptotic post-buckling analysis of simply supported column

Assume a (bifurcational) flexural deflection mode $v(x) = v_0 \sin(\pi x/\ell)$

$$\boxed{\Delta\Pi \approx \frac{1}{2}\int_0^\ell EIv''^2[1+\frac{1}{2}v'^2]^2 \mathrm{d}x - \frac{1}{2}P\int_0^\ell (v')^2 \mathrm{d}x,}$$

$$\Delta\Pi = -\frac{\pi^2}{4}P\ell\left(\frac{v_0}{\ell}\right)^2 + \frac{\pi^2 EI}{\ell^2}\cdot\frac{\pi^2}{128}\left(\frac{v_0}{\ell}\right)^2\cdot\ell\left[32+8\pi^2\left(\frac{v_0}{\ell}\right)^2+\pi^4\left(\frac{v_0}{\ell}\right)^4\right]$$

$$\Downarrow \quad = -\frac{\pi^2}{4}P\ell\delta^2 + P_E\cdot\frac{\pi^2\ell}{128}\delta^2\left[32+8\pi^2\delta^2+\pi^4\delta^4\right] \equiv \Delta\Pi(\delta,\lambda;\ell),$$

$$\boxed{\delta(\Delta\Pi(v_0;P)) = 0 \implies \mathrm{d}\Delta\Pi(v_0;P)/\mathrm{d}v_0 = 0 \implies}$$

$$\Downarrow \quad \implies P = \frac{\pi^2 EI}{\ell^2} + \frac{1}{2}\frac{\pi^2 EI}{\ell^2}\cdot\pi^2\left(\frac{v_0}{\ell}\right)^2 + \frac{3}{32}\frac{\pi^2 EI}{\ell^2}\cdot\pi^4\left(\frac{v_0}{\ell}\right)^4$$

$$P = P_E\left[1+\frac{1}{2}\cdot\pi^2\left(\frac{v_0}{\ell}\right)^2+\frac{3}{32}\cdot\pi^4\left(\frac{v_0}{\ell}\right)^4\right].$$

The asymptotic force-displacement relation

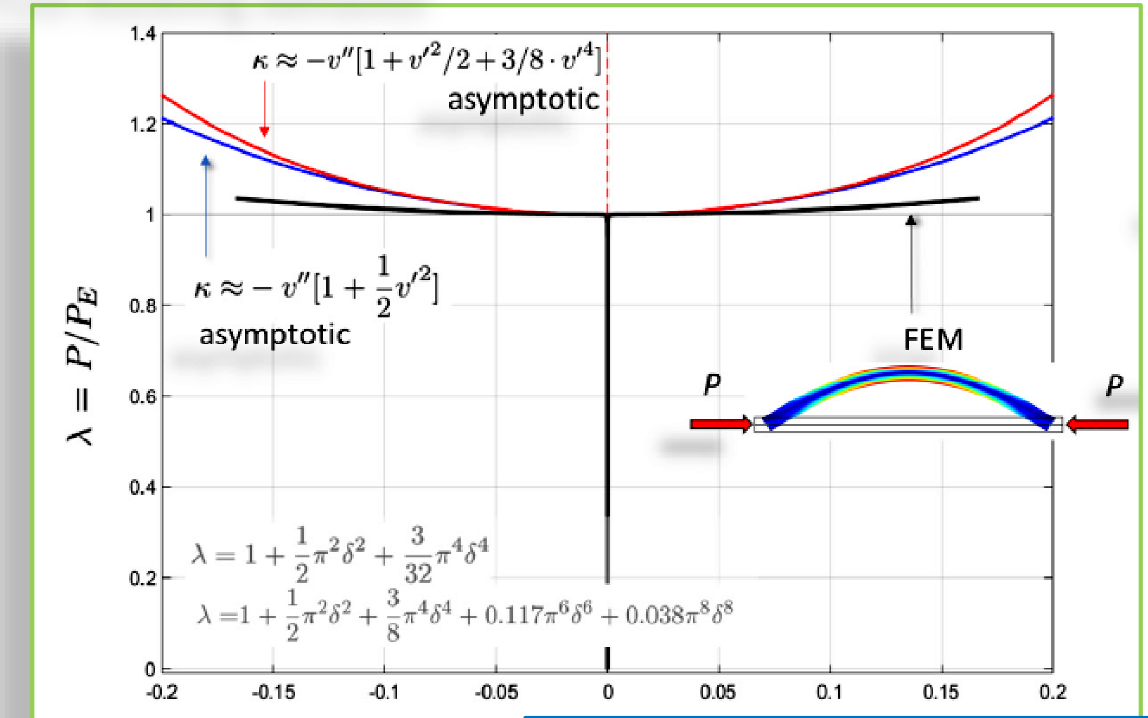
$$\boxed{\lambda \approx 1 + \frac{1}{2}\pi^2\delta^2 + \frac{3}{32}\pi^4\delta^4 = 1 + \frac{1}{2}\pi^2\delta^2\left[1+\frac{2\cdot 3}{32}\pi^2\delta^2\right]}$$

$\delta = v_0/\ell \qquad \lambda = P/P_E$

Taylor expansions with only two terms

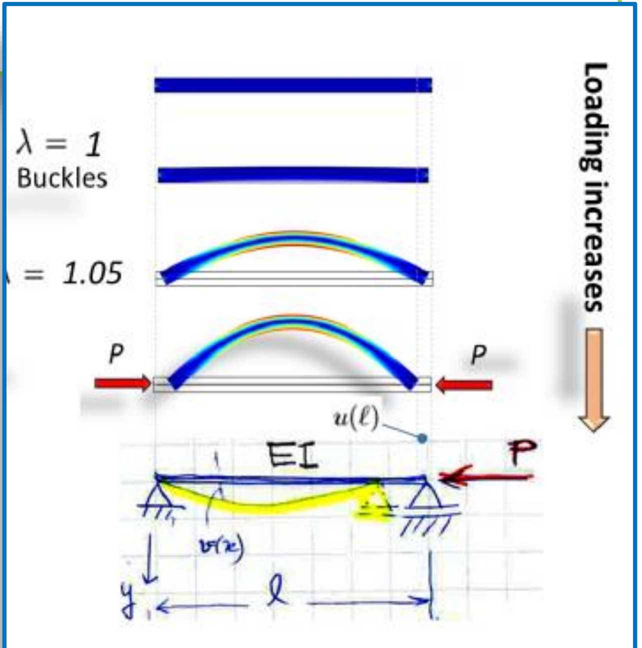
Matlab symbolic toolbox,

Post-buckling behavior



$\delta = v_0/\ell \longrightarrow$

FE-Post-buckling behavior ⟶

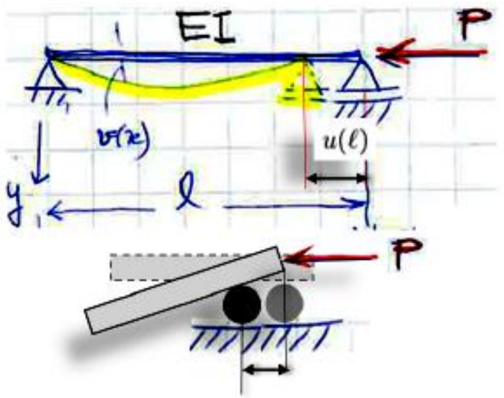


Asymptotic post-buckling analysis of simply supported column

The asymptotic post-buckling analysis provides also the value of column shortening and rotations at buckling

$$u(\ell) \approx \frac{\ell}{2}\left(\frac{P}{P_E}-1\right)\cdot(P \geq P_E)+\frac{P_E \ell}{EA},$$

logical proposition $(P \geq P_E) = 1$ when `true`, otherwise, zero.



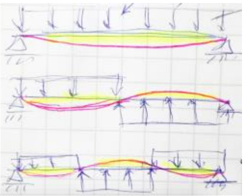
$$u(\ell) \approx \frac{\pi^2 \ell}{4}\left(\frac{v_0}{\ell}\right)^2+\frac{P_E \ell}{EA}$$

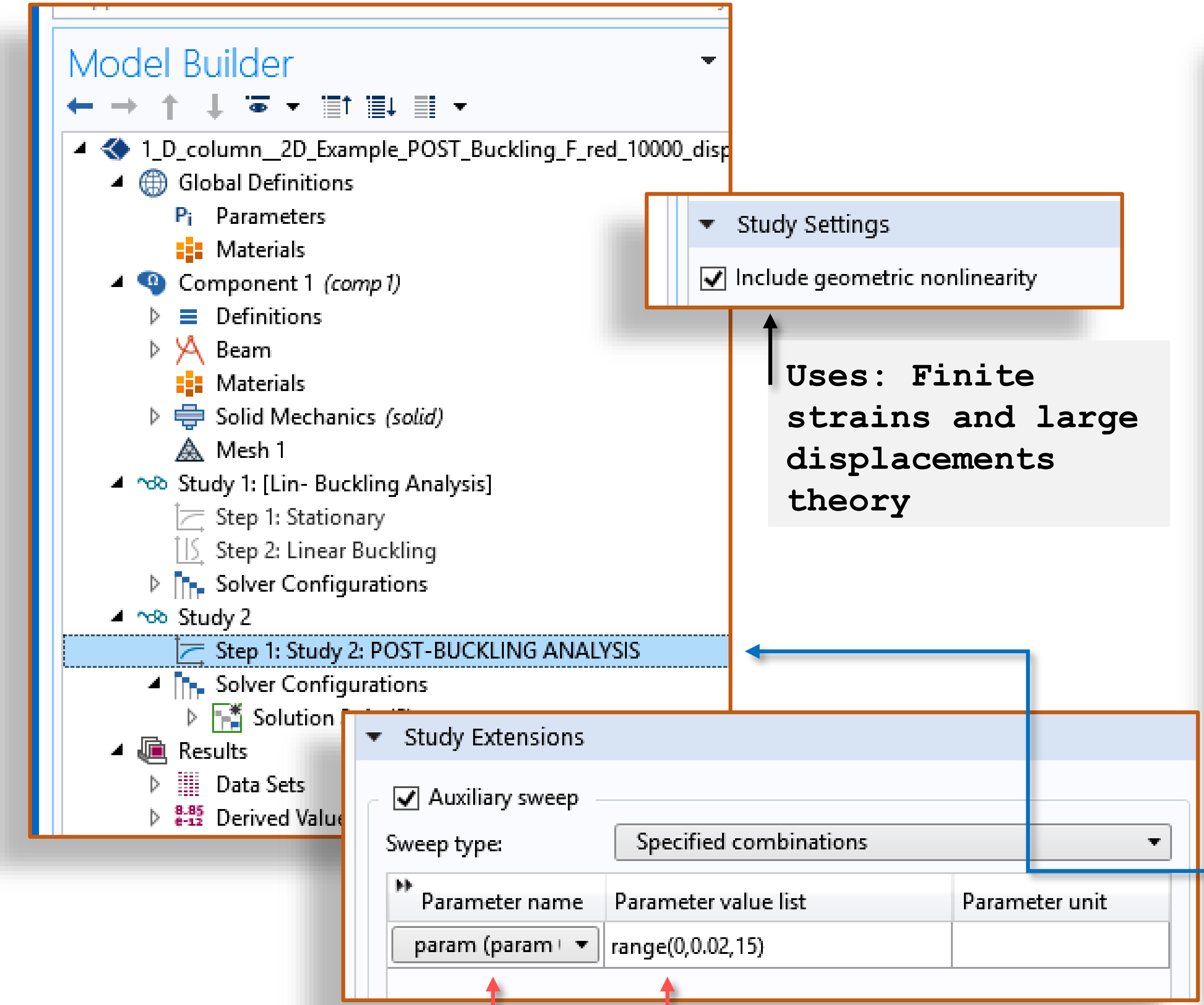
$$u(\ell) \approx \frac{\ell}{2}\left(\frac{P}{P_E}-1\right)\cdot(P \geq P_E)+\frac{P_E \ell}{EA}$$

Roller ’buckling’ displacement.

FE-based post-buckling analysis of axially compressed column

- Perturbed with tiny transversal distributed load
- Can also be given as initial shape imperfection





Model Builder

- 1_D_column__2D_Example_POST_Buckling_F_red_10000_disp
  - Global Definitions
    - Parameters
    - Materials
  - Component 1 *(comp1)*
    - Definitions
    - Beam
    - Materials
    - Solid Mechanics *(solid)*
    - Mesh 1
  - Study 1: [Lin- Buckling Analysis]
    - Step 1: Stationary
    - Step 2: Linear Buckling
    - Solver Configurations
  - Study 2
    - Step 1: Study 2: POST-BUCKLING ANALYSIS
    - Solver Configurations
      - Solution
  - Results
    - Data Sets
    - Derived Value

Study Settings

☑ Include geometric nonlinearity

`Uses: Finite strains and large displacements theory`

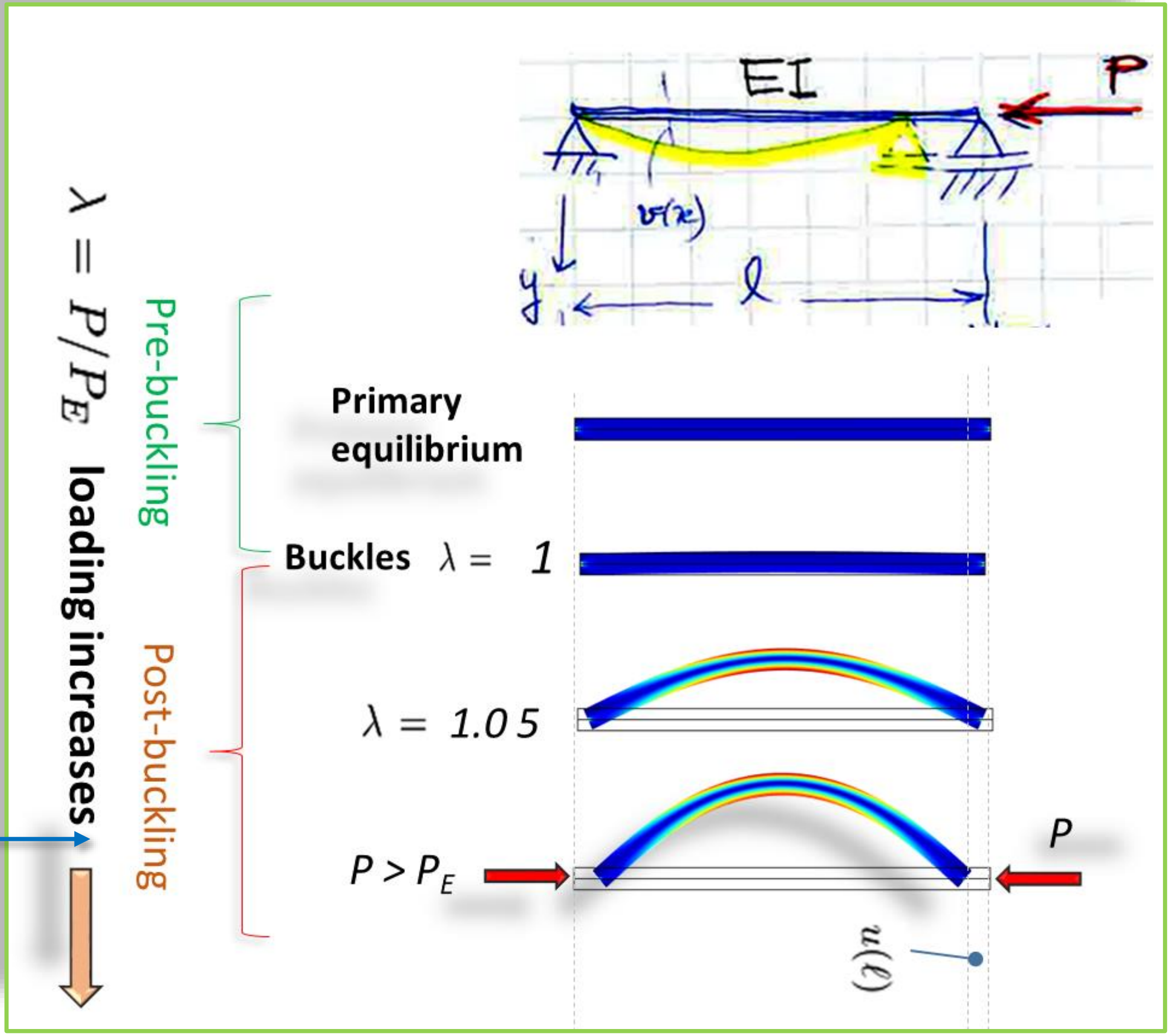
Study Extensions

☑ Auxiliary sweep

Sweep type: Specified combinations

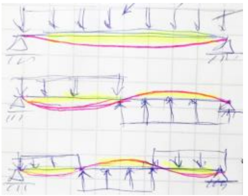
Parameter name	Parameter value list	Parameter unit
param (param	range(0,0.02,15)	

$$\boxed{\lambda = P/P_E}$$

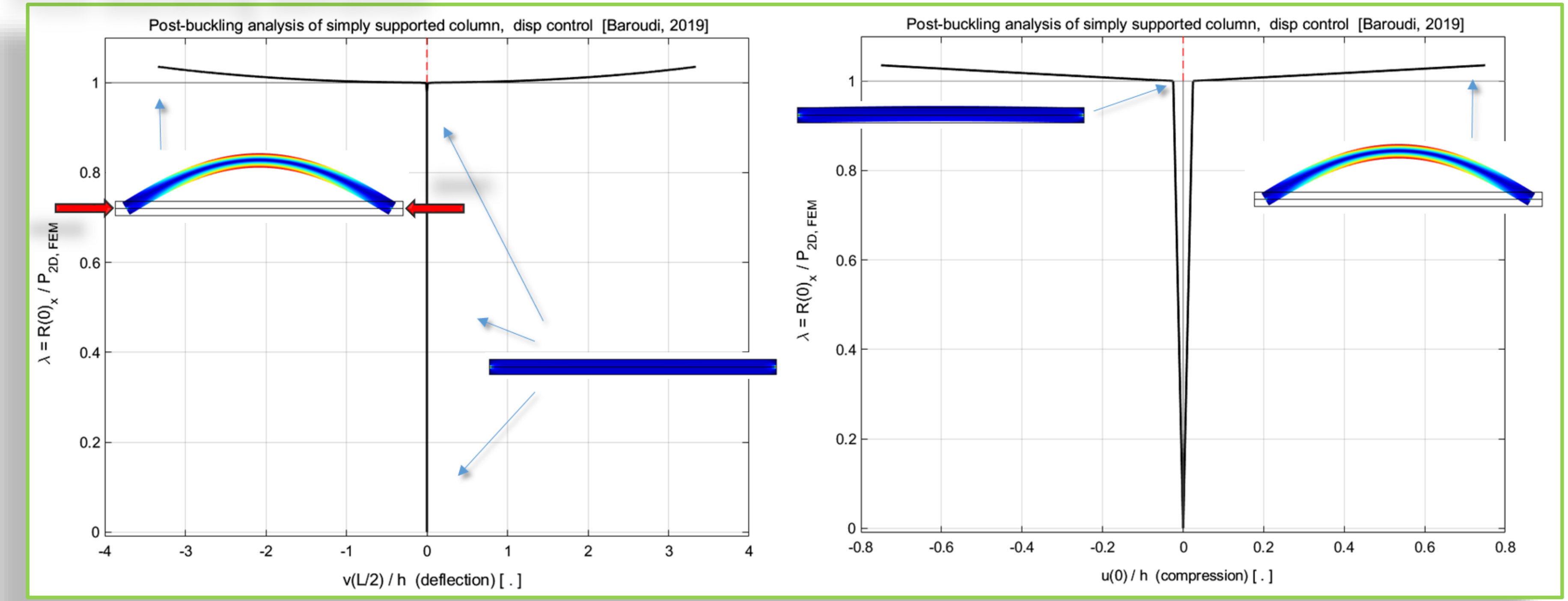


FE-based post-buckling analysis of axially compressed column

- Perturbed with tiny distributed load
- Can also be given as initial shape imperfection

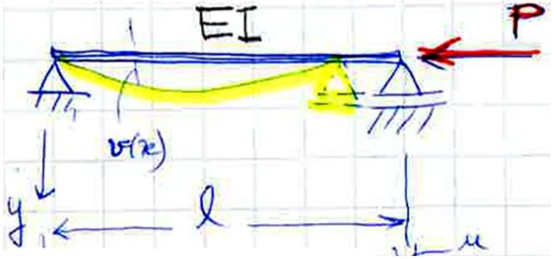


Post-buckling behavior



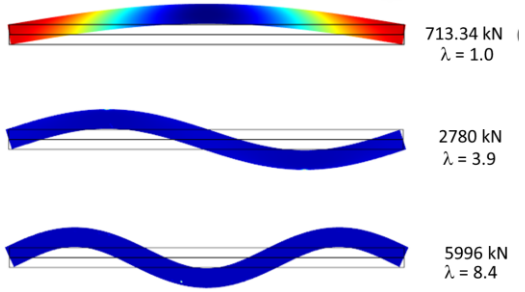
Flexural deflection v(L/2) / h

Axial shortening u / h

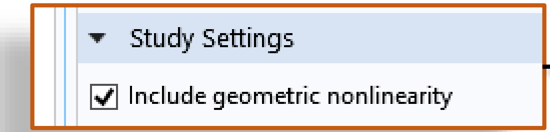


P_E = 719.66 kN (analytical 1D)

Linear buckling analysis



- at least, up-to the first mode is stable
- very shallow shape… no much increase in load bearing capacity



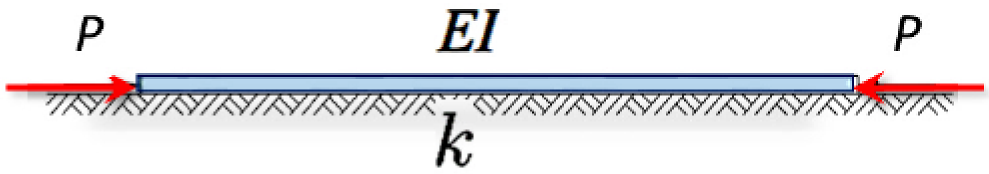
Study Settings

☑ Include geometric nonlinearity

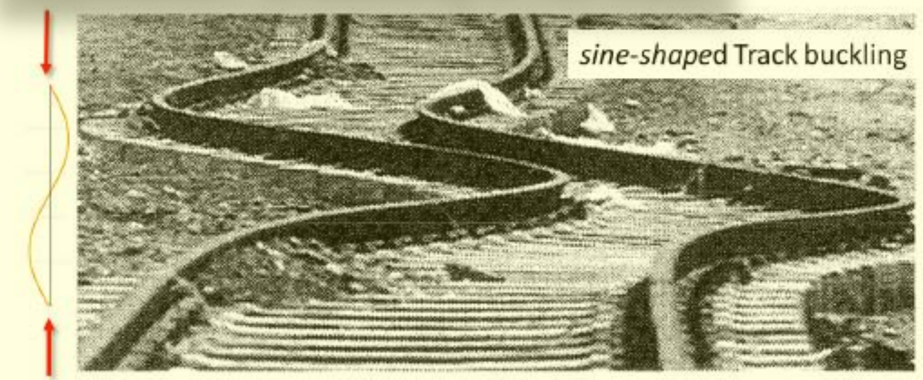
$$\epsilon^*_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$$

Uses: Finite strains and large displacements theory

Buckling of columns on elastic foundation



Application: 2) Buckling of rail track



Buckled rail track. Note the *sine-shaped* buckles

Linear Buckling analysis

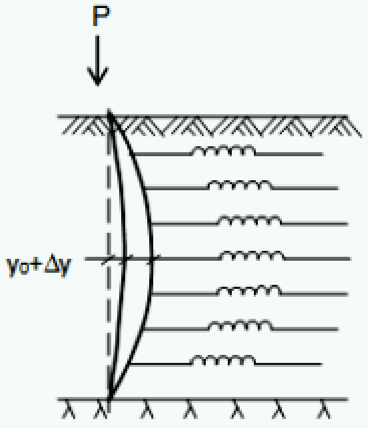
Sensitivity to imperfections
Post-buckling analysis
(Non-linear Buckling analysis)

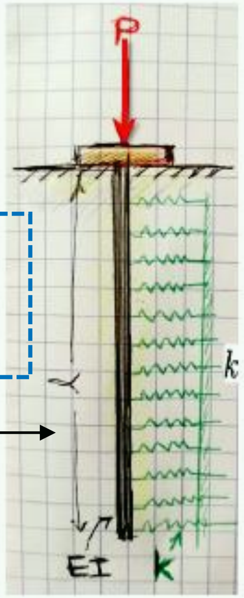
Application: 1) Buckling in pile design

Winkler model

$$r(x) = kv(x).$$

k, [N/m.m]





Shematic of foundation pile under axial thrust which is elastically restrained by the soil (geotechnical design; Eurocode 7).

Liikenneviraston ohjeita 13/2017
Eurokoodin soveltamisohje
Geotekninen suunnittelu – NCCI 7 (21.4.2017)

5.3.4 Nurjahduskestävyys STR/GEO

Nurjahdustarkastelu voidaan suorittaa rakennemallilla, jossa maan paalua tukeva vaikutus kuvataan jousilla.

Rakennemallissa (yleensä FEM) tulee paalun alkukaarevuus ja kuorman epäkeskisyys mallintaa. Jos paalu mallinnetaan suorana ja kuorma keskeisenä se ei laskennallisesti nurjahda.

Cf. **Eurocode 7**

Buckling of columns on elastic foundation

$$v(0) = v''(0) = 0,$$
$$v(\ell) = v''(\ell) = 0,$$

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^\ell EI[v''(x)]^2 + k[v(x)]^2\mathrm{d}x - P\int_0^\ell \frac{1}{2}[v'(x)]^2\mathrm{d}x.}$$

Euler-Bernoulli beam

← Can be used to find approximate solutions **Rayleigh-Ritz**

$$\boxed{\text{energy criterion } \delta(\Delta\Pi) = 0}$$

$$\Downarrow$$

$$\int_0^\ell EIv''\delta v'' + kv\delta v\mathrm{d}x - P\int_0^\ell v'\delta v'\mathrm{d}x = 0,\ \forall\delta v$$

new term

$$\delta\left(\frac{1}{2}\int_0^\ell kv(x)^2\mathrm{d}x\right) = \int_0^\ell \underbrace{kv}_{\text{new add to ODE}} \delta v\mathrm{d}x.$$

$$\Downarrow$$

which becomes after twice integration by parts

$$\int_0^\ell \underbrace{[EIv^{(4)} + \boxed{kv} + Pv'']}_{=0}\delta v\mathrm{d}x + [\underbrace{EIv''}_{-M}\delta v']_0^\ell - [(\underbrace{EIv'''}_{-Q} + Pv')\delta v]_0^\ell = 0,\ \forall\delta v$$

`Field equation`

`Boundary conditions`

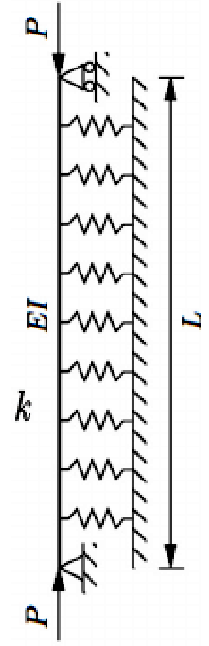
$$\Downarrow$$

The linearised buckling equation

$$\boxed{EIv^{(4)} + Pv'' + \boxed{kv} = 0}.$$

&

← Can be used to find exact solutions



Shematic of simply supported axially compressed column on elastic foundation.

`Solutions?`

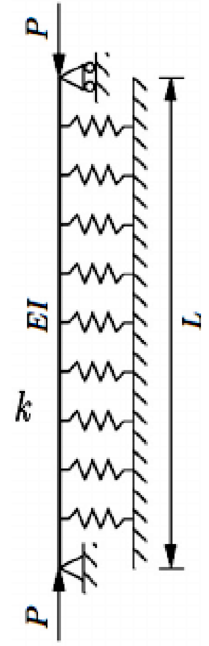
Buckling of columns on elastic foundation

$$v(0) = v''(0) = 0,$$
$$v(\ell) = v''(\ell) = 0,$$

$$\boxed{\Delta\Pi = \frac{1}{2}\int_0^\ell EI[v''(x)]^2 + k[v(x)]^2\mathrm{d}x - P\int_0^\ell \frac{1}{2}[v'(x)]^2\mathrm{d}x.}$$

Euler-Bernoulli beam

Shematic of simply supported axially compressed column on elastic foundation.



energy criterion $\delta(\Delta\Pi) = 0$

$$\Downarrow$$

$$\int_0^\ell EIv''\delta v'' + kv\delta v\mathrm{d}x - P\int_0^\ell v'\delta v'\mathrm{d}x = 0,\ \forall\delta v$$

new term

$$\delta\left(\frac{1}{2}\int_0^\ell kv(x)^2\mathrm{d}x\right) = \int_0^\ell \underbrace{kv}_{\text{new add to ODE}} \delta v\mathrm{d}x.$$

$$\Downarrow$$

which becomes after twice integration by parts

$$\int_0^\ell \underbrace{[EIv^{(4)} + \boxed{kv} + Pv'']}_{=0}\delta v\mathrm{d}x + [\underbrace{EIv''}_{-M}\delta v']_0^\ell - [(\underbrace{EIv'''}_{-Q} + Pv')\delta v]_0^\ell = 0,\ \forall\delta v$$

Field equation

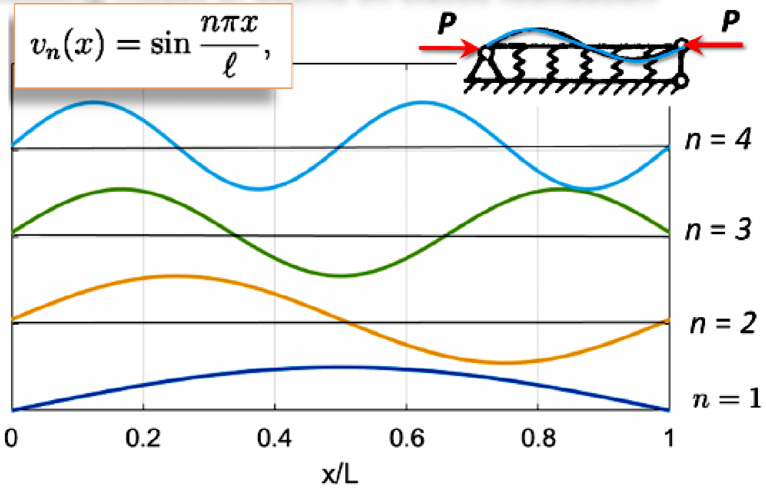
Boundary conditions

$$\Downarrow$$

&

The linearised buckling equation

$$\boxed{EIv^{(4)} + Pv'' + \boxed{kv} = 0}.$$


Buckling modes of beams on elastic foundation

$$v_n(x) = \sin\frac{n\pi x}{\ell},$$

Buckling of columns on elastic foundation

The linearised buckling equation

$$\boxed{EIv^{(4)} + Pv'' + kv = 0}.$$

& Boundary conditions

$$v(0) = v''(0) = 0,$$
$$v(\ell) = v''(\ell) = 0,$$

$$\Downarrow$$

The following trial satisfies the differential equations & the boundary conditions

$$v_n(x) = \sin\frac{n\pi x}{\ell}, \quad n = 1, 2, 3, \ldots$$

$$\Downarrow$$

The critical load is now

$$P_n = \left(\frac{n\pi}{\ell}\right)^2 EI + \left(\frac{\ell}{n\pi}\right)^2 k$$

$$= n^2 \underbrace{\left[\frac{\pi^2 EI}{\ell^2}\right]}_{P_\mathrm{E}} + \frac{1}{n^2}\frac{k\ell^2}{\pi^2} > n^2 P_\mathrm{E}$$

$$= \left[\frac{\pi^2 EI}{\ell^2}\right]\left[n^2 + \frac{1}{n^2}\underbrace{\frac{k}{EI}\left(\frac{\ell}{\pi}\right)^4}_{\equiv\beta}\right], \; n = 1, 2, 3, \ldots$$

$$= P_\mathrm{E}\left[n^2 + \frac{\beta}{n^2}\right].$$

which represents, graphically, a set of straight lines for $n = 1, 2, 3, \ldots$, etc. in function of the relative stiffness β. The graph $\bar{P}_n - \beta$ shows the lowest values for $\bar{P}_n$ which correspond to the critical loads as function of the parameter β.

$$\Longrightarrow$$

$$\beta = \frac{k\ell^4}{\pi^4 EI},$$

$$\bar{P}_n = \frac{P_n \ell^2}{\pi^2 EI} \equiv \frac{P_n}{P_\mathrm{E}}$$

The buckling load is the smallest critical load:

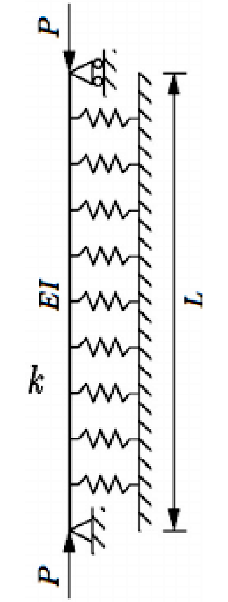
The smallest critical load $P_{cr} = P_n$ depends on the half-wave number n.

$$\frac{\mathrm{d}P_n}{\mathrm{d}n} = 0 \implies n^2 = \sqrt{\beta},$$

$$P_{cr} = 2P_\mathrm{E}\sqrt{\beta} = 2\sqrt{kEI}.$$

`Constraint: half-wave number` n `should be an integer`

Indeed, this is a *limit* for 'long' beams for which $\bar{\ell} \equiv \beta^{1/4} \geq 3$



The buckling load:

'Long' beams:

$$P_{cr} \approx 2\sqrt{kEI}$$

$$\bar{\ell} \equiv \beta^{1/4} \geq 3$$

Beams of arbitrary length:

$$\boxed{P_{cr} = k_{cr}\sqrt{kEI}}$$

Buckling load

Buckling coefficient (see graph next slide)

depends on

$$\beta = \frac{k\ell^4}{\pi^4 EI}$$

What is the corresponding buckling mode?

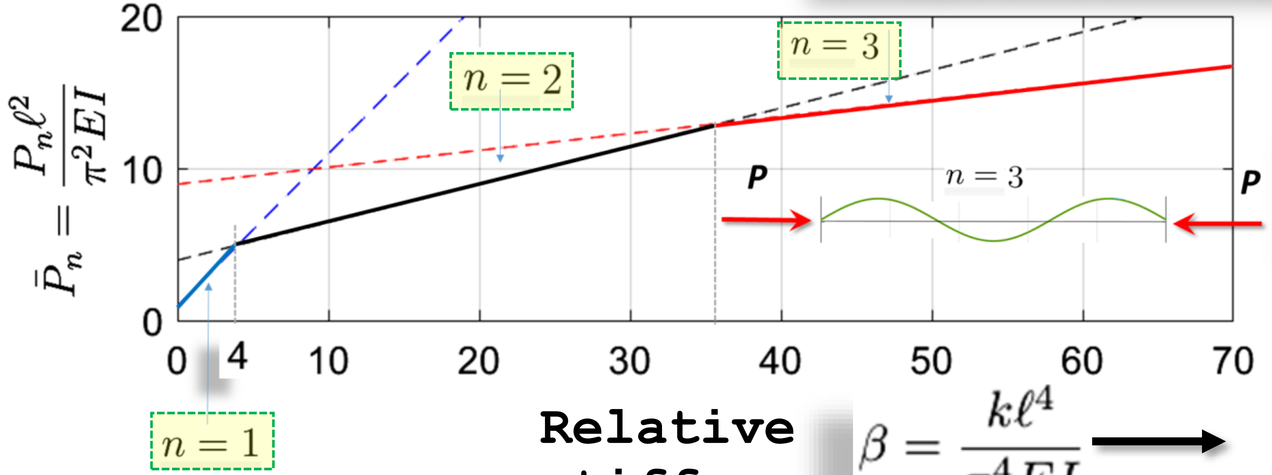
Buckling of columns on elastic foundation

What is the corresponding buckling mode?

Attention: The buckling mode corresponding to the buckling load does not always the first mode

$$\bar{P}_n = n^2 + \frac{\beta}{n^2}$$

Relative buckling load

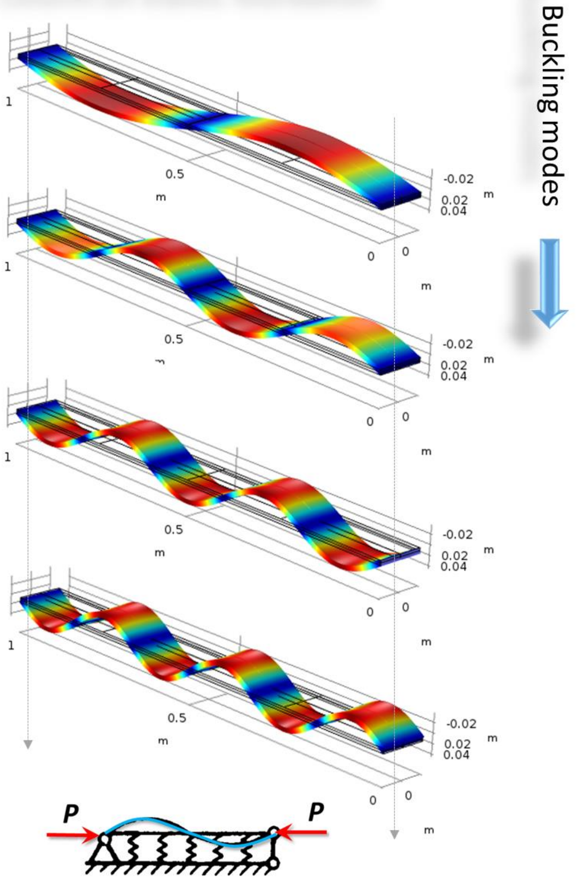


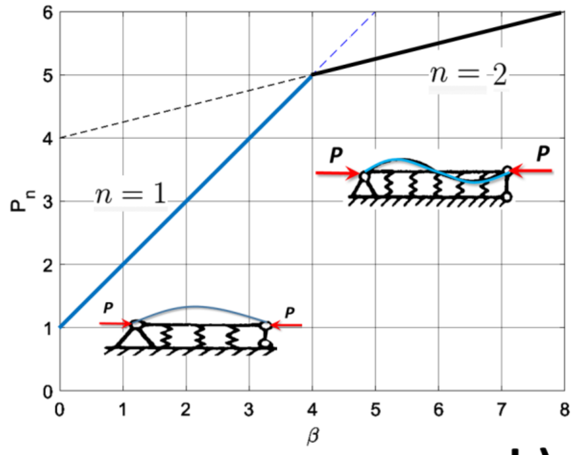
Relative stiffness $\beta = \frac{k\ell^4}{\pi^4 EI}$

depends on

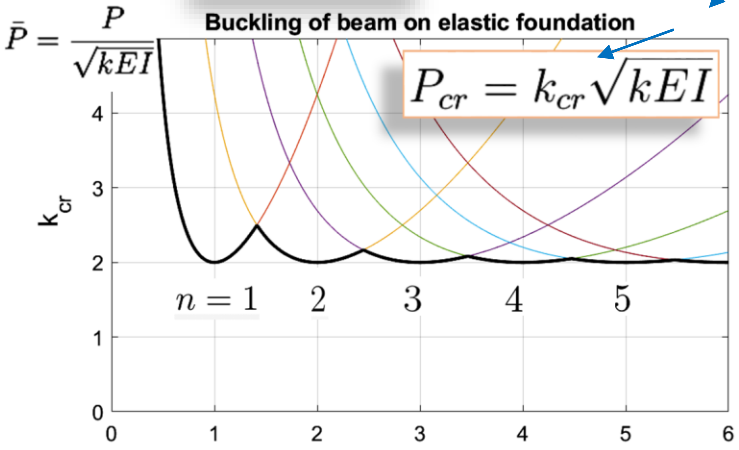
$$\beta = \frac{k\ell^4}{\pi^4 EI}$$

Buckling of axially compressed column on elastic foundation





b)



$$\bar{P} = \frac{P}{\sqrt{kEI}}$$

$$P_{cr} = k_{cr}\sqrt{kEI}$$

$$\bar{\ell} = \frac{\ell}{\pi}\left[\frac{k}{EI}\right]^{1/4}$$

c)

$$P_{cr} = k_{cr}\sqrt{kEI}$$

Buckling load

Buckling coefficient

depends on

$$\beta = \frac{k\ell^4}{\pi^4 EI}$$

Buckling of a column on elastic foundation - a summary

Other types of boundary conditions

- **For general types of BCs** one should obtain a complete solution of the **ODE**

$$v^{(4)} + \frac{P}{EI}v'' + \frac{k}{EI}v = 0$$

$$v^{(4)} + \lambda_P^2 v'' + \frac{\beta_k^4}{4}v = 0$$

$$\lambda_P^2 \equiv P/EI\,(= p^2)$$

$$\beta_k^4 \equiv 4k/EI\,(= 4b^4)$$

The general solution $v(x) = Ae^{rx}$

- $\lambda_P > \beta_k,$

$$v(x) = C_1 \cos px + C_2 \sin px + C_3 \cos qx + C_4 \sin qx$$

$$p = \tfrac{1}{2}\sqrt{\lambda_P^2 + \beta_k^2} + \tfrac{1}{2}\sqrt{\lambda_P^2 - \beta_k^2} \;\;\&\;\; q = \tfrac{1}{2}\sqrt{\lambda_P^2 + \beta_k^2} - \tfrac{1}{2}\sqrt{\lambda_P^2 - \beta_k^2}$$

- $\lambda_P < \beta_k,$

$$v(x) = C_1 \cosh px + C_2 \sinh px + C_3 \cosh qx + C_4 \sinh qx$$

$$p = \tfrac{1}{2}\sqrt{\lambda_P^2 + \beta_k^2} + \tfrac{1}{2}\sqrt{\beta_k^2 - \lambda_P^2} \;\;\&\;\; q = \tfrac{1}{2}\sqrt{\lambda_P^2 + \beta_k^2} - \tfrac{1}{2}\sqrt{\beta_k^2 - \lambda_P^2}$$

- $\lambda_P = \beta_k,$

$$v(x) = (C_1 + C_2 x)\cos\left(\lambda_k/\sqrt{2}\right) + (C_3 + C_4 x)\sin\left(\lambda_k/\sqrt{2}\right)$$

- **For general types of BCs** one should obtain a complete solution of the **ODE**

$$\beta_k^4 \equiv 4k/EI\,(=4b^4) \qquad \lambda_P^2 \equiv P/EI\,(=p^2)$$

$$\boxed{\begin{aligned} v^{(4)} + \frac{P}{EI}v'' + \frac{k}{EI}v = 0 \\ v^{(4)} + \lambda_P^2 v'' + \frac{\beta_k^4}{4}v = 0 \end{aligned}}$$

$$\boxed{v(x) = C_1\cos k_1x + C_2\sin k_1x + C_3\cos k_2x + C_4\sin k_2x}$$

$$(k_1,k_2) = \sqrt{a^2 \pm \sqrt{\Delta}}$$

$$v(-L) = v(L) = 0,$$
$$v'(-L) = v'(L) = 0.$$

$$\begin{bmatrix} \cos k_1L & \sin k_1L & \cos k_2L & \sin k_2L \\ \cos k_1L & -\sin k_1L & \cos k_2L & -\sin k_2L \\ -k_1\sin k_1L & k_1\cos k_1L & -k_2\sin k_2L & k_2\cos k_2L \\ k_1\sin k_1L & k_1\cos k_1L & k_2\sin k_2L & k_2\cos k_2L \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{P_{cr} = \mu \cdot 2\sqrt{kEI}},$$

$$\beta = \frac{k\ell^4}{\pi^4 EI}$$

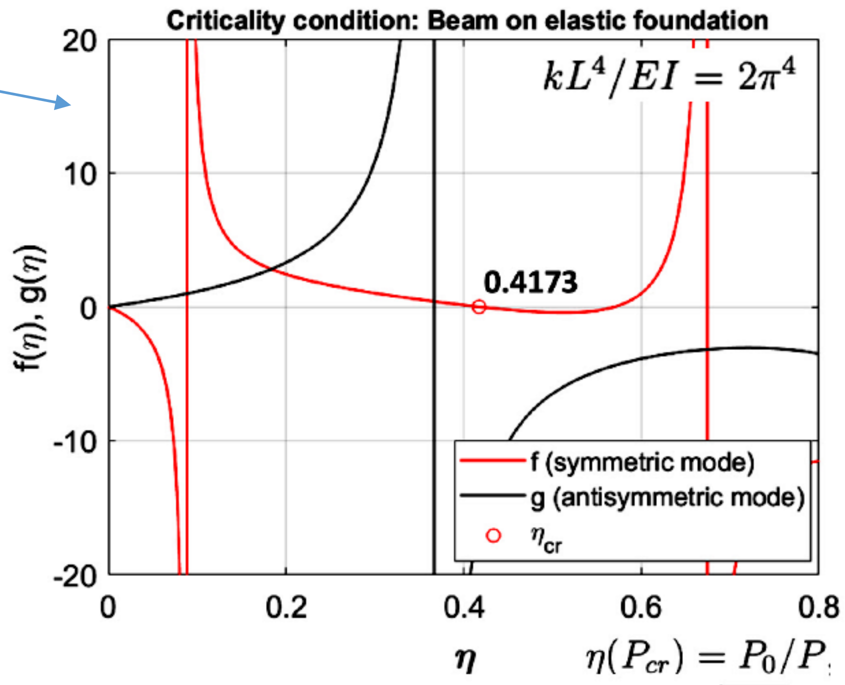
To obtain from the smallest zero of the determinant

Buckling load (symmetric mode)

$$\boxed{P_{cr} = P_0/\eta_{cr} \approx \underbrace{2.4}_{\mu}\cdot\underbrace{2\sqrt{kEI}}_{P_0},}$$

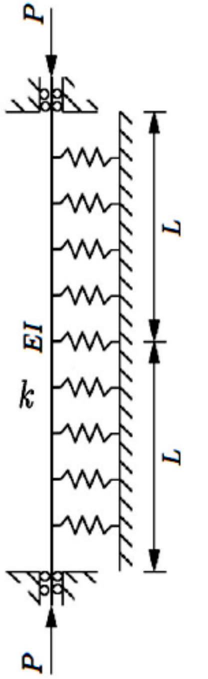
Let's fix the value In this example: $kL^4/EI = 2\pi^4.$

- One should consider, separately, symmetric and asymmetric buckling
- The smallest critical load → buckling load



$\eta(P_{cr}) = P_0/P$

$P_0 = 2\sqrt{kEI}$



The zeros of the determinant for the buckling of a column on elastic foundation.

Read the details in the pdf-notes I provided

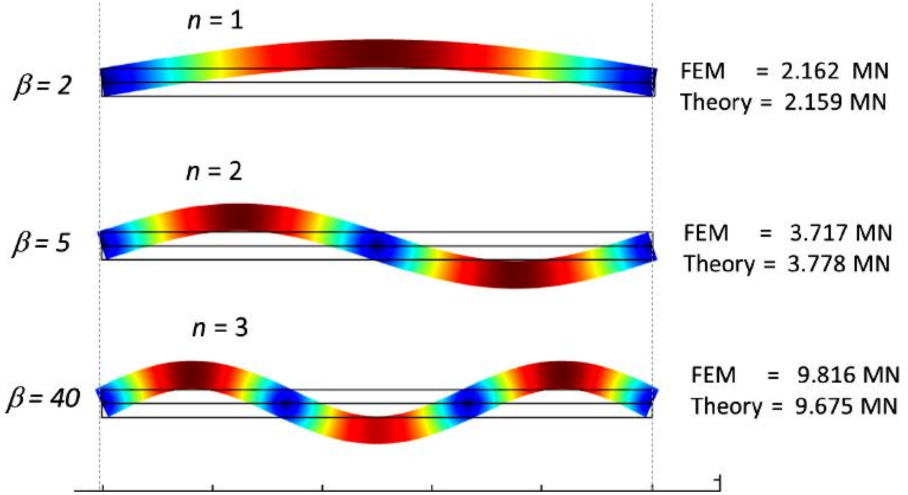
Buckling of columns on elastic foundation

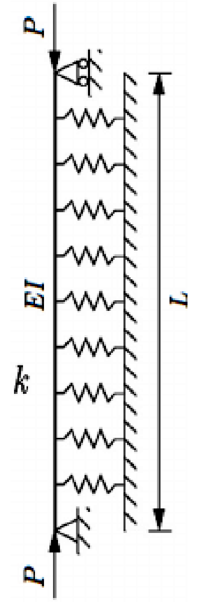
$$v(0) = v''(0) = 0,$$
$$v(\ell) = v''(\ell) = 0,$$

The linearised buckling equation

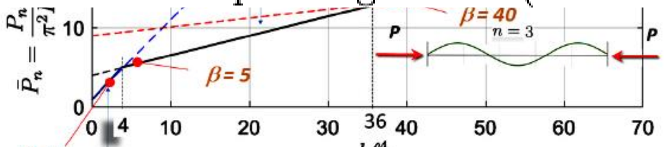
$$\boxed{EIv^{(4)} + Pv'' + kv = 0}.$$

& Boundary conditions





In the following, for illustrative pedagogical purposes, we analyse a simply supported column on elastic foundation with centric axial compressive load P. Simulation data: $\ell = 1$ m, $b = \ell/10$, $h = 50$ mm. $E = 70$ GPa ($\nu = 0.33$). We investigate, how the relative 'stiffness number' $\beta \equiv k\ell^2/(\pi^4 EI)$ determine the number n of half-waves of the buckling modes corresponding to the (smallest) buckling load P_{cr},



Linear FE-buckling analysis. Buckling of axially compressed

Table 1.1: FE-linear buckling analysis. The loads are given in [MN] units.

β	$\bar{\ell}$	n	$P_{cr}^{\text{lim.}}$	k_{cr}	$P_{cr}^{(\text{theor.})}$	P_{cr}^{FEM}	$P_{cr}^{(\text{theor.})}/P_E$	k [N/m^2]
2	1.189	1	2.04	2.121	2.159	2.162	3	14.2
5	1.495	2	3.22	2.348	3.778	3.717	5.3	35.5
40	2.515	3	9.10	2.126	9.675	9.816	13.4	284.1

The buckling load:

$$\boxed{P_{cr} = k_{cr}\sqrt{kEI}}$$

Buckling load

Buckling coefficient

(see graph next slide)

depends on

$$\boxed{\beta = \frac{k\ell^4}{\pi^4 EI}}$$

What is the corresponding buckling mode?

Post-buckling analysis of columns on elastic foundation

FE-based post-buckling analysis

Buckles here (2D elasticity solution with tiny initial imperfection)

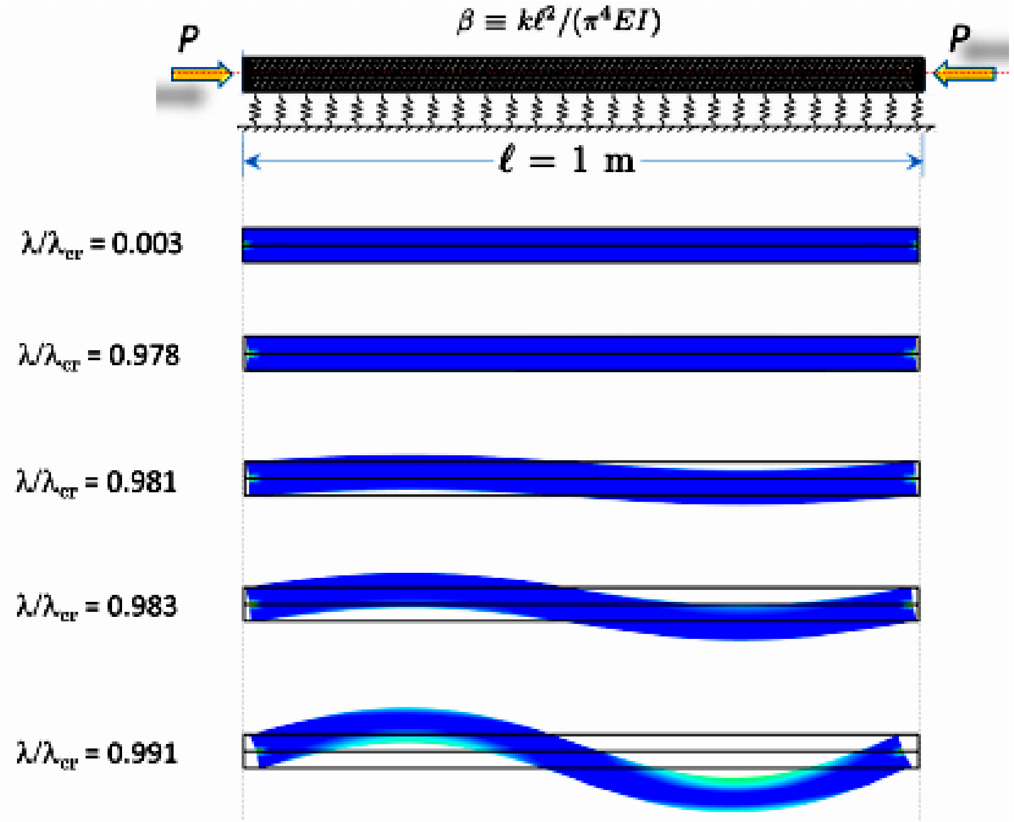
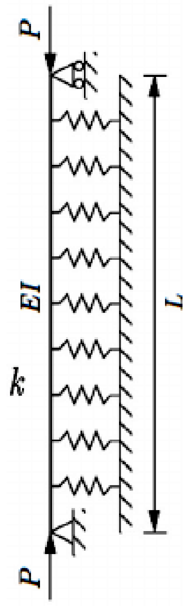


Figure 1.89: Post-buckling displacements in 1:1 scale (FE simulation). The perturbation scale $\epsilon = 1/1000$. After $\lambda/\lambda_{cr,FE} > 0.991$, the behaviour seems (in this simulation) to become unstable and could not be captured because of force control approach used (I will do a displacement control soon). ($EI = 72917$ N.m^2, $\beta = 5$ ($n = 2$)), theoretical 1-D value for $P_{cr} = 3.778$ MN (2D-elasticity FE based linear buckling analysis gave $P_{cr,FE} = 3.720$ MN). .

The column-beam is simply supported

(kuvasta puuttuu nivelet)

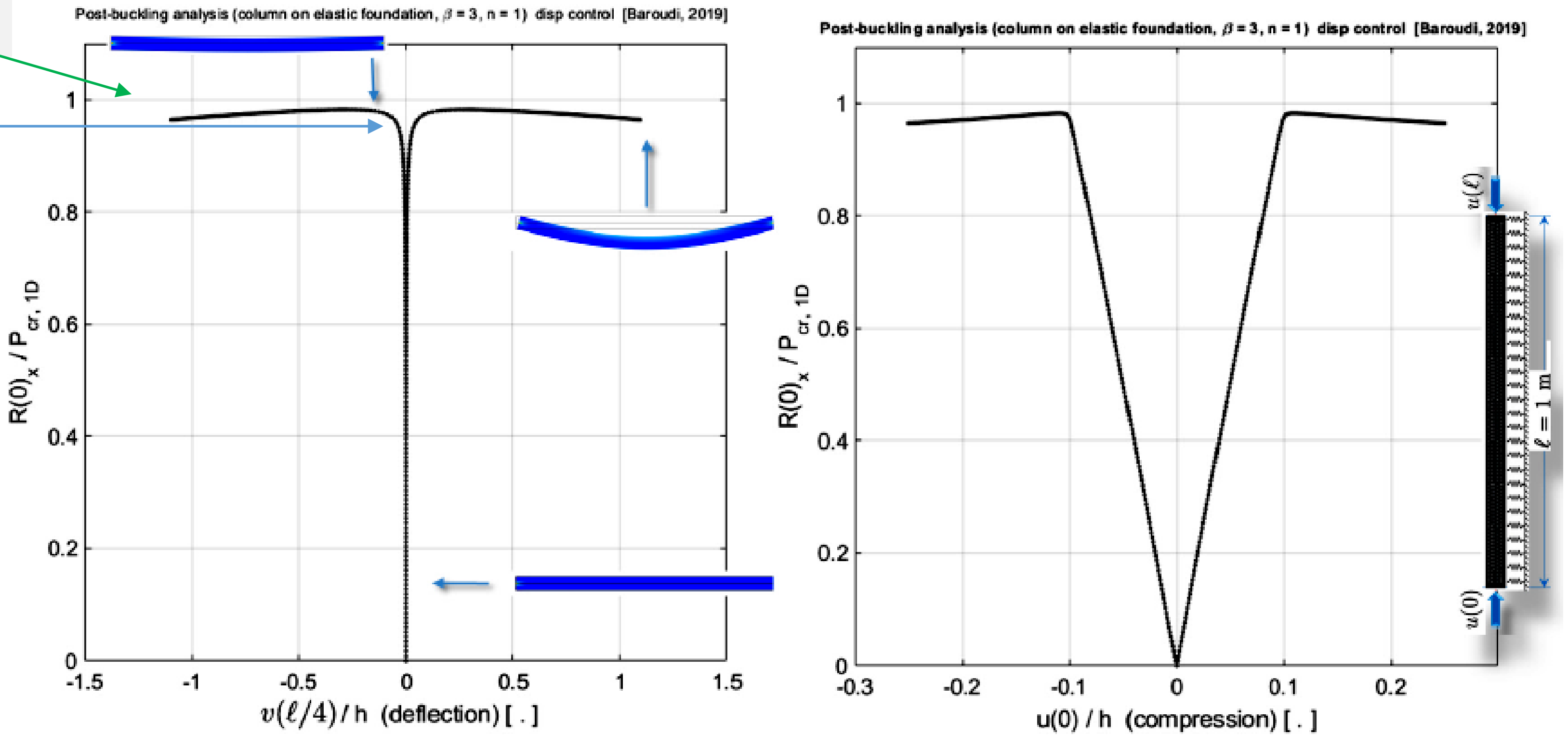


Effect of foundation stiffness on post-buckling behaviour

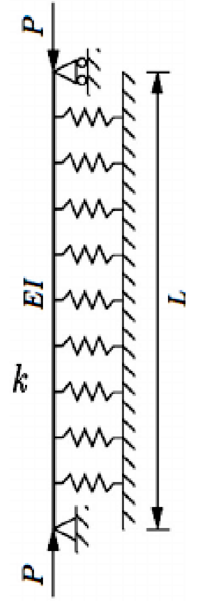
Buckles here (1D theoretical ideal solution)

Buckles here (2D elasticity solution with imperfection)

Post-buckling of beam on elastic foundation (displacement control)



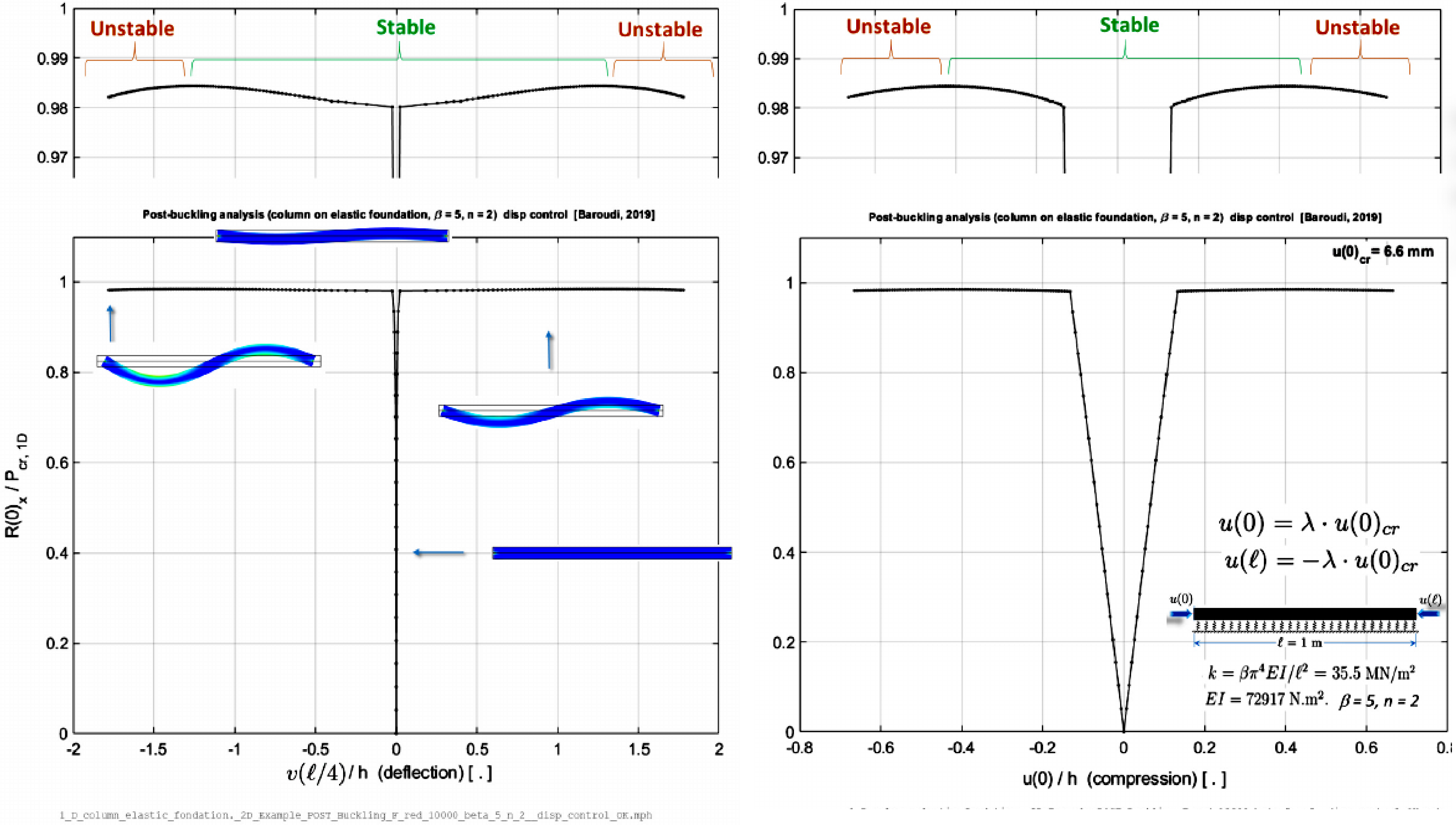
The column-beam is simply supported
(kuvasta puuttuu nivelet)



1 D column elastic fondation. 2D Example POST Buckling F red 10000 beta 3 n 1 disp control OK.mph

Figure 1.93: Post-buckling equilibrium paths (FE-simulation, displacement-control) of a uniformly compressed column on elastic foundation. The ends-load is centric. The parameters ℓ, k and EI are such that $\beta = 3$ and the initial post-buckling mode corresponds to one-half waves ($n = 1$). The perturbation scale for the transverse loads was $\epsilon = 1/1000$.

Effect of foundation stiffness on post-buckling behaviour



The column-beam is simply supported

(kuvasta puuttuu nivelet)

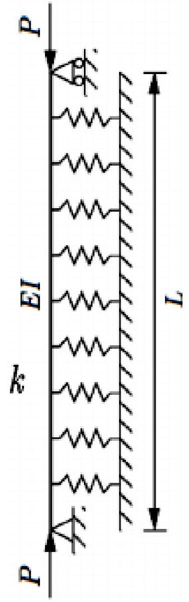


Figure 1.91: Post-buckling equilibrium paths; $v(\ell/4)$ *versus* P/P_{cr}, (FE-simulation, displacement-control). The parameters ℓ, k and EI are such that $\beta = 5$ and the initial post-buckling mode corresponds to two-half waves ($n = 2$). The perturbation scale for the transverse loads was $\epsilon = 1/10000$. (the post-buckled displacements are in scale 1:1 in the deformed column).

Discrete energy method - FEM

The starting point for deriving the elementary matrices above is the total potential energy functional (1.233) or more directly, its variation which is known as *Virtual Work Principle*. The idea is the write the variation of the total functional as a sum over the elements

$$v^{(e)}(x) = \sum_{i=1}^{M} \phi_i(x) a_i^{(e)} \equiv \mathbf{N}(x)\mathbf{a}^{(e)},$$

$$\mathbf{a}^{(e)} = \begin{bmatrix} v_1 & \theta_1 & v_2 & \theta_2. \end{bmatrix}^{\mathrm{T}}$$

$$\delta v(x) = \mathbf{N}(x)\delta\mathbf{a}^{(e)},$$

$N_j(x)$ are the shape functions

$$\delta(\Delta\Pi) = \sum_{e=1}^{N}\left[\int_0^{\ell^{(e)}} EIv''(x)\delta v'' + kv(x)\delta v(x)\mathrm{d}x - P^{(e)}\int_0^{\ell^{(e)}} v'(x)\delta v'(x)\mathrm{d}x\right] = 0$$

$$= \sum_{e=1}^{N}(\delta\mathbf{a}^{(e)})^{\mathrm{T}}\left[\underbrace{\int_0^{\ell^{(e)}} \mathbf{N}''^{\mathrm{T}}(x)\cdot EI\cdot\mathbf{N}''(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{L}}^{(B)}} + \underbrace{\int_0^{\ell^{(e)}} \mathbf{N}^{\mathrm{T}}(x)\cdot k\cdot\mathbf{N}(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{L}}^{(F)}} + \right.$$

$$\left.\underbrace{-\int_0^{\ell^{(e)}} \mathbf{N}'^{\mathrm{T}}(x)\cdot P^{(e)}\cdot\mathbf{N}'(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{G}}}\right]\mathbf{a}^{(e)} = 0,\ \forall\delta\mathbf{a}^{(e)}$$

where $P^{(e)} = -N^0(x)$ and $N^0(x)$ being the membrane stress-resultant

Discrete energy method - FEM

The starting point for deriving the elementary matrices above is the total potential energy functional (1.233) or more directly, its variation which is known as *Virtual Work Principle*. The idea is the write the variation of the total functional as a sum over the elements

$$\delta(\Delta\Pi) = \sum_{e=1}^{N} \left[\int_0^{\ell^{(e)}} EIv''(x)\delta v'' + kv(x)\delta v(x)\mathrm{d}x - P^{(e)} \int_0^{\ell^{(e)}} v'(x)\delta v'(x)\mathrm{d}x \right] = 0$$

$$= \sum_{e=1}^{N} (\delta \mathbf{a}^{(e)})^{\mathrm{T}} \left[\underbrace{\int_0^{\ell^{(e)}} \mathbf{N}''^{\mathrm{T}}(x) \cdot EI \cdot \mathbf{N}''(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{L}}^{(B)}} + \underbrace{\int_0^{\ell^{(e)}} \mathbf{N}^{\mathrm{T}}(x) \cdot k \cdot \mathbf{N}(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{L}}^{(F)}} + \right.$$

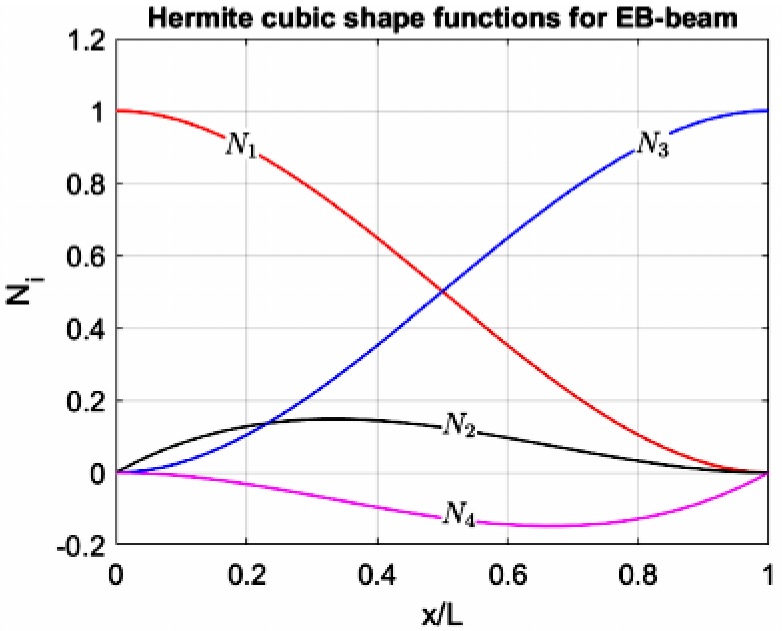
$$\left. \underbrace{- \int_0^{\ell^{(e)}} \mathbf{N}'^{\mathrm{T}}(x) \cdot P^{(e)} \cdot \mathbf{N}'(x)\mathrm{d}x}_{\mathbf{K}_{\mathrm{G}}} \right] \mathbf{a}^{(e)} = 0, \ \forall \delta \mathbf{a}^{(e)}$$

where $P^{(e)} = -N^0(x)$ and $N^0(x)$ being the membrane stress-resultant

Discrete energy method - FEM

Euler-Bernoulli beam element

$$N_1(x) = 1 - 3(x/\ell)^2 + 2(x/\ell)^3,$$
$$N_2(x) = x(1 - x/\ell)^2,$$
$$N_3(x) = 3(x/\ell)^2 - 2(x/\ell)^3,$$
$$N_4(x) = x((x/\ell)^2 - x/\ell)$$



linearised stiffness matrix for bending

$$\mathbf{K}_{\mathrm{L}}^{(B)} = \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix}$$

consistent stiffness matrix from the elastic foundation

$$\mathbf{K}_{\mathrm{L}}^{(F)} = \frac{k\ell}{70}\begin{bmatrix} 26 & 11\ell/3 & 9 & -13\ell/6 \\ 11\ell/3 & 2\ell^2/3 & 13\ell/6 & -\ell^2/2 \\ 9 & 13\ell/6 & 26 & -11\ell/3 \\ -13\ell/6 & -\ell^2/2 & -11\ell/3 & 2\ell^2/3 \end{bmatrix}$$

Diagonalized foundation stiffness matrix:

$$\mathbf{K}_{\mathrm{L}}^{(F)} = \frac{k\ell}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

geometric elementary matrix is

$$\mathbf{K}_{\mathrm{G}} = -\frac{P}{30\ell}\begin{bmatrix} 36 & 3\ell & -36 & 3\ell \\ 3\ell & 4\ell^2 & -3\ell & -\ell^2 \\ -36 & -3\ell & 36 & -3\ell \\ 3\ell & -\ell^2 & -3\ell & 4\ell^2 \end{bmatrix}$$

compression load $P = -N^0(x) > 0.$

$$\mathbf{K}_{\mathrm{L}}^{(\mathbf{B})} = \int_0^{\ell^{(e)}} \mathbf{N}''^{\mathrm{T}}(x) \cdot EI \cdot \mathbf{N}''(x)\mathrm{d}x,$$
$$\mathbf{K}_{\mathrm{L}}^{(F)} = \int_0^{\ell^{(e)}} \mathbf{N}^{\mathrm{T}}(x) \cdot k \cdot \mathbf{N}(x)\mathrm{d}x,$$
$$\mathbf{K}_{\mathrm{G}} = -\int_0^{\ell^{(e)}} \mathbf{N}'^{\mathrm{T}}(x) \cdot P^{(e)} \cdot \mathbf{N}'(x)\mathrm{d}x.$$

[A result from FEA] The convergence rate *k* for Euler-Bernoulli beam element for the Eigen-values is *k* = 4

Application example

$$\mathbf{a} = [\phi_1, \phi_2]^{\mathrm{T}}$$

DO: Determine the critical load and the corresponding mode by the “handy-FE” method (stiffness method)

Assembly:

$$K_{11} = K_{44}^{(1)} = \frac{EI}{\ell^3}4\ell^2 - \frac{P}{30\ell}4\ell^2 \qquad P^{(1)} = P$$

$$K_{12} = K_{21} = K_{42}^{(1)} = \frac{EI}{\ell^3}2\ell^2 + \frac{P}{30\ell}\ell^2 \qquad P^{(2)} = 3P$$

$$K_{22} = K_{22}^{(1)} + K_{44}^{(2)} = \frac{EI}{\ell^3}4\ell^2 - \frac{P}{30\ell}4\ell^2 + \frac{2EI}{\ell^3}4\ell^2 - \frac{3P}{30\ell}4\ell^2$$

The global linearised stiffness and geometric matrices

$$\Downarrow \quad \mathbf{K}_{\mathrm{L}} = \frac{2EI}{\ell}\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}, \qquad \mathbf{K}_{\mathrm{G}} = -\frac{P\ell}{30}\begin{bmatrix} 4 & -1 \\ -1 & 16 \end{bmatrix}$$

$$\left(\frac{2EI}{\ell}\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix} - \frac{P\ell}{30}\begin{bmatrix} 4 & -1 \\ -1 & 16 \end{bmatrix}\right)\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

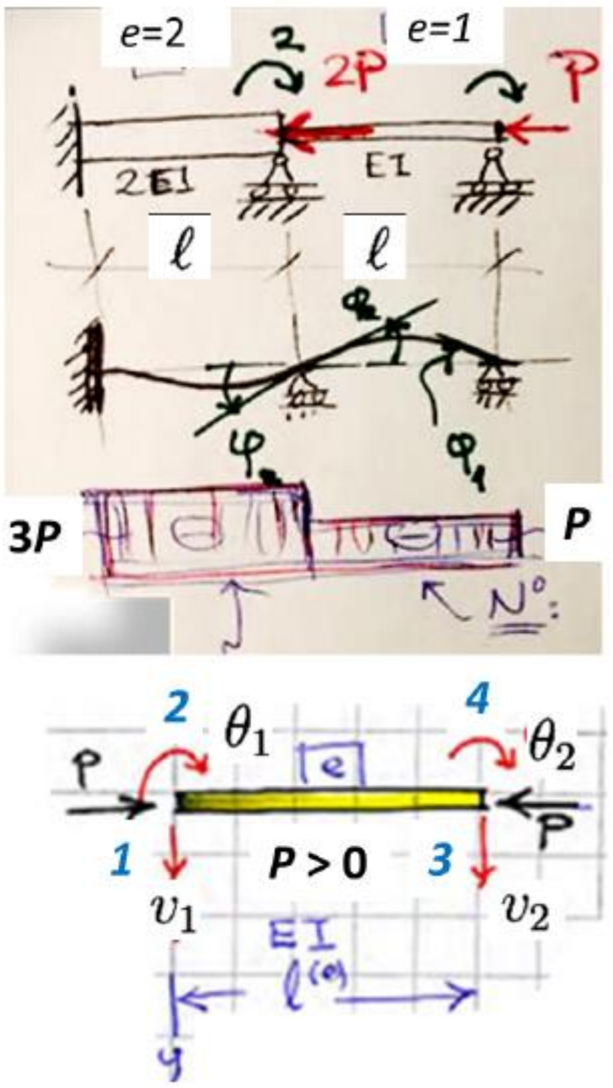
$$P_{1,cr} = 1.62\pi^2\frac{EI}{\ell^2} \qquad \phi_1 = \begin{bmatrix} 0.266 \\ -0.196 \end{bmatrix}$$

$$P_{2,cr} = 3.97\pi^2\frac{EI}{\ell^2} \qquad \phi_2 = \begin{bmatrix} -0.428 \\ -0.159 \end{bmatrix}$$

Buckling
Load & mode

Buckled state

Initial membrane stress (pre-buckling)



NB. On should refine the FE-mesh until convergence …

Accelerating convergence

refine the FE-mesh until convergence …

About convergence … and Richardson extrapolation toward the limit

- Assume we have an priori knowledge on the **convergence rate** of some **quantity** (can be always estimated)

λ

e.g.. buckling load

The numerical solution is proportional to

Convergence rate

Ch^k

Positive constant

h step-size or characteristic mesh-size (length of the largest element)

- The above **extrapolated solution** is **much closer to the exact** one than the solutions 1 and 2

Two solution with two different mesh-size: $h_2 < h_1$

$$\begin{cases} \lambda_1 = \lambda_{ex} + Ch_1^k = \lambda(h_1) \\ \lambda_2 = \lambda_{ex} + Ch_2^k = \lambda(h_2) \end{cases}$$

Richardson extrapolation: is a **sequence convergence acceleration** method

$$\lambda_{ex} = \frac{\lambda_2 - \lambda_1 \left(\frac{h_2}{h_1}\right)^k}{1 - \left(\frac{h_2}{h_1}\right)^k} \Longleftarrow$$

Extrapolated value

$$C = (\lambda_1 - \lambda_{ex})h_1^{-k}$$

$$\lambda_2 = \lambda_{ex} + (\lambda_1 - \lambda_{ex})\left(\frac{h_2}{h_1}\right)^k$$

[A result from FEA] The convergence rate k for Euler-Bernoulli beam element for the Eigen-values is $k = 4$. (We can also estimate k from log-log plot of convergence rate (graph of changes in lambda *versus* changes in h)

NB. This extrapolated value is much more accurate than if would refine *substantially* the mesh further

Physical discrete model based post-buckling analysis

Simplified model of elastically restrained column

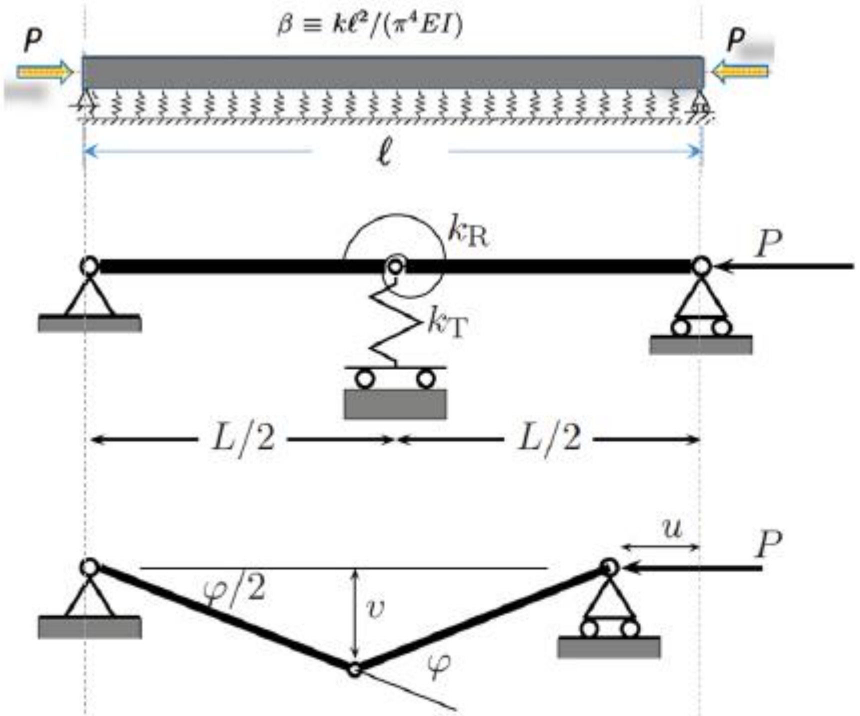
translational spring

$$k_T = k\ell/2$$

rotational spring

$$k_R = 1/4\pi^2 EI/\ell$$

$$\beta = k\ell^4/[\pi^2 EI]$$



$$u = 2 \cdot \frac{L}{2}(1 - \cos(\varphi/2)), \qquad v = \frac{L}{2}\sin(\varphi/2)$$

Solution

$$u = 2\cdot\frac{L}{2}(1-\cos(\varphi/2)), \qquad v = \frac{L}{2}\sin(\varphi/2)$$

$$\Pi = \frac{1}{2}k_{\mathrm{R}}\varphi^2 + \frac{1}{2}k_{\mathrm{T}}v^2 - Pu$$

$$= \frac{1}{8}\pi^2\frac{EI}{L}\varphi^2 + \frac{1}{16}\beta\pi^2\frac{EI}{L}\sin^2(\varphi/2) - PL(1-\cos(\varphi/2))$$

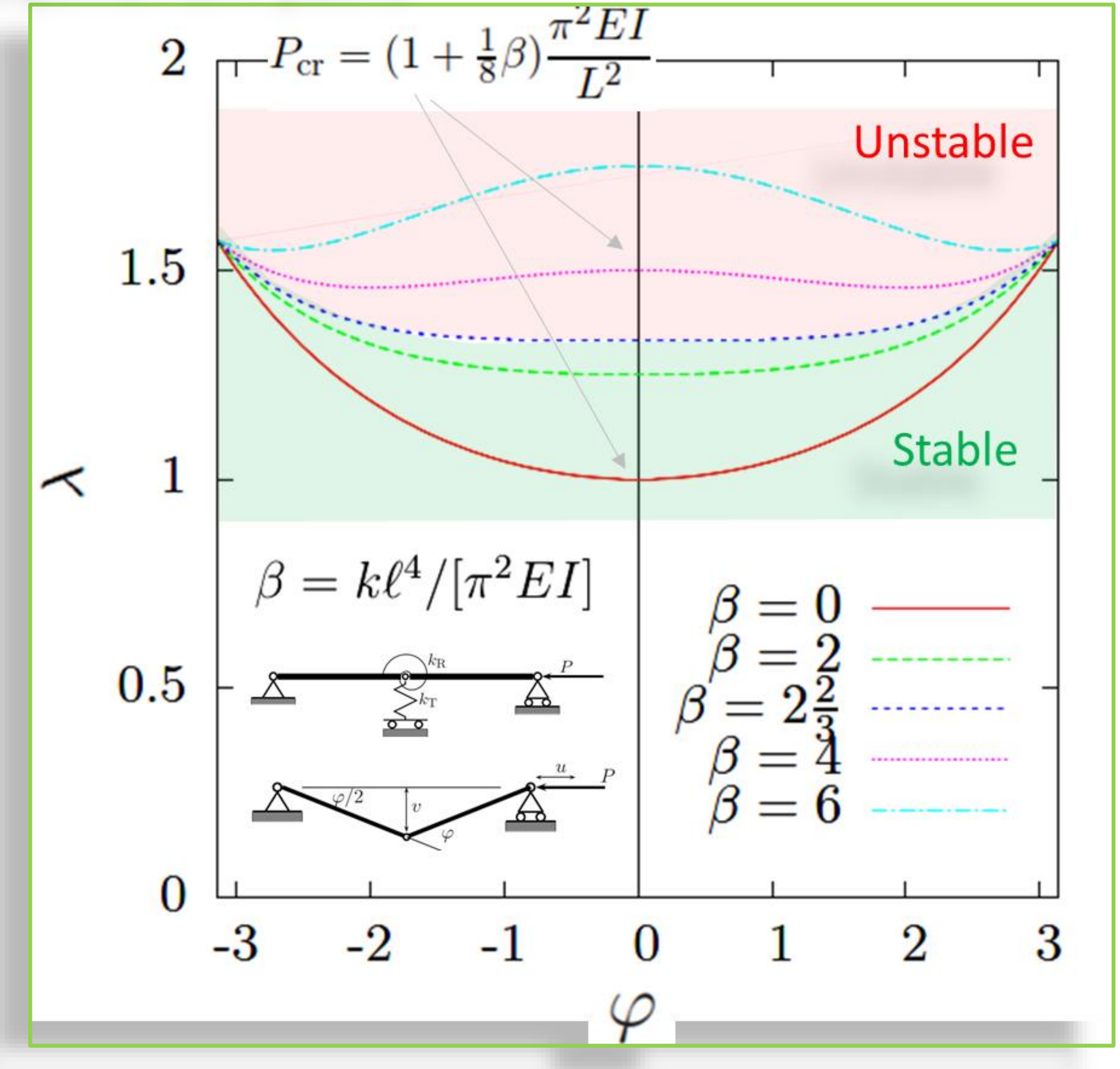
$$\varphi = 0$$

$$\lambda = \frac{\varphi/2}{\sin(\varphi/2)} + \frac{1}{8}\beta\cos(\varphi/2) \qquad \varphi = 0,\ \lambda = 1 + \tfrac{1}{8}\beta,$$

$$\boxed{P_{cr}(\beta) = \underbrace{(1+\frac{\beta}{8})}_{\lambda}\frac{\pi^2 EI}{\ell^2}}.$$

$$P = \lambda\frac{\pi^2 EI}{L^2}$$

Equilibrium paths



$$\frac{\mathrm{d}^2\tilde{\Pi}}{\mathrm{d}\varphi^2} = \frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{1}{4}\varphi + \frac{1}{16}\beta\sin(\varphi/2)\cos(\varphi/2) - \frac{1}{2}\lambda\sin(\varphi/2)\right)$$

(Ref: This problem is provided by R. Kouhia)

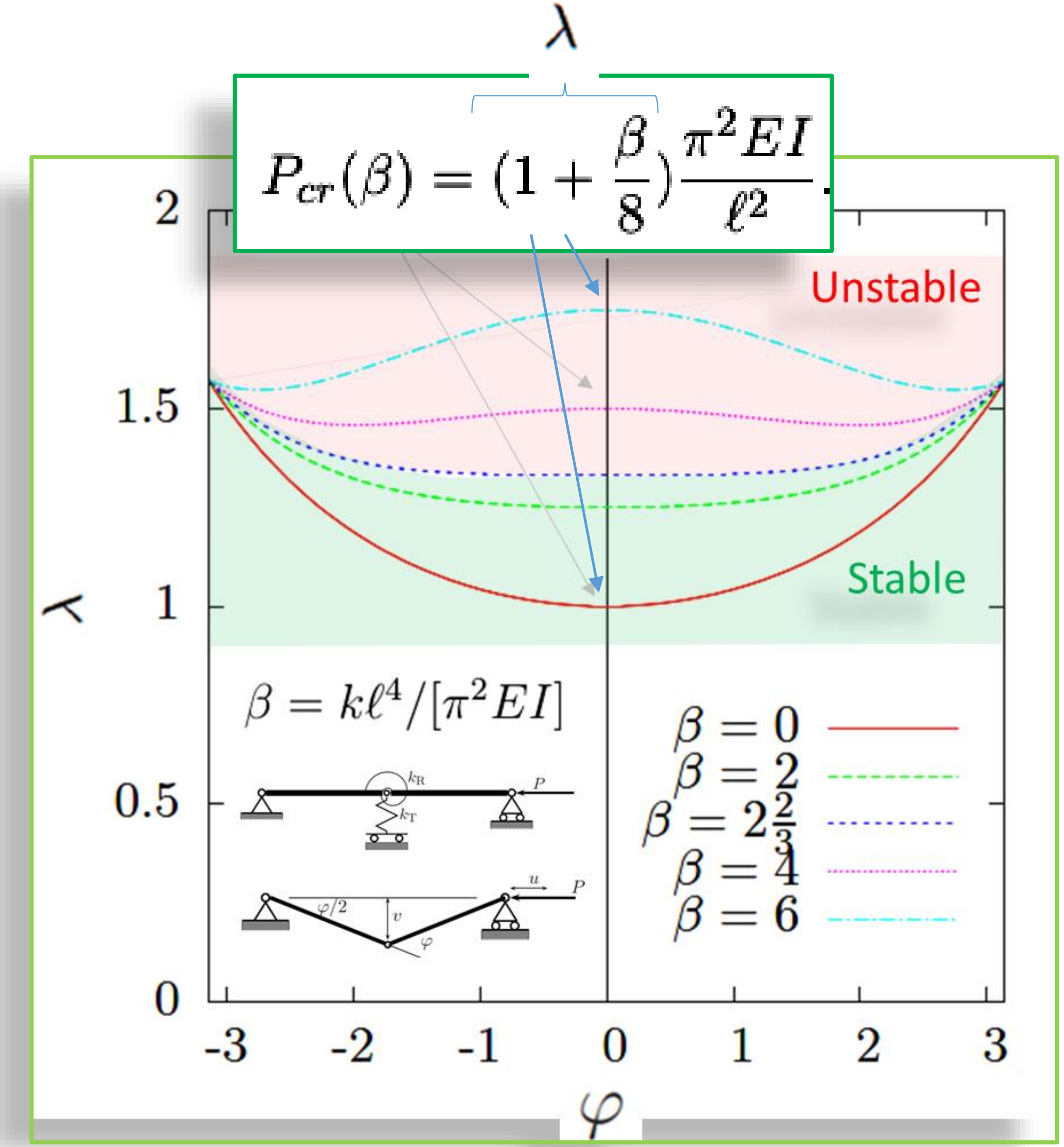
Solution

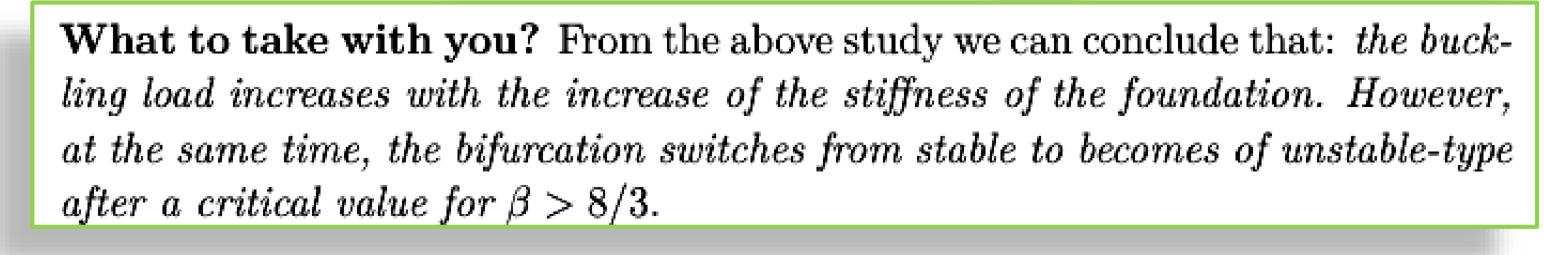
Equilibrium paths

From this study we conclude that

- the buckling load increases with the increase of the stiffness of the foundation.
- **However**, at the same time, the **bifurcation** switches from stable to becomes of unstable after a critical value $\beta > 8/3$

$$\beta = k\ell^4/[\pi^2 EI]$$

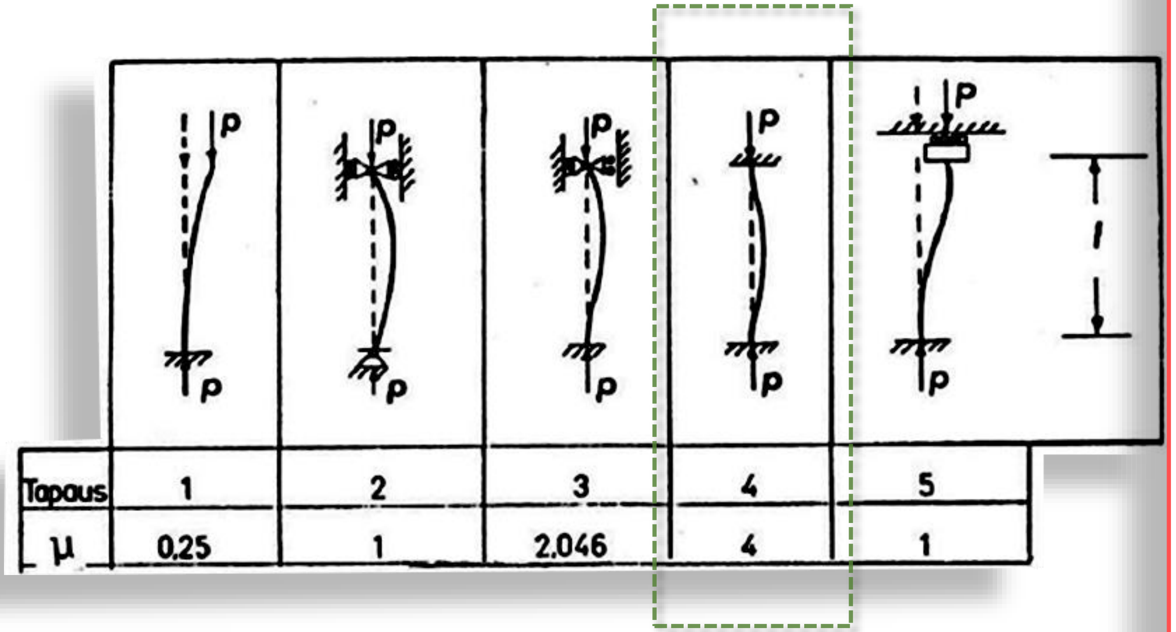

$$P_{cr}(\beta) = (\overbrace{1 + \frac{\beta}{8})}^{\lambda}\frac{\pi^2 EI}{\ell^2}.$$


What to take with you? From the above study we can conclude that: *the buckling load increases with the increase of the stiffness of the foundation. However, at the same time, the bifurcation switches from stable to becomes of unstable-type after a critical value for* $\beta > 8/3$.

Euler's basic buckling cases

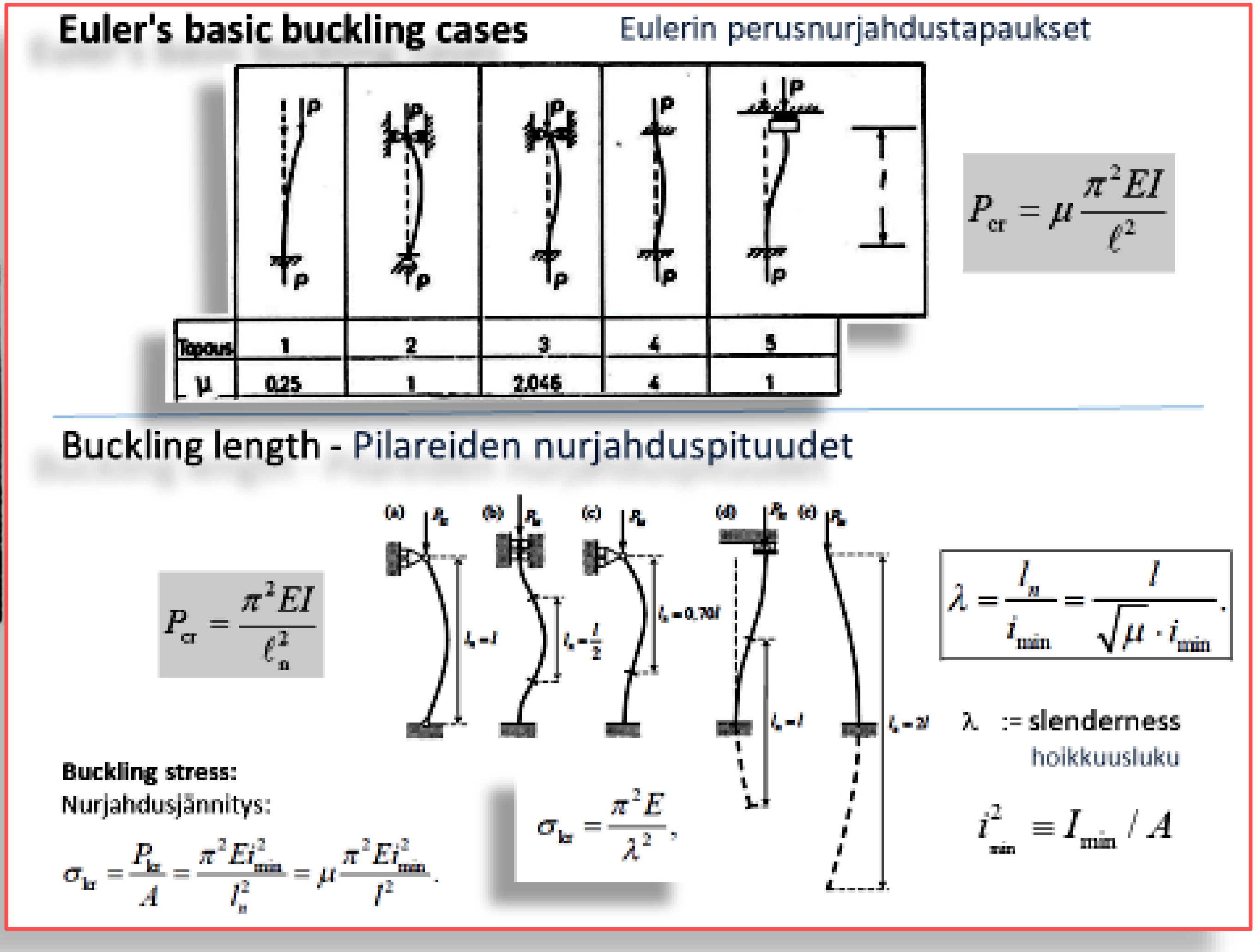
Eulerin perusnurjahdustapaukset

$$P_{\mathrm{cr}} = \mu \frac{\pi^2 EI}{\ell^2}$$



Tapous	1	2	3	4	5
μ	0.25	1	2.046	4	1

$$P_{\mathrm{cr}} = 4 \frac{\pi^2 EI}{\ell^2}$$



Euler's basic buckling cases

Eulerin perusnurjahdustapaukset

Tapous	1	2	3	4	5
μ	0.25	1	2.046	4	1

$$P_{\mathrm{cr}} = \mu \frac{\pi^2 EI}{\ell^2}$$

Buckling length - Pilareiden nurjahduspituudet

$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{\ell_n^2}$$

$$\lambda = \frac{l_n}{i_{\min}} = \frac{l}{\sqrt{\mu} \cdot i_{\min}}.$$

λ := slenderness

hoikkuusluku

$$i_{\min}^2 = I_{\min} / A$$

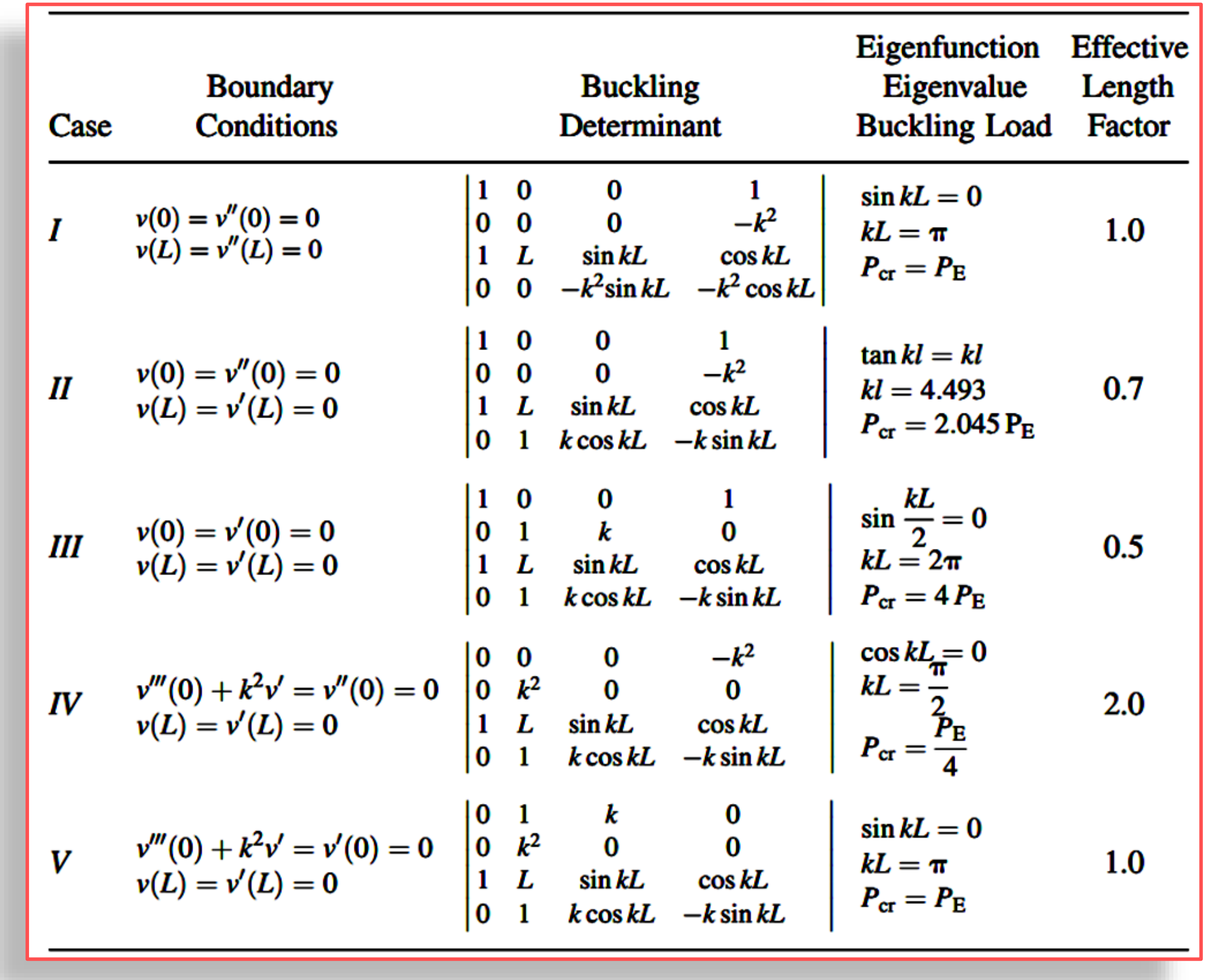
Buckling stress:

Nurjahdusjännitys:

$$\sigma_{kr} = \frac{P_{kr}}{A} = \frac{\pi^2 E i_{\min}^2}{l_n^2} = \mu \frac{\pi^2 E i_{\min}^2}{l^2}.$$

$$\sigma_{kr} = \frac{\pi^2 E}{\lambda^2},$$

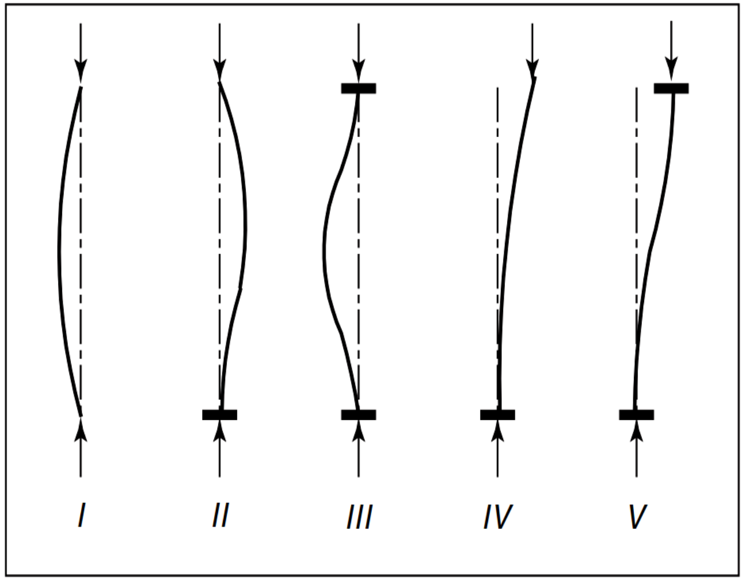
Five Fundamental Cases of Column Buckling



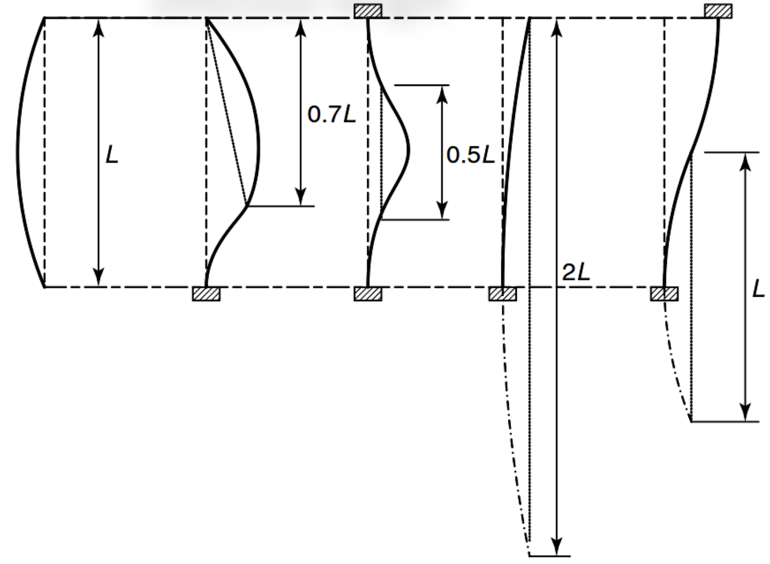
Case	Boundary Conditions	Buckling Determinant	Eigenfunction Eigenvalue Buckling Load	Effective Length Factor
I	$v(0) = v''(0) = 0$ $v(L) = v''(L) = 0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 0 & -k^2 \sin kL & -k^2 \cos kL \end{vmatrix}$	$\sin kL = 0$ $kL = \pi$ $P_{cr} = P_E$	1.0
II	$v(0) = v''(0) = 0$ $v(L) = v'(L) = 0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -k^2 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k \cos kL & -k \sin kL \end{vmatrix}$	$\tan kl = kl$ $kl = 4.493$ $P_{cr} = 2.045\, P_E$	0.7
III	$v(0) = v'(0) = 0$ $v(L) = v'(L) = 0$	$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & k & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k \cos kL & -k \sin kL \end{vmatrix}$	$\sin \frac{kL}{2} = 0$ $kL = 2\pi$ $P_{cr} = 4\, P_E$	0.5
IV	$v'''(0) + k^2 v' = v''(0) = 0$ $v(L) = v'(L) = 0$	$\begin{vmatrix} 0 & 0 & 0 & -k^2 \\ 0 & k^2 & 0 & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k \cos kL & -k \sin kL \end{vmatrix}$	$\cos kL = 0$ $kL = \frac{\pi}{2}$ $P_{cr} = \frac{P_E}{4}$	2.0
V	$v'''(0) + k^2 v' = v'(0) = 0$ $v(L) = v'(L) = 0$	$\begin{vmatrix} 0 & 1 & k & 0 \\ 0 & k^2 & 0 & 0 \\ 1 & L & \sin kL & \cos kL \\ 0 & 1 & k \cos kL & -k \sin kL \end{vmatrix}$	$\sin kL = 0$ $kL = \pi$ $P_{cr} = P_E$	1.0

Adapted from the reference:
STRUCTURAL STABILITY OF STEEL: CONCEPTS AND APPLICATIONS FOR STRUCTURAL ENGINEERS. THEODORE V. GALAMBOS ANDREA E. SUROVEK
JOHN WILEY & SONS, INC.

Elementary buckling cases



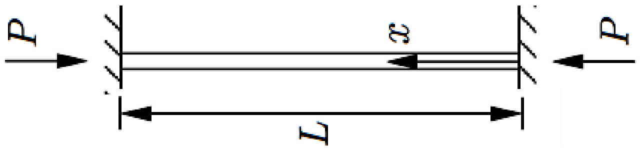
Geometric interpretation of the effective length



Example – rigidly fixed ends column

$$\boxed{v(x) = A\sin kx + B\cos kx + Cx + D}$$

$$v'(x) = Ak\cos kx - Bk\sin kx + C.$$



$$v(0) = v'(0) = v(L) = v'(L) = 0.$$

$$\begin{aligned} B + D &= 0, \\ kA + C &= 0, \\ A\sin kL + B\cos kL + CL + D &= 0, \\ kA\cos kL - kB\sin kL + C &= 0. \end{aligned}$$

$$\boxed{\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ \sin kL & \cos kL & L & 1 \\ k\cos kL & -k\sin kL & 1 & 0 \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

Cf.

$$\boxed{\Longrightarrow \mathbf{Hq} = \mathbf{0}\,,}$$

$$\boxed{\det\{\mathbf{H}\} = 0}$$

Non-trivial solution: the determinant vanishes: $\boxed{\det\{\mathbf{H}\} = 0}$

$$\boxed{4k\sin\frac{kL}{2}(\sin\frac{kL}{2} - \frac{kL}{2}\cos\frac{kL}{2}) = 0.}$$

$\Longrightarrow$ Criticality: $\sin\frac{kL}{2} = 0$ or $\tan\frac{kL}{2} = \frac{kL}{2}$,

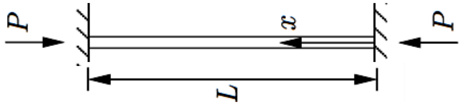
The zeros of the determinant: $\frac{kL}{2} = n\pi\,,\quad n = 1, 2, \ldots\quad,$ $\qquad \frac{kL}{2} \approx 4.493.$

The critical load is the smallest: $k_1 = \frac{2\pi}{L},\quad (n = 1),$ $\boxed{P_1 \equiv P_{kr} = \frac{4\pi^2 EI}{L^2}.}$

The critical load from the Euler's 'Table' : $\boxed{P_{\text{cr}} = 4\frac{\pi^2 EI}{\ell^2}}$

Adapted from ref: prof. Tuomala M.

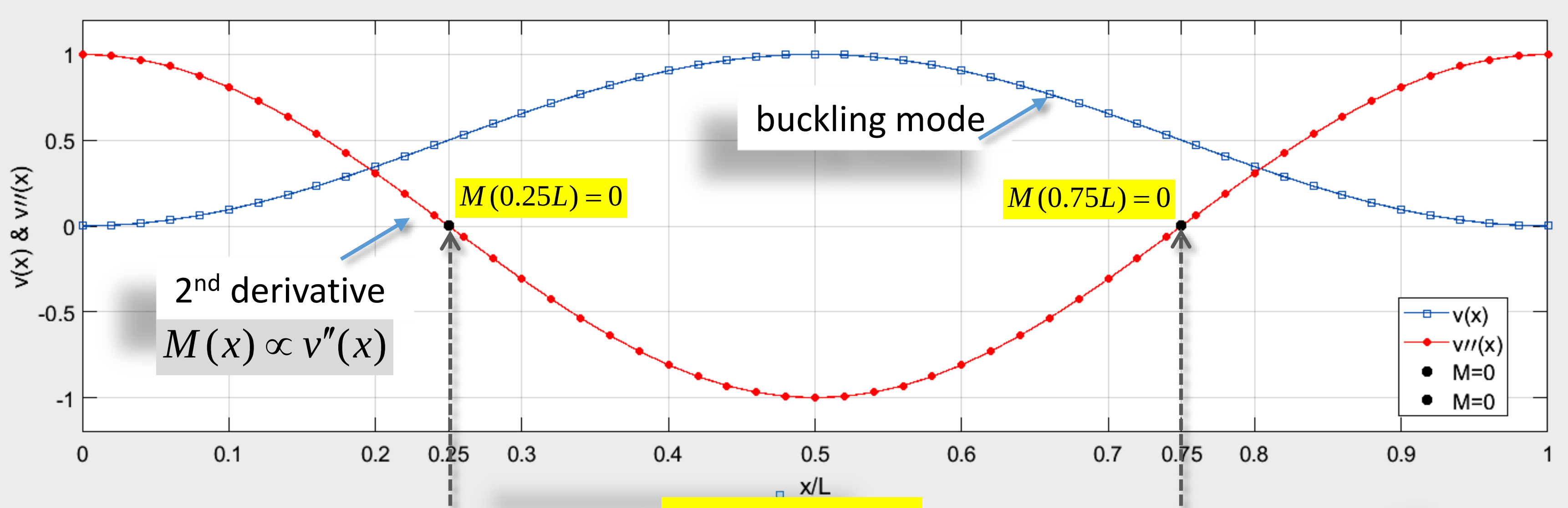
Examples – what is the buckling length?



corresponding buckling mode:

critical load: $P_{\mathrm{cr}} = 4\dfrac{\pi^2 EI}{L^2}$ $\quad P_1 \equiv P_{kr} = \dfrac{4\pi^2 EI}{L^2}.$

$$v(x) = B(\cos\frac{2\pi x}{L} - 1).$$



buckling mode

$M(0.25L) = 0$

$M(0.75L) = 0$

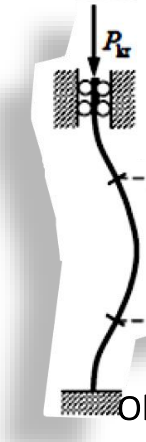
2nd derivative

$M(x) \propto v''(x)$

$\ell_{\text{buckling}} = L/2$

buckling length

$$P_{\mathrm{cr}} = 4\pi^2 EI / L^2 = \pi^2 EI / \ell^2_{\text{buckling}} \Rightarrow \ell_{\text{buckling}} = L/2$$

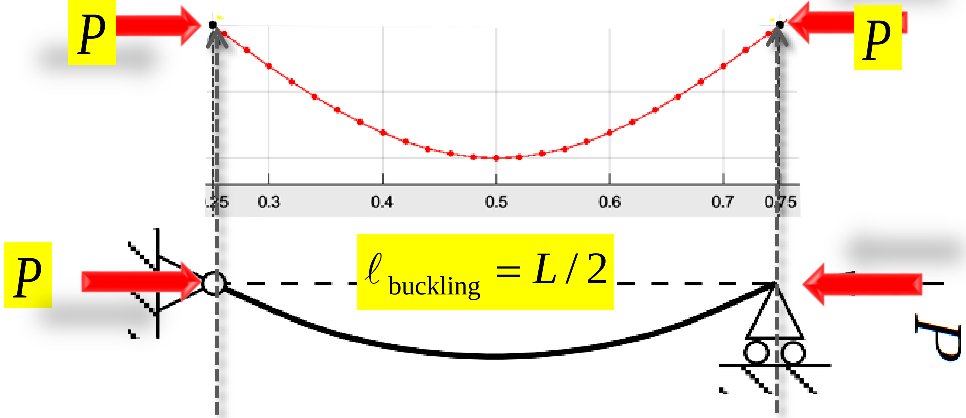


Given by tables

$l_n = \dfrac{l}{2}$

$\ell_{\text{buckling}} = L/2$

Obtained from theory



$\ell_{\text{buckling}} = L/2$

$$\ell_{\text{buckling}} = \frac{1}{\sqrt{\mu}} L$$

Appendix

Stability theorem of Lagrange-Dirichlet

Self-reading

RECALL

Lagrange-Dirichlet Theorem: *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total potential energy of the system has a strict minimum (i.e., is positive-definite).*

$\Longrightarrow$

Trefftz condition

for stability of an equilibrium:

$$\begin{cases} \delta^2\Pi(u) > 0, & \text{stable,} \\ \delta^2\Pi(u) = 0, & \text{neutral,} \\ \delta^2\Pi(u) < 0, & \text{unstable.} \end{cases}$$

- Is a *global energy criterion for stability*
- will be used systematically to derive the all the equations of stability (loss) we need for all elastic structure s

$\Downarrow$

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$

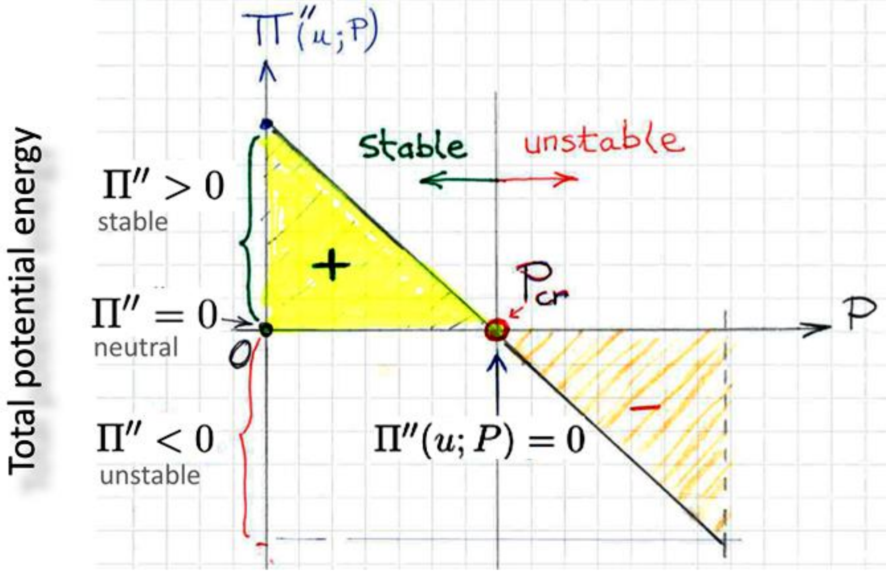
Lagrange

Dirichlet

Lagrange-Dirichlet theorem and investigate the sign of the increment

$$\boxed{\Delta\Pi = \delta\Pi + \delta^2\Pi + \delta^3\Pi + \delta^4\Pi + \ldots}$$

(More general than Trefftz)



Trefftz is a particular case where the total potential energy increment is expanded only up-to its quadratic terms between the initial and perturbed states

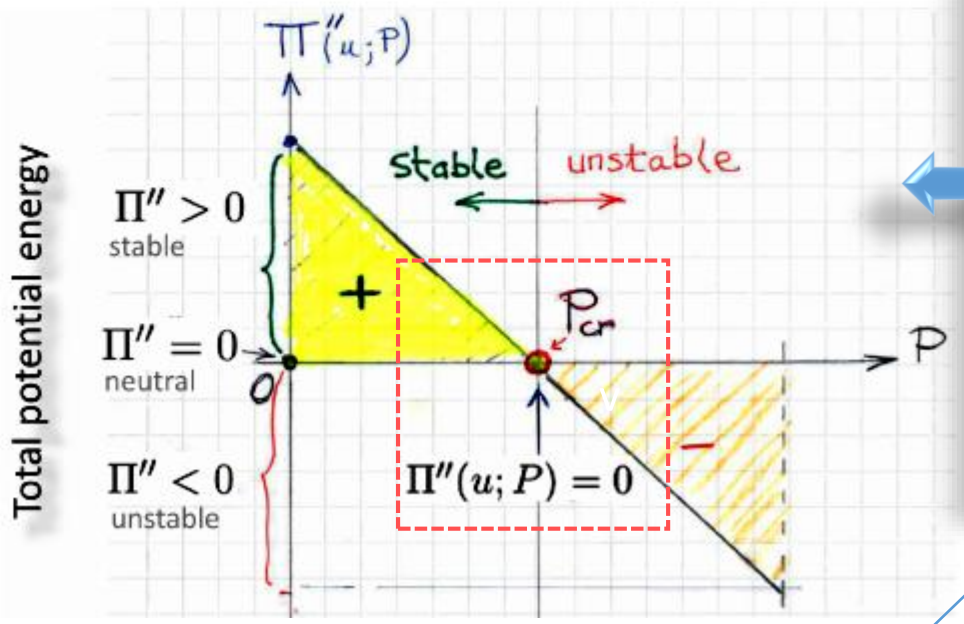
The criteria of loss of stability

RECALL

This is a **Taylor expansion** of a function

$$\underbrace{\Pi(\mathbf{q}^0+\delta\mathbf{q})}_{\mathbf{q}} = \Pi(\mathbf{q}^0) + \sum_{i=1}^{N} \frac{\partial \Pi}{\partial q_i}\Big|_{\mathbf{q}^0} \cdot \delta q_i + \frac{1}{2!}\sum_{i,j=1}^{N} \underbrace{\frac{\partial^2 \Pi}{\partial q_i \partial q_j}\Big|_{\mathbf{q}^0}}_{\equiv \mathbf{H}(\mathbf{q}^0)} \cdot \delta q_i \delta q_j + \ldots$$

$$\approx \Pi(\mathbf{q}^0) + \underbrace{\underbrace{[\nabla\Pi(\mathbf{q}^0)]^{\mathrm{T}}}_{=0,\ \text{equilibrium}} \delta\mathbf{q}}_{\equiv\delta\Pi} + \underbrace{\boxed{\frac{1}{2!}\delta\mathbf{q}^{\mathrm{T}}[\mathbf{H}(\mathbf{q}^0)]\delta\mathbf{q}}}_{\equiv\delta^2\Pi} + \mathcal{O}(\|\delta\mathbf{q}\|^3),$$



at equilibrium ($\delta\Pi = 0$).

$$\boxed{\Delta\Pi = \boxed{\delta^2\Pi} + \mathcal{O}(\|\delta\mathbf{q}\|^3) \sim \frac{1}{2!}\delta\mathbf{q}^{\mathrm{T}}[\mathbf{H}(\mathbf{q}^0)]\delta\mathbf{q}}$$

More suitable form for finite number of dofs and continuous case

Leading term for sign change in the increment of total potential energy

Lagrange-Dirichlet Theorem: *Assuming the continuity of the total potential energy, the equilibrium of a system containing only conservative and dissipative forces is stable if the total po*

A **Taylor expansion** of a function

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

Self-reading

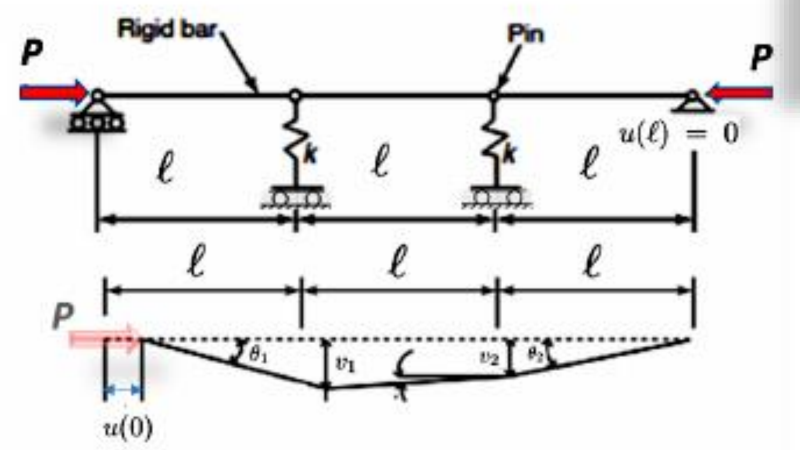
$$\begin{cases} \delta^2\Pi(u) > 0, & \text{stable,} \\ \boxed{\delta^2\Pi(u) = 0, \quad \text{neutral,}} & \\ \delta^2\Pi(u) < 0, & \text{unstable.} \end{cases}$$

$$\boxed{\Pi''(u;P) = 0 \text{ or more generally, } \delta(\Delta\Pi) = 0,}$$

About the criteria of loss of stability – Example with two dofs

RECALL

$$\Delta\Pi(v_1, v_2) = \frac{1}{2}k(v_1^2 + v_2^2) - P\ell\left[\frac{1}{2}\left(\frac{v_1}{\ell}\right)^2 + \frac{1}{2}\left(\frac{v_2 - v_1}{\ell}\right)^2 + \frac{1}{2}\left(\frac{v_2}{\ell}\right)^2\right]$$



1.42: A simple system having two degrees of freedom.

Requiring the neutral equilibrium condition $\delta(\Delta\Pi) = 0$ (for loss of stability)

$$\Delta\Pi(v_1, v_2) = \frac{1}{2}\begin{bmatrix} v_1 & v_2 \end{bmatrix} \underbrace{\left(\underbrace{\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}}_{\mathbf{K}} - \underbrace{\frac{P}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{\mathbf{S}(P)} \right)}_{\mathbf{H}(0,0)} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{1.68}$$

So, one obtains the *quadratic form*

$$\Delta\Pi(\mathbf{q}) = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{q}, \tag{1.69}$$

where $\mathbf{q}$ being a tiny deviation from trivial equilibrium configuration $\mathbf{q}^0 = \mathbf{0}$ and

$$\mathbf{H} = \begin{bmatrix} \lambda - 2P & P \\ P & \lambda - 2P \end{bmatrix}. \tag{1.70}$$

We can also write directly the loss of stability condition in its *variational form* $\delta(\Delta\Pi) = 0$ and obtain

$$\delta(\Delta\Pi) = \frac{1}{2}\delta\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{q} + \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{H}\delta\mathbf{q} = \delta\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{q} = 0, \forall\delta\mathbf{q} \implies \tag{1.71}$$

$$\implies \mathbf{H}\mathbf{q} = \mathbf{0}\text{, which is linear Eigen-value problem.} \tag{1.72}$$

Note that the coefficient matrix of the associated Eigen-value problem (Equation 1.66) is the same[60] than our Hessian matrix So loss of stability occurs when

$$\Pi'' = 0 \sim \det\{\mathbf{H}\} = 0 \tag{1.73}$$

Self-reading

Energy criteria for determination of instability of elastic structures

Self-reading

RECALL

Let's illustrate mathematically the basic stability types

- stable
- unstable
- Indifferent

keeping a simplified example of the rigid ball (null strain energy)

The total potential energy of the system $\Pi(x) = \Pi_0 + mgax^2$

Initial total potential energy

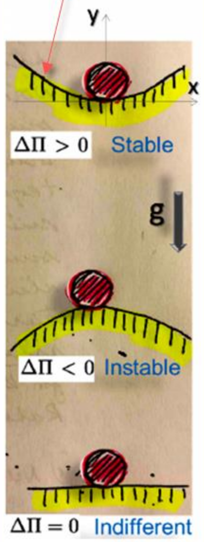
potential energy of gravitation

Geometry locally approximated $y(x) = ax^2$

$a > 0$ for convex,

$a < 0$ for concave

$a = 0$ for the neutral



Three various types of equilibrium configurations.

perturbed equilibrium position

$$\Pi(x_0+\delta x) = \Pi(x_0) + \underbrace{\frac{\mathrm{d}\Pi(x)}{\mathrm{d}x}|_{x_0}\delta x}_{\delta\Pi|_{x_0}} + \frac{1}{2}\underbrace{\boxed{\frac{\mathrm{d}^2\Pi(x)}{\mathrm{d}x^2}}|_{x_0}(\delta x)^2}_{\delta^2\Pi|_{x_0}} + \frac{1}{3!}\underbrace{\frac{\mathrm{d}^3\Pi(x)}{\mathrm{d}x^3}|_{x_0}(\delta x)^3}_{\delta^3\Pi|_{x_0}} + \dots$$

Initial equilibrium position

$$\equiv \Pi(x_0) + \delta\Pi|_{x_0} + \frac{1}{2}\delta^2\Pi|_{x_0} + \frac{1}{3!}\delta^3\Pi|_{x_0} + \dots$$

Since x_0 is an equilibrium then $\delta\Pi|_{x_0} = 0$.

$$\Longrightarrow \quad \boxed{\Delta\Pi = \Pi(x_0+\delta x) - \Pi(x_0) = \frac{1}{2}\boxed{\delta^2\Pi|_{x_0}} + \frac{1}{3!}\delta^3\Pi|_{x_0} + \dots}$$

The sign will provides us the nature of **stability**

Equilibrium? Yes.
But, is it **stable**? No.

$\Pi'' = 2mga.$

… or equivalently

$$\begin{cases} \Pi'' > 0, & \text{stable,} \\ \Pi'' = 0, & \text{neutral,} \\ \Pi'' < 0, & \text{unstable.} \end{cases}$$

`The idea is the make the study of stability in terms of` *`variational calculus`*

Energy criteria for determination of instability of elastic structures

Self-reading

RECALL

First, keep only up-to the second order[21] term:

$$\Delta\Pi = \frac{1}{2}\boxed{\frac{\mathrm{d}^2\Pi(x)}{\mathrm{d}x^2}\Big|_{x_0}}(\delta x)^2 = mga(\delta x)^2 + O(\delta x)^3.$$

Consequently, the initial equilibrium x_0 is stable when $a > 0$ (locally convex surface), unstable for $a < 0$ (locally concave surface) and indifferent when $a = 0$.

Bellow follows a résumé: At the critical points (equilibrium points), studying the sign of the increment of total potential energy $\Delta\Pi$, makes it possible to make statements on the nature of the actual equilibrium:

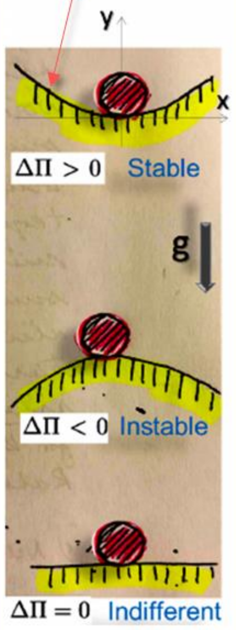
1. **stable**: (stabiili) $\Delta\Pi > 0$
2. **indifferent** : (indiferentti) $\Delta\Pi = 0$. Often, the total potential energy increment $\Delta\Pi$ is expanded to second order only (squares of small displacements). In this case, $\Delta\Pi = 0$ and therefore, higher order terms should be included in the Taylor expansion to decide of the sign of $\Delta\Pi$ to disclose the character of indifferent equilibrium.
3. **unstable**: (labiili, epästabiili) $\Delta\Pi < 0$

Geometry locally approximated $y(x) = ax^2$

$a > 0$ for convex,

$a < 0$ for concave

$a = 0$ for the neutral

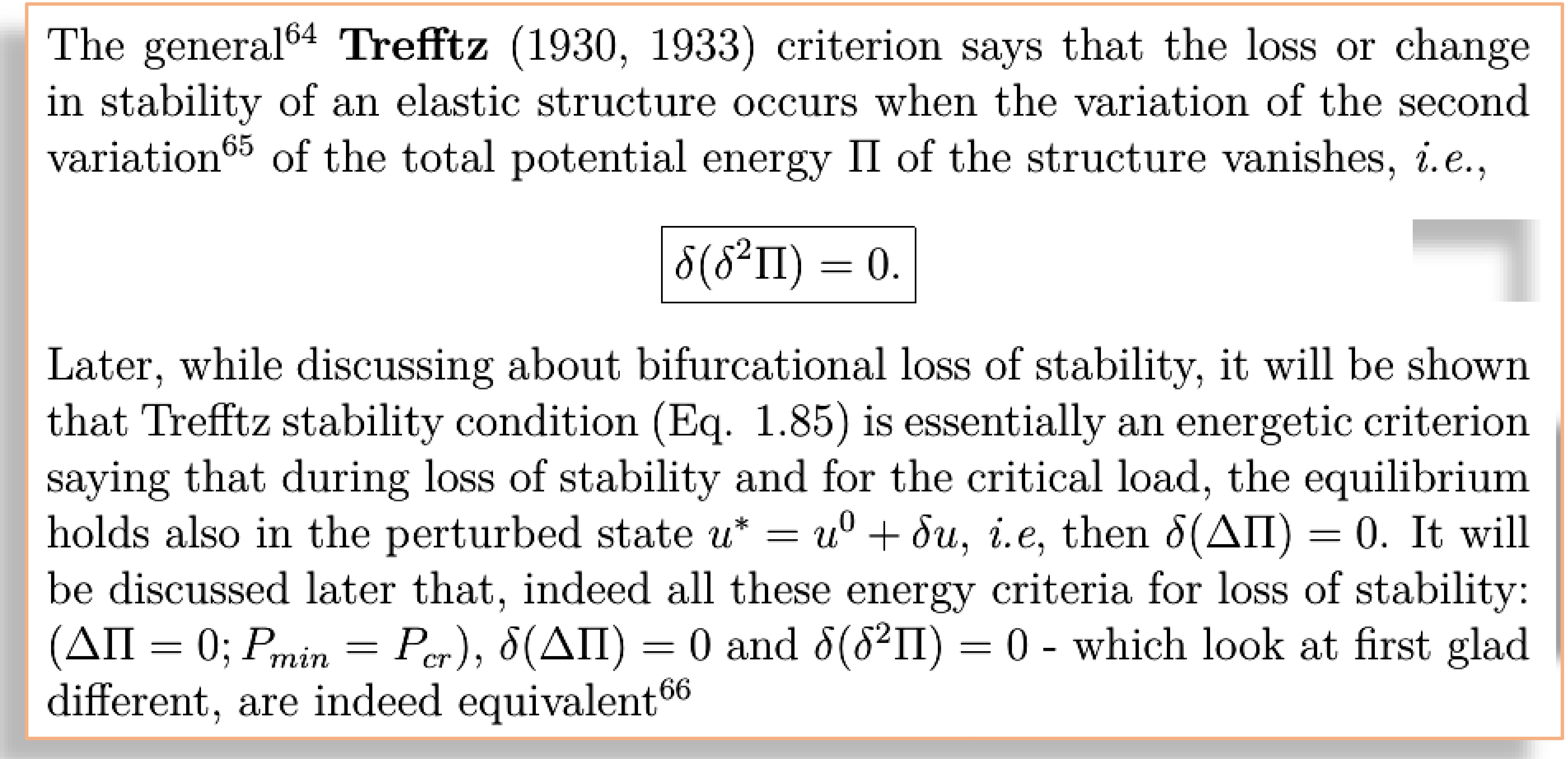


Three various types of equilibrium configurations.

Equilibrium? Yes.
But, is it **stable**? No.

Energy criteria for determination **loss of stability** of elastic structures

Self-reading

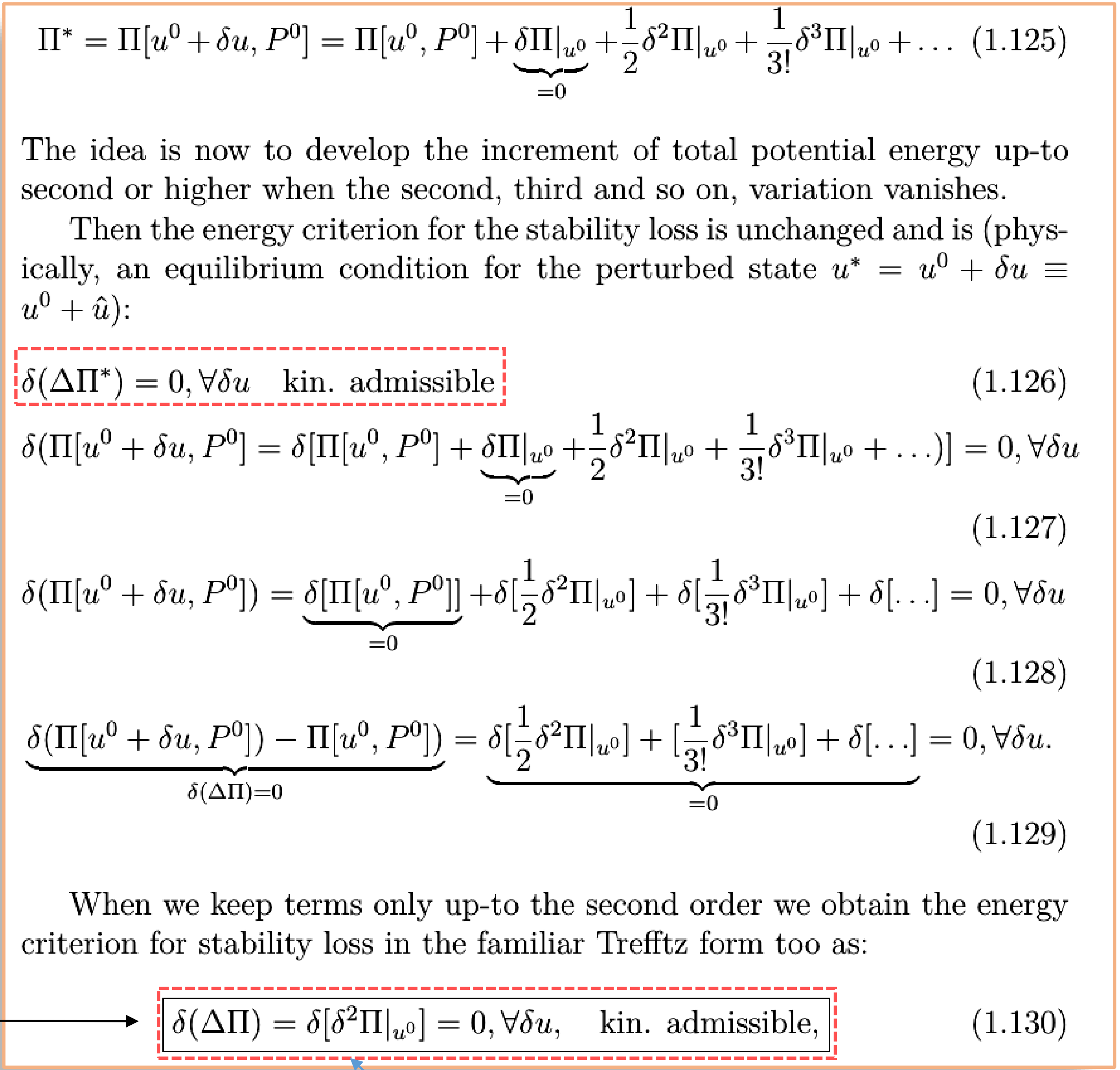


The general[64] **Trefftz** (1930, 1933) criterion says that the loss or change in stability of an elastic structure occurs when the variation of the second variation[65] of the total potential energy Π of the structure vanishes, *i.e.*,

$$\boxed{\delta(\delta^2\Pi) = 0.}$$

Later, while discussing about bifurcational loss of stability, it will be shown that Trefftz stability condition (Eq. 1.85) is essentially an energetic criterion saying that during loss of stability and for the critical load, the equilibrium holds also in the perturbed state $u^* = u^0 + \delta u$, *i.e*, then $\delta(\Delta\Pi) = 0$. It will be discussed later that, indeed all these energy criteria for loss of stability: $(\Delta\Pi = 0; P_{min} = P_{cr})$, $\delta(\Delta\Pi) = 0$ and $\delta(\delta^2\Pi) = 0$ - which look at first glad different, are indeed equivalent[66]

Self-reading



$$\Pi^* = \Pi[u^0+\delta u, P^0] = \Pi[u^0, P^0] + \underbrace{\delta\Pi|_{u^0}}_{=0} + \frac{1}{2}\delta^2\Pi|_{u^0} + \frac{1}{3!}\delta^3\Pi|_{u^0} + \dots \tag{1.125}$$

The idea is now to develop the increment of total potential energy up-to second or higher when the second, third and so on, variation vanishes.

Then the energy criterion for the stability loss is unchanged and is (physically, an equilibrium condition for the perturbed state $u^* = u^0 + \delta u \equiv u^0 + \hat{u}$):

$$\boxed{\delta(\Delta\Pi^*) = 0, \forall\delta u \quad \text{kin. admissible}} \tag{1.126}$$

$$\delta(\Pi[u^0+\delta u, P^0] = \delta[\Pi[u^0, P^0] + \underbrace{\delta\Pi|_{u^0}}_{=0} + \frac{1}{2}\delta^2\Pi|_{u^0} + \frac{1}{3!}\delta^3\Pi|_{u^0} + \dots)] = 0, \forall\delta u \tag{1.127}$$

$$\delta(\Pi[u^0+\delta u, P^0]) = \underbrace{\delta[\Pi[u^0, P^0]]}_{=0} + \delta[\frac{1}{2}\delta^2\Pi|_{u^0}] + \delta[\frac{1}{3!}\delta^3\Pi|_{u^0}] + \delta[\dots] = 0, \forall\delta u \tag{1.128}$$

$$\underbrace{\delta(\Pi[u^0+\delta u, P^0]) - \Pi[u^0, P^0])}_{\delta(\Delta\Pi)=0} = \underbrace{\delta[\frac{1}{2}\delta^2\Pi|_{u^0}] + [\frac{1}{3!}\delta^3\Pi|_{u^0}] + \delta[\dots]}_{=0} = 0, \forall\delta u. \tag{1.129}$$

When we keep terms only up-to the second order we obtain the energy criterion for stability loss in the familiar Trefftz form too as:

$$\boxed{\boxed{\delta(\Delta\Pi) = \delta[\delta^2\Pi|_{u^0}] = 0, \forall\delta u, \quad \text{kin. admissible,}}} \tag{1.130}$$

We will use systematically this more general energy criterion:

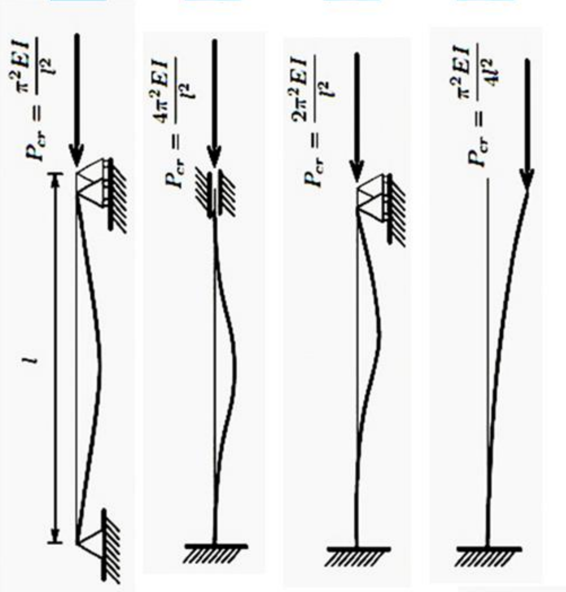
Trefftz stability loss criteria in its canonical form

Effects of boundary conditions – experimental evidence for Euler's buckling formulas

PINNED-PINNED **FIXED-FIXED** **FIXED-PINNED** **FIXED-FREE**

1 4 2 1/4

PINNED-PINNED **FIXED-FIXED** **FIXED-PINNED** **FIXED-FREE**



$$P_{cr} = \frac{\pi^2 EI}{l^2} \quad \frac{4\pi^2 EI}{l^2} \quad \frac{2\pi^2 EI}{l^2} \quad \frac{\pi^2 EI}{4l^2}$$

1 4 2 1/4

Change of total potential energy – example of a buckling cantilever

Bryan form

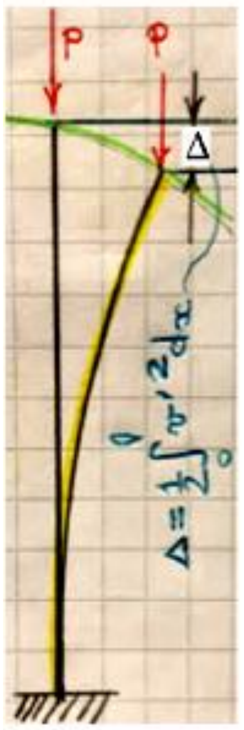
$$\boxed{\Delta\Pi = \frac{1}{2}\int_V {\epsilon_1}^{\mathrm{T}}\mathbf{E}\epsilon_1 \mathrm{d}V + \int_V {\epsilon_2}^{\mathrm{T}}\sigma^0 \mathrm{d}V.}$$

+ increment of work of external work not accounted in by the work of initial stresses

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2\mathrm{d}x + \int_0^\ell \sigma_x^0 A \underbrace{\left[\frac{1}{2}(v')^2\right]}\mathrm{d}x,$$

$$\Delta\Pi = \frac{1}{2}\int_0^\ell EI(v'')^2\mathrm{d}x - P\underbrace{\int_0^\ell \left[\frac{1}{2}(v')^2\right]\mathrm{d}x}_{\Delta},$$

$$\Delta V = -\Delta W_{\text{ext}} = -P\int_0^\ell \left[\frac{1}{2}(v')^2\right]\mathrm{d}x$$



$$\mathbf{u}^* = \mathbf{u}^0 + \delta\mathbf{u}$$

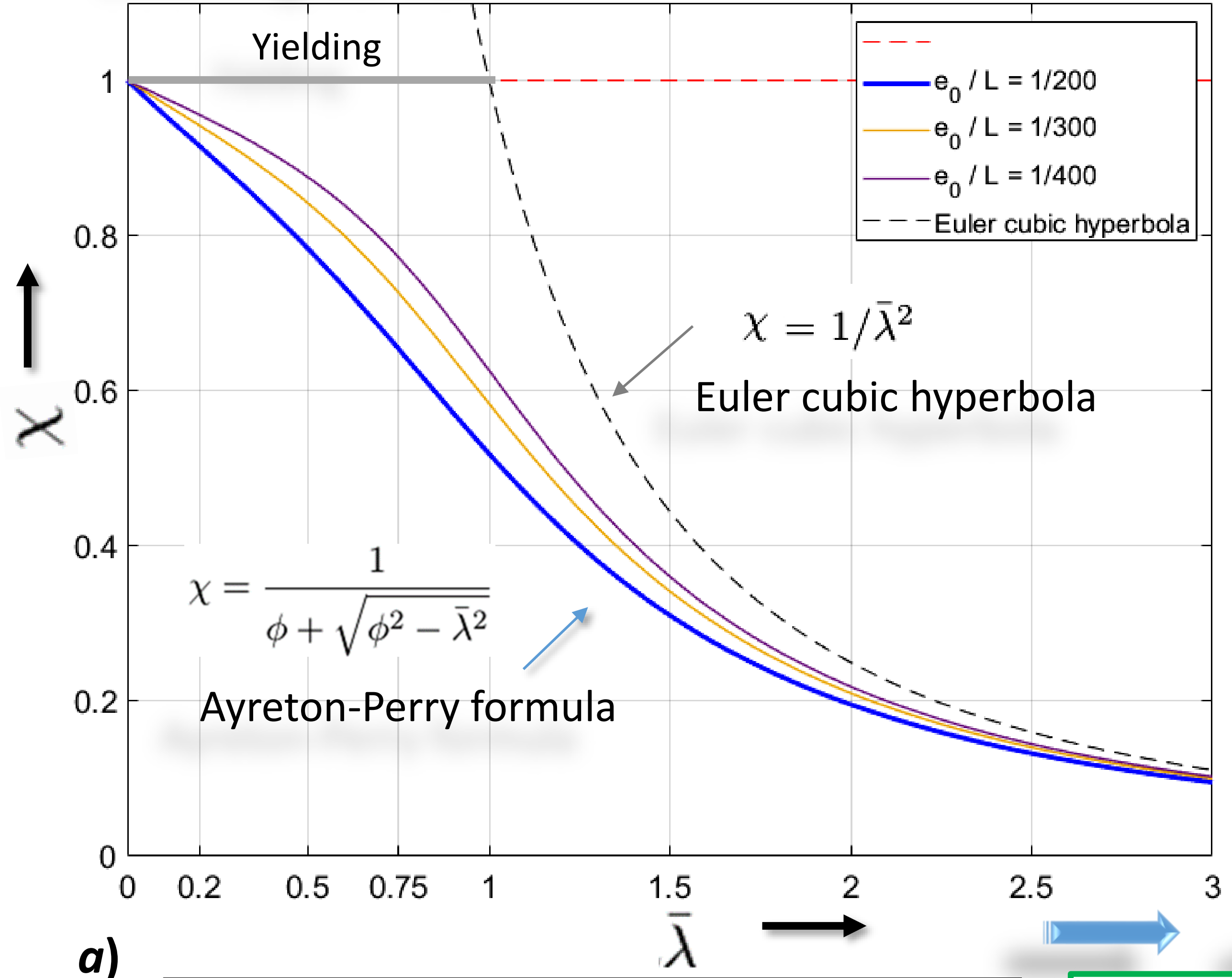
$$\epsilon_{ij}^* = \frac{1}{2}(\underbrace{u_{i,j} + u_{j,i}}_{\text{Linear part}} + u_{k,i}u_{k,j})$$

Non-linear part

Ayreton-Perry design formula

Steel $\ell = 2\text{m},$ $a = [0.446,\ 0.297,\ 0.223],$ $i = 0.1714\ \text{m},\ h = 0.2\ \text{m},$ $e_0/\ell = [1/400,\ 1/300,\ 1/400]$

Buckling curves



$\chi = 1/\bar{\lambda}^2$

Euler cubic hyperbola

$$\chi = \frac{1}{\phi + \sqrt{\phi^2 - \bar{\lambda}^2}}$$

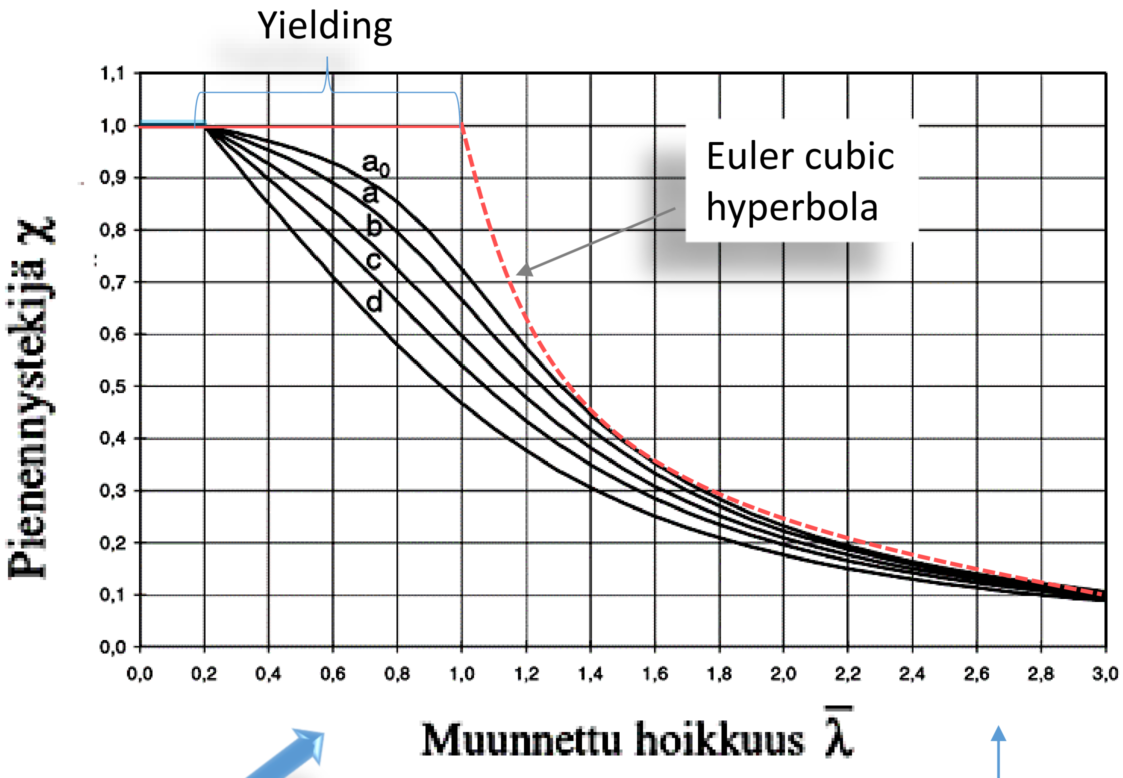
Ayreton-Perry formula

a)

$$\chi = \frac{1}{\phi + \sqrt{\phi^2 - \bar{\lambda}^2}}, \quad \text{where} \ \ \phi = \frac{1}{2}\left[1 + a\bar{\lambda} + \bar{\lambda}^2\right]$$

$$\boxed{N_s = P \leq N_R = \chi \cdot \frac{\sigma_y A}{\gamma}}$$

Eurocode buckling curves



Non-dimensional slenderness

b)

$$a\bar{\lambda} = [e_0 h/2]/i^2$$

$$a = \pi\sqrt{E/\sigma_y}\,\frac{e_0}{\ell}\,\frac{h/2}{i}$$

$$\chi = P/P_y.$$

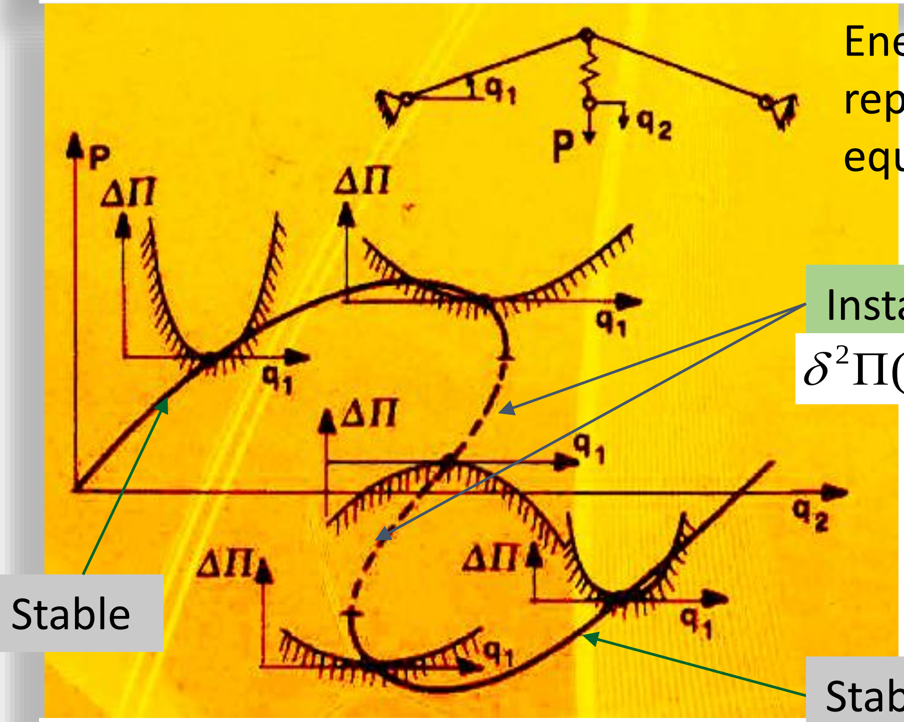
Adapted from Eurocode 3

Equilibrium path, Stability, Instability

Examples – snap-through

Note that loss of stability may happen also without bifurcation through limit points as here

Slide from "Beams and Frames – course'



Energy space representation and equilibrium paths

Instable

$\delta^2\Pi(q_1,q_2) < 0$

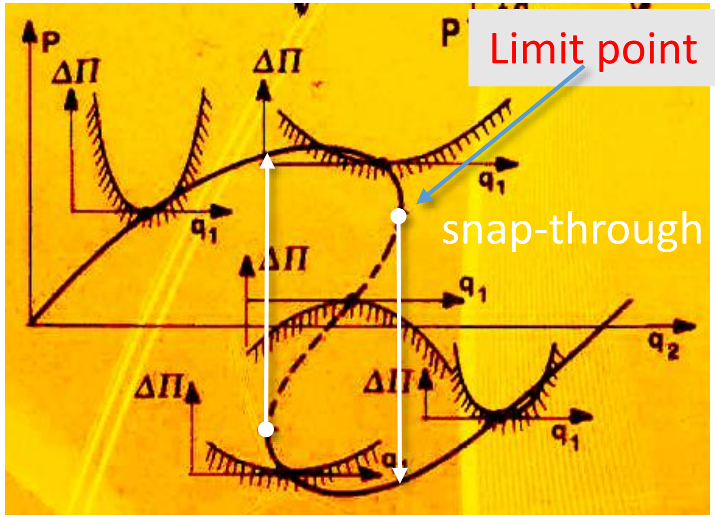
Stable

Stable

$\delta^2\Pi(q_1,q_2) > 0$

Ref: Bazant's classical textbook on stability

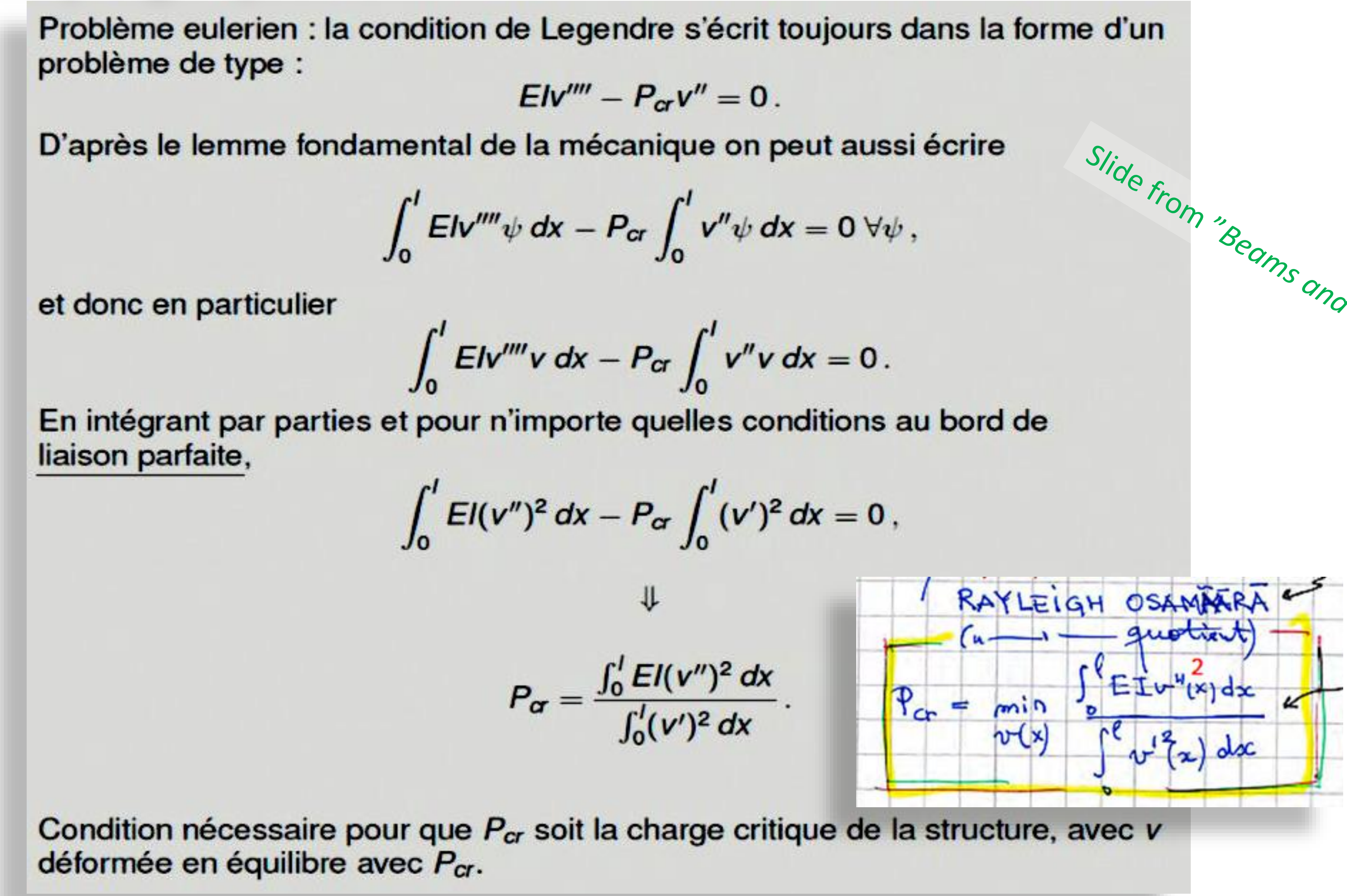
$$\Delta\Pi = \delta\Pi + \delta^2\Pi + \ldots$$



Limit point

snap-through

The Rayleigh-quotient



Problème eulerien : la condition de Legendre s'écrit toujours dans la forme d'un problème de type :

$$EIv'''' - P_{cr}v'' = 0\,.$$

D'après le lemme fondamental de la mécanique on peut aussi écrire

$$\int_0^l EIv''''\psi\,dx - P_{cr}\int_0^l v''\psi\,dx = 0\ \forall\psi\,,$$

et donc en particulier

$$\int_0^l EIv''''v\,dx - P_{cr}\int_0^l v''v\,dx = 0\,.$$

En intégrant par parties et pour n'importe quelles conditions au bord de liaison parfaite,

$$\int_0^l EI(v'')^2\,dx - P_{cr}\int_0^l (v')^2\,dx = 0\,,$$

$$\Downarrow$$

$$P_{cr} = \frac{\int_0^l EI(v'')^2\,dx}{\int_0^l (v')^2\,dx}\,.$$

Condition nécessaire pour que P_{cr} soit la charge critique de la structure, avec v déformée en équilibre avec P_{cr}.

RAYLEIGH OSAMĀRĀ

(u—— , —— quotient)

$$P_{cr} = \min_{v(x)} \frac{\int_0^l EIv''^{2}(x)\,dx}{\int_0^l v'^{2}(x)\,dx}$$

Slide from "Beams and Frames – course'

https://eductv.enpc.fr/videos/mecanique-des-structures-seance-8/ (5.10.2017)

Homework?

slope-deflection method

Show the above result.

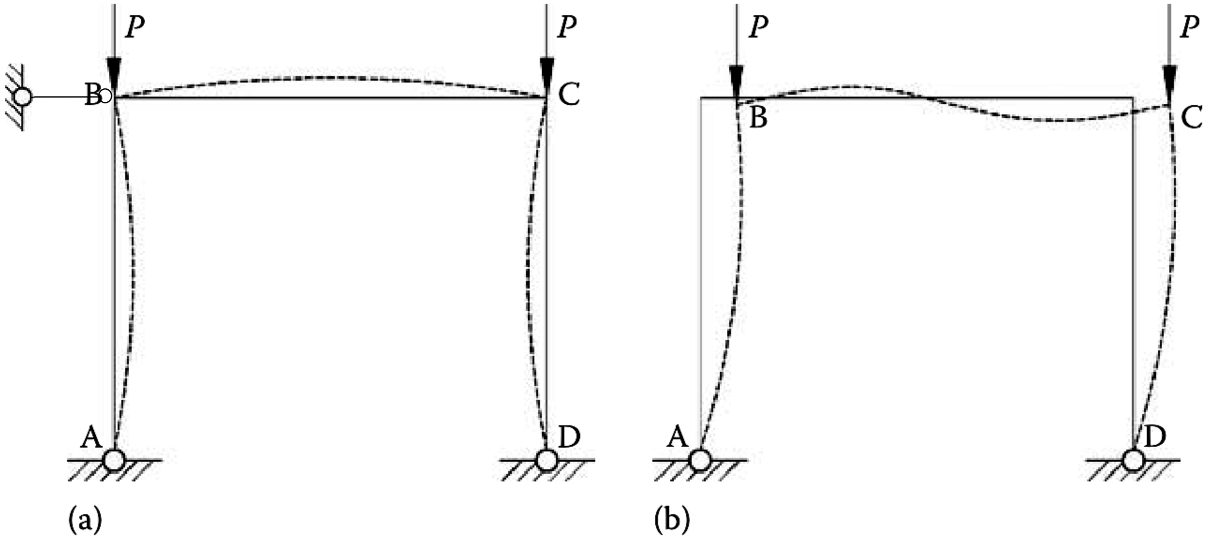
All beams and columns elements bending rigidities are equal. The height and the span are equal too.

Hint: you can assume the symmetric and the anti-symmetric modes of buckling. Think how this hypothesis can simplifies or reduces your problem.

Slide from "Beams and Frames – course'

$$P_{cr} = 12.9 \frac{EI}{L^2}$$

$$P_{cr} = 1.82 \frac{EI}{L^2}$$



framed systems: (a) no side-sway and (b) side-sway allowed

Slope-deflection method

Stiffness coefficients and Berry's stability functions [1]

Slides from "Beams and Frames – course'

- **The geometrically non-linear problem** (Called also sometime the *stress-problem*): The equilibrium equation should be written in the deformed configuration. The stiffness matrix is now non-linear. As for bending without axial load, we here solve the BVP with given four boundary conditions at the two nodes (or ends) of the beam where nodal deflections and rotations are given. Solving for the bending moment at end 1, one obtains again the stiffness-equations of the well known & versatile *slope-deflection method*
- Now, in the slope-deflection method the stiffness coefficients are magnified by a factor depending on member compressive/tensional load which are called Stability or Berry's functions.

---

[1] Berry, A. (1916). The Calculation of Stresses in Aeroplane Spars. *Transactions of the Royal Aeronautical Society, 1*.

Slope-deflection method – Stiffness-equation

The stiffness equations of the slope-deflection method with axial load

$$M_{ij} = A_{ij}(P)\varphi_{ij} + B_{ij}(P)\varphi_{ij} - C_{ij}(P)\psi_{ij} + MK_{ij}(P)$$

$$ij = \{\ 12,\ 21\}$$

Stiffness-coefficients and loading terms depend on the member axial force

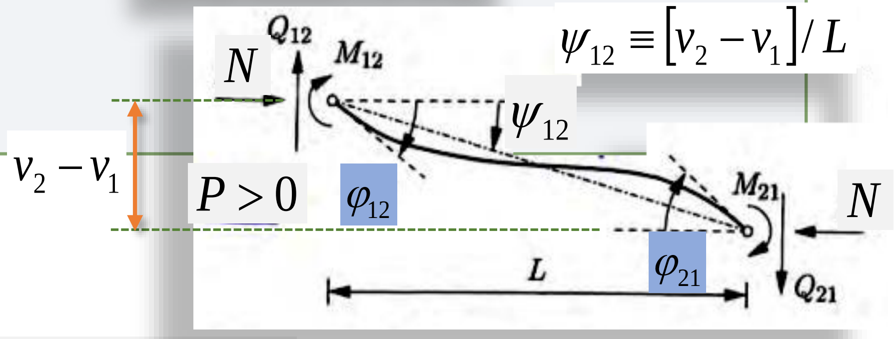
$$\lambda \equiv kL \equiv L\sqrt{\frac{P}{EI}}$$

$$M_{12} = A_{12}^{0}(kL)\varphi_{12} - C_{12}^{0}(kL)\psi_{21} + MK_{12}^{0}(kL)$$

Stiffness-coefficients are symmetric with respect to *i* and *j*

Compression : P > 0

$$\psi_{12} \equiv [v_2 - v_1]/L$$



$$N \equiv -N_{12} = P > 0$$

Case of compression : P > 0

Member axial force can be compressive or tensile. The stiffness-coefficients are different in compression and in tension.

The stiffness coefficients – axial compression and bending

Compression : P > 0 $\psi_{12} \equiv [v_2 - v_1]/\ell$

NB. Notation: $y \equiv v$ $\theta \equiv \varphi$

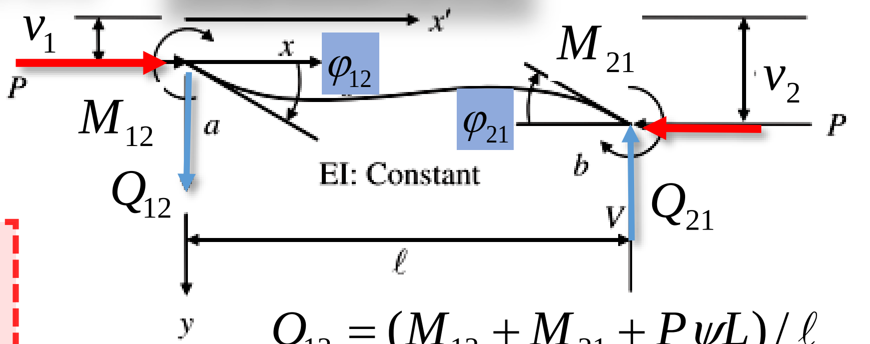
$$v^{(4)}(x) + k^2 v''(x) = 0$$

$$v(x) = A\sin(kx) + B\cos(kx) + Cx + D$$

Boundary conditions:

$$v(0) = v_1 = 0 \quad v(\ell) = v_2 \equiv \psi_{12}\ell = v_2 - v_1 \equiv \Delta$$

$$v'(0) = \varphi_{12} \quad \text{and} \quad v'(\ell) = \varphi_{21}$$



$$Q_{12} = (M_{12} + M_{21} + P\psi L)/\ell$$

$$\Delta = \psi_{12}\ell = v_2 - v_1 = v_2 - 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin\beta & \cos\beta & \ell & 1 \\ k & 0 & 1 & 0 \\ k\cos\beta & -k\sin\beta & 1 & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ \Delta \\ \varphi_{12} \\ \varphi_{21} \end{Bmatrix}$$

However, it is more practical to express the stiffness coefficients in terms of Berry's functions as we did till now.

$$\Downarrow$$

$$M_{12} = M(0) = -EIv''(0) = EIBk^2$$

$$\beta \equiv k\ell \equiv \lambda$$

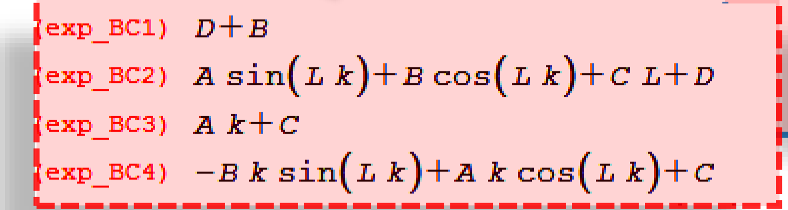
$$= \left[\frac{EIk^2}{k(2\cos\beta + \beta\sin\beta - 2)}\right]\left[(\beta\cos\beta - \sin\beta)\varphi_{12} + (\sin\beta - \beta)\varphi_{21} + (k - k\cos\beta)\Delta\right]$$

$$= \left[\frac{EI\beta}{\ell(2\cos\beta + \beta\sin\beta - 2)}\right]\left[(\beta\cos\beta - \sin\beta)\varphi_{12} + (\sin\beta - \beta)\varphi_{21} + (\beta - \beta\cos\beta)\frac{\Delta}{\ell}\right]$$

$$A_{12}(k\ell) = M(0, k\ell)\downarrow$$

$$A_{12}(\lambda) = \frac{\lambda(\lambda\cos\lambda - \sin\lambda)}{2\cos\lambda + \lambda\sin\lambda - 2}$$

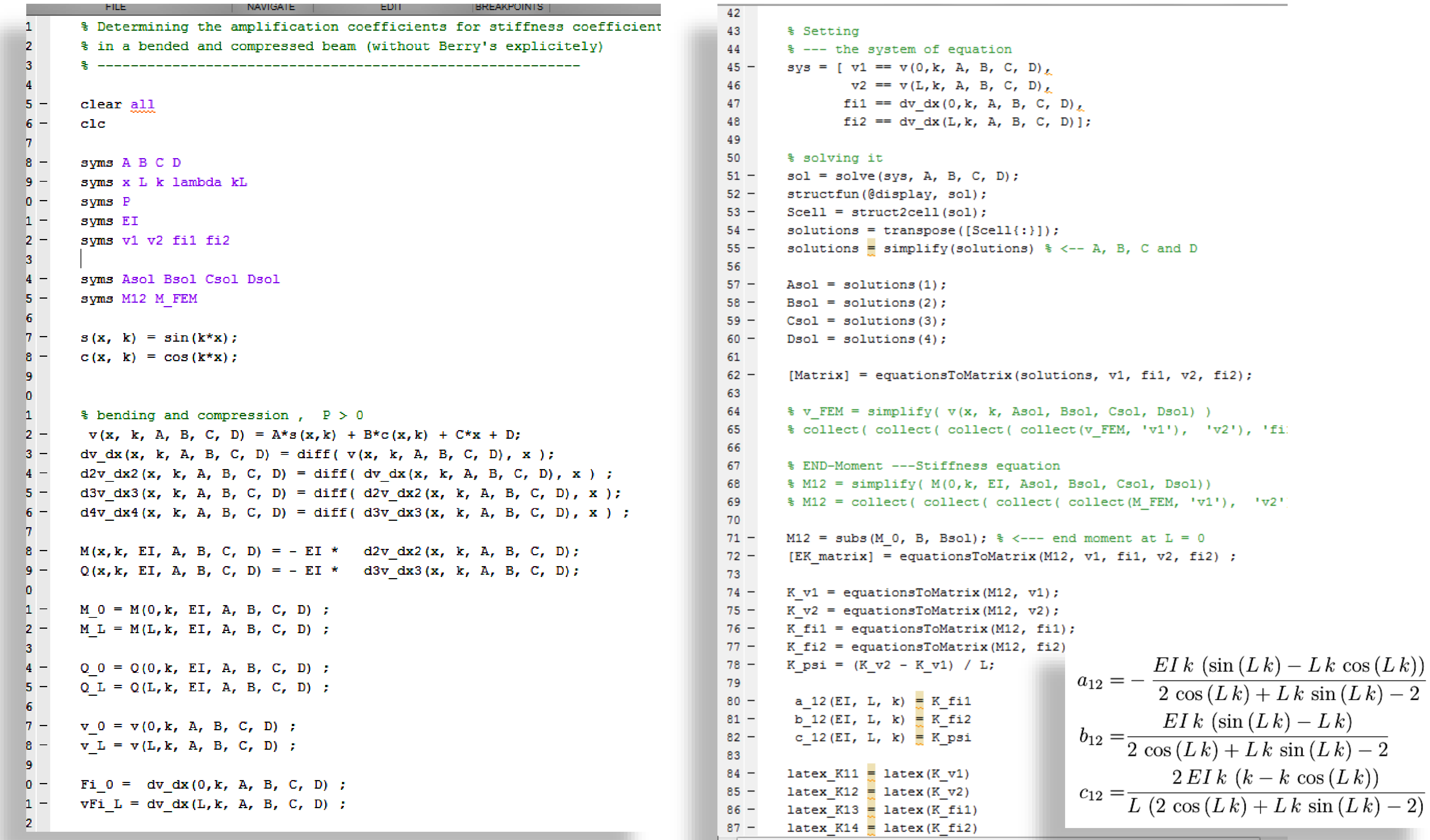
$$\sin\beta = 2\sin(\beta/2)\cos(\beta/2)$$



```
(exp_BC1) D+B
(exp_BC2) A sin(L k)+B cos(L k)+C L+D
(exp_BC3) A k+C
(exp_BC4) -B k sin(L k)+A k cos(L k)+C
```

We have earlier established these eqs previously when using Maxima

$$\beta \equiv k\ell \equiv \lambda$$



```
% Determining the amplification coefficients for stiffness coefficient
% in a bended and compressed beam (without Berry's explicitely)
% --------------------------------------------------------------

clear all
clc

syms A B C D
syms x L k lambda kL
syms P
syms EI
syms v1 v2 fi1 fi2

syms Asol Bsol Csol Dsol
syms M12 M_FEM

s(x, k) = sin(k*x);
c(x, k) = cos(k*x);


% bending and compression ,  P > 0
 v(x, k, A, B, C, D) = A*s(x,k) + B*c(x,k) + C*x + D;
dv_dx(x, k, A, B, C, D) = diff( v(x, k, A, B, C, D), x );
d2v_dx2(x, k, A, B, C, D) = diff( dv_dx(x, k, A, B, C, D), x ) ;
d3v_dx3(x, k, A, B, C, D) = diff( d2v_dx2(x, k, A, B, C, D), x );
d4v_dx4(x, k, A, B, C, D) = diff( d3v_dx3(x, k, A, B, C, D), x ) ;

M(x,k, EI, A, B, C, D) = - EI *   d2v_dx2(x, k, A, B, C, D);
Q(x,k, EI, A, B, C, D) = - EI *   d3v_dx3(x, k, A, B, C, D);

M_0 = M(0,k, EI, A, B, C, D) ;
M_L = M(L,k, EI, A, B, C, D) ;

Q_0 = Q(0,k, EI, A, B, C, D) ;
Q_L = Q(L,k, EI, A, B, C, D) ;

v_0 = v(0,k, A, B, C, D) ;
v_L = v(L,k, A, B, C, D) ;

Fi_0 =  dv_dx(0,k, A, B, C, D) ;
vFi_L = dv_dx(L,k, A, B, C, D) ;

```

```

% Setting
% --- the system of equation
sys = [ v1 == v(0,k, A, B, C, D),
        v2 == v(L,k, A, B, C, D),
       fi1 == dv_dx(0,k, A, B, C, D),
       fi2 == dv_dx(L,k, A, B, C, D)];

% solving it
sol = solve(sys, A, B, C, D);
structfun(@display, sol);
Scell = struct2cell(sol);
solutions = transpose([Scell{:}]);
solutions = simplify(solutions) % <-- A, B, C and D

Asol = solutions(1);
Bsol = solutions(2);
Csol = solutions(3);
Dsol = solutions(4);

[Matrix] = equationsToMatrix(solutions, v1, fi1, v2, fi2);

% v_FEM = simplify( v(x, k, Asol, Bsol, Csol, Dsol) )
% collect( collect( collect( collect(v_FEM, 'v1'),  'v2'), 'fi

% END-Moment ---Stiffness equation
% M12 = simplify( M(0,k, EI, Asol, Bsol, Csol, Dsol))
% M12 = collect( collect( collect( collect(M_FEM, 'v1'),  'v2'

M12 = subs(M_0, B, Bsol); % <--- end moment at L = 0
[EK_matrix] = equationsToMatrix(M12, v1, fi1, v2, fi2) ;

K_v1 = equationsToMatrix(M12, v1);
K_v2 = equationsToMatrix(M12, v2);
K_fi1 = equationsToMatrix(M12, fi1);
K_fi2 = equationsToMatrix(M12, fi2)
K_psi = (K_v2 - K_v1) / L;

 a_12(EI, L, k) = K_fi1
 b_12(EI, L, k) = K_fi2
 c_12(EI, L, k) = K_psi

latex_K11 = latex(K_v1)
latex_K12 = latex(K_v2)
latex_K13 = latex(K_fi1)
latex_K14 = latex(K_fi2)
```

$$a_{12} = -\frac{EI\,k\,\left(\sin\left(L\,k\right) - L\,k\,\cos\left(L\,k\right)\right)}{2\,\cos\left(L\,k\right) + L\,k\,\sin\left(L\,k\right) - 2}$$

$$b_{12} = \frac{EI\,k\,\left(\sin\left(L\,k\right) - L\,k\right)}{2\,\cos\left(L\,k\right) + L\,k\,\sin\left(L\,k\right) - 2}$$

$$c_{12} = \frac{2\,EI\,k\,\left(k - k\,\cos\left(L\,k\right)\right)}{L\,\left(2\,\cos\left(L\,k\right) + L\,k\,\sin\left(L\,k\right) - 2\right)}$$

```
a_12(EI, L, k) = -(EI*k*(sin(L*k) - L*k*cos(L*k))) / (2*cos(L*k) + L*k*sin(L*k) - 2)
b_12(EI, L, k) = (EI*k*(sin(L*k) - L*k)) / (2*cos(L*k) + L*k*sin(L*k) - 2)
c_12(EI, L, k) =  (2*EI*k*(k - k*cos(L*k))) / (L*(2*cos(L*k) + L*k*sin(L*k) - 2))
```

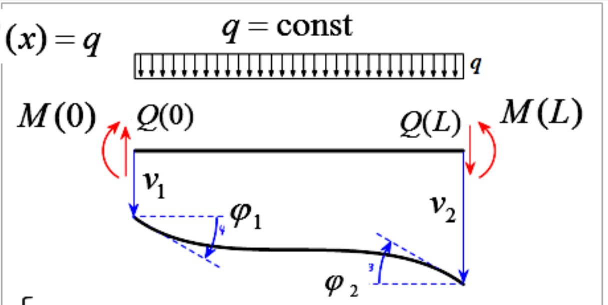
$$A_{12}(\lambda) = \frac{\lambda(\lambda\cos\lambda - \sin\lambda)}{2\cos\lambda + \lambda\sin\lambda - 2}$$

Recall: Full displacement method with zero axial force

The slope-deflection method – Stiffness matrix (no axial load)

Euler-Bernoulli beam element:

$$EIv^{(4)}(x) = q$$



$$\begin{cases} v(0) = v_1 \\ \varphi(0) = v'(0) = \varphi_1 \end{cases} \qquad \begin{cases} v(L) = v_2 \\ \varphi(L) = v'(L) = \varphi_2 \end{cases}$$

$$\begin{bmatrix} Q(0) \\ M(0) \\ Q(L) \\ M(L) \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} -12 & -6L & 12 & -6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ -6L & -2L^2 & 6L & -4L^2 \end{bmatrix} \begin{bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{bmatrix} + qL \begin{bmatrix} +1/2 \\ -1/12 \\ -1/2 \\ -L/12 \end{bmatrix}$$

Geometrically nonlinear stiffness equation
(raw # 2 from the stiffness matrix)

$$M(0) = \frac{EI}{L}\left[4 \cdot S_1(\lambda) \quad 2 \cdot S_2(\lambda) \quad 6 \cdot S_3(\lambda)\right] \cdot \begin{bmatrix} \varphi_{12} \\ \varphi_{21} \\ \psi_{12} \end{bmatrix} + S_0(\lambda) \cdot \overline{M}_{12}$$

The stiffness equations of the slope-deflection method & zero axial load.

$$M_{12} = \frac{4EI}{L}\varphi_{12} + \frac{2EI}{L}\varphi_{21} - \frac{6EI}{L}\psi_{12} + \overline{M}_{12}$$

$$M_{21} = \frac{4EI}{L}\varphi_{21} + \frac{2EI}{L}\varphi_{12} - \frac{6EI}{L}\psi_{12} + \overline{M}_{21}$$

$$\psi_{12} \equiv [v_2 - v_1]/L$$

The stiffness equations of the slope-deflection method **with axial load**

$$M_{12} = S_1(\lambda) \cdot \frac{4EI}{L}\varphi_{12} + S_2(\lambda) \cdot \frac{2EI}{L}\varphi_{21} - S_3(\lambda) \cdot \frac{6EI}{L}\psi_{12} + S_0(\lambda) \cdot \overline{M}_{12}$$

$S_i(\lambda)$: Amplification factor functions depending NONLINEARLY on axial load.

These stiffness coefficients are the elements of the elementary (GL) geometrically nonlinear stiffness matrix

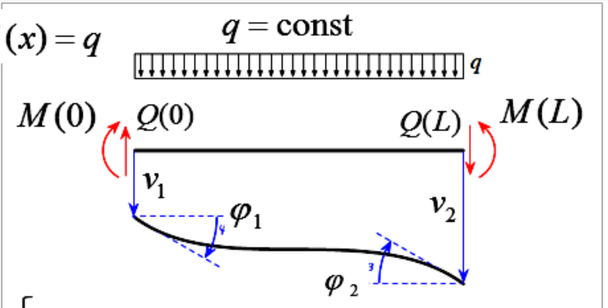
These are called Berry's stability functions. They are obtained from solutions of the geometrically nonlinear problem of combined bending and axial load for a beam.

Dimensionless axial load

Recall: Full displacement method with zero axial force

Euler-Bernoulli beam element:

$$EIv^{(4)}(x)=q$$



$$\begin{cases} v(0)=v_1 \\ \varphi(0)=v'(0)=\varphi_1 \end{cases} \qquad \begin{cases} v(L)=v_2 \\ \varphi(L)=v'(L)=\varphi_2 \end{cases}$$

$\psi_{12}\equiv[v_2-v_1]/L-$relative sway

The slope-deflection method – Stiffness matrix (no axial load)

$$\begin{bmatrix} Q(0) \\ \boxed{M(0)} \\ Q(L) \\ M(L) \end{bmatrix}=\frac{EI}{L^3}\begin{bmatrix} -12 & -6L & 12 & -6L \\ \boxed{6L} & \boxed{4L^2} & \boxed{-6L} & \boxed{2L^2} \\ -12 & -6L & 12 & -6L \\ -6L & -2L^2 & 6L & -4L^2 \end{bmatrix}\begin{bmatrix} \boxed{v_1} \\ \boxed{\varphi_1} \\ \boxed{v_2} \\ \boxed{\varphi_2} \end{bmatrix}+qL\begin{bmatrix} +1/2 \\ \boxed{-1/12} \\ -1/2 \\ -L/12 \end{bmatrix}$$

Geometrically nonlinear stiffness equation
(raw # 2 from the stiffness matrix)

$$M(0)=\frac{EI}{L}\left[4\cdot S_1(\lambda)\quad 2\cdot S_2(\lambda)\quad 6\cdot S_3(\lambda)\right]\cdot\begin{bmatrix} \varphi_{12} \\ \varphi_{21} \\ \psi_{12} \end{bmatrix}+S_0(\lambda)\cdot\overline{M}_{12}$$

The stiffness equations of the slope-deflection method with axial load

$$\mathbf{K}^{(e)}\mathbf{u}^{(e)}=\frac{EI}{L}\begin{bmatrix} * & * & * & * \\ 6S_3\cdot c_{12} & 4S_1a_{12} & -6S_3c_{12} & 2S_2b_{12} \\ * & * & * & * \\ * & * & * & * \end{bmatrix}\begin{bmatrix} v_1/L \\ \varphi_1 \\ v_2/L \\ \varphi_2 \end{bmatrix}$$

$$\text{case}: P>0\ \ (\text{compression}) \qquad \lambda\equiv kL$$

$$A_{ij}=\boxed{\frac{3\psi(\lambda)}{\mathrm{D}(\lambda)}}\cdot\frac{4EI}{L} \qquad B_{ij}=\boxed{\frac{3\phi(\lambda)}{\mathrm{D}(\lambda)}}\cdot\frac{2EI}{L}$$

$$\equiv S_1(\lambda) \qquad \equiv S_3(\lambda) \qquad \equiv S_2(\lambda)$$

$$C_{ij}=A_{ij}+B_{ij}=\boxed{\frac{\phi(\lambda)+2\psi(\lambda)}{\mathrm{D}(\lambda)}}\cdot\frac{6EI}{L} \qquad \mathrm{D}(\lambda)\equiv 4\psi^2(\lambda)-\phi^2(\lambda)$$

$$\lambda\equiv kL \qquad P=k^2EI \qquad \psi_{12}\equiv[v_2-v_1]/L-\text{relative sway}$$

$$\begin{array}{cccc} A_{ij}(\lambda) & B_{ij}(\lambda) & C_{ij}(\lambda) & \overline{M}_{ij}(\lambda) \end{array}$$

$$M_{12}=\boxed{S_1(\lambda)\cdot\frac{4EI}{L}}\varphi_{12}+\boxed{S_2(\lambda)\cdot\frac{2EI}{L}}\varphi_{21}-\boxed{S_3(\lambda)\cdot\frac{6EI}{L}}\psi_{12}+\boxed{S_0(\lambda)\cdot\overline{M}_{12}}$$

$S_i(\lambda)$: Amplification factor functions depending NONLINEARLY on axial load.

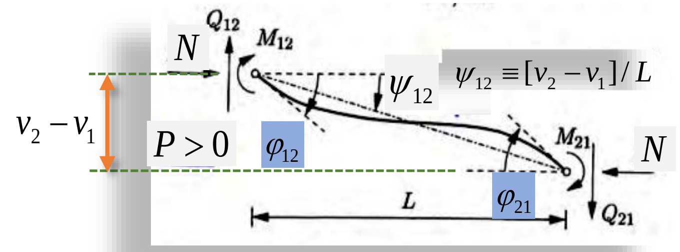
These stiffness coefficients are the elements of the elementary (GL) geometrically nonlinear stiffness matrix

Dimensionless axial load

These are called Berry's stability functions. They are obtained from solutions of the geometrically nonlinear problem of combined bending and axial load for a beam.

Full displacement method

Euler-Bernoulli beam element



$N \equiv -N_{12} = P > 0$ Case of compression : P > 0

$$\mathbf{K}_{\mathrm{NL}}^{(e)}\mathbf{u}^{(e)} = \frac{EI}{L}\begin{bmatrix} * & * & * & * \\ 6S_3 & 4S_1 & -6S_3 & 2S_2 \\ * & * & * & * \\ * & * & * & * \end{bmatrix}\begin{bmatrix} v_1/L \\ \varphi_1 \\ v_2/L \\ \varphi_2 \end{bmatrix}$$

case : $P > 0$ (compression)

$$\lambda \equiv kL$$

$$A_{ij} = \underbrace{\boxed{\frac{3\psi(\lambda)}{\mathrm{D}(\lambda)}}}_{\equiv S_1(\lambda)} \cdot \frac{4EI}{L}, \qquad B_{ij} = \underbrace{\boxed{\frac{3\phi(\lambda)}{\mathrm{D}(\lambda)}}}_{\equiv S_2(\lambda)} \frac{2EI}{L}$$

$$C_{ij} = A_{ij} + B_{ij} = \underbrace{\boxed{\frac{\phi(\lambda) + 2\psi(\lambda)}{\mathrm{D}(\lambda)}}}_{\equiv S_3(\lambda)} \frac{6EI}{L} \qquad \mathrm{D}(\lambda) \equiv 4\psi^2(\lambda) - \phi^2(\lambda)$$

The slope-deflection method – Stiffness matrix (no axial load)

$$\mathbf{K}_{\mathrm{L}}^{(e)}\mathbf{u}^{(e)} = \begin{bmatrix} * & * & * & * \\ c_{ij} & a_{ij} & -c_{ij} & b_{ij} \\ * & * & * & * \\ * & * & * & * \end{bmatrix}\begin{bmatrix} v_1/L \\ \varphi_1 \\ v_2/L \\ \varphi_2 \end{bmatrix}$$

$$\begin{bmatrix} Q(0) \\ \boxed{M(0)} \\ Q(L) \\ M(L) \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} -12 & -6L & 12 & -6L \\ \boxed{6L} & \boxed{4L^2} & \boxed{-6L} & \boxed{2L^2} \\ -12 & -6L & 12 & -6L \\ -6L & -2L^2 & 6L & -4L^2 \end{bmatrix}\begin{bmatrix} \boxed{v_1} \\ \boxed{\varphi_1} \\ \boxed{v_2} \\ \boxed{\varphi_2} \end{bmatrix} + qL\begin{bmatrix} +1/2 \\ \boxed{-1/12} \\ -1/2 \\ -L/12 \end{bmatrix}$$

$$a_{ij} = \frac{4EI}{L}$$

$$b_{ij} = \frac{2EI}{L}$$

$$c_{ij} = \frac{6EI}{L}$$

Geometrically nonlinear stiffness equation (raw # 2 from the stiffness matrix) when accounting effect of axial force

$$\lambda \equiv kL$$

$$M(0) = \frac{EI}{L}\left[4 \cdot S_1(\lambda) \quad 2 \cdot S_2(\lambda) \quad 6 \cdot S_3(\lambda)\right] \cdot \begin{bmatrix} \varphi_{12} \\ \varphi_{21} \\ \psi_{12} \end{bmatrix} + S_0(\lambda) \cdot \overline{M}(0)$$

The stiffness equations of the slope-deflection method with axial load

$$\lambda \equiv kL \qquad P = k^2 EI$$

$$M_{12} = \boxed{S_1(\lambda) \cdot \frac{4EI}{L}}\varphi_{12} + \boxed{S_2(\lambda) \cdot \frac{2EI}{L}}\varphi_{21} - \boxed{S_3(\lambda) \cdot \frac{6EI}{L}}\psi_{12} + \boxed{S_0(\lambda) \cdot \overline{M}_{12}}$$

$S_i(\lambda)$:

Dimensionless axial load

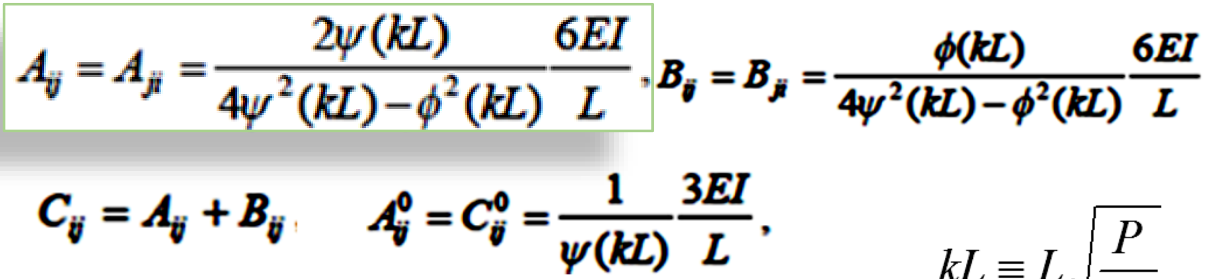
Amplification factor functions depending NONLINEARLY on axial load.

These stiffness coefficients are the elements of the elementary (GL) geometrically nonlinear stiffness matrix

These are called Berry's stability functions. They are obtained from solutions of the geometrically nonlinear problem of combined bending and axial load for a beam.

The stiffness coefficients – axial compression/tension and bending

Beam-column with constant flexural rigidity:

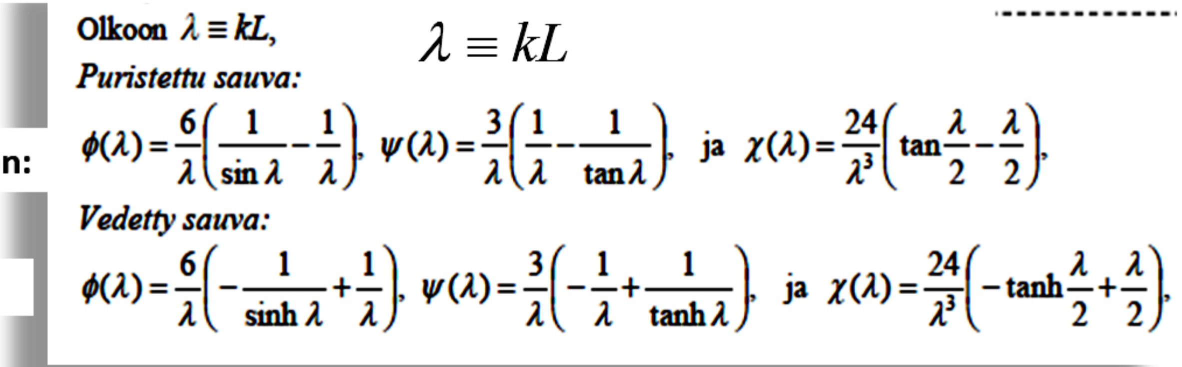


$$A_{ij} = A_{ji} = \frac{2\psi(kL)}{4\psi^2(kL)-\phi^2(kL)}\frac{6EI}{L}, B_{ij} = B_{ji} = \frac{\phi(kL)}{4\psi^2(kL)-\phi^2(kL)}\frac{6EI}{L}$$

$$C_{ij} = A_{ij} + B_{ij} \qquad A^0_{ij} = C^0_{ij} = \frac{1}{\psi(kL)}\frac{3EI}{L},$$

$$kL \equiv L\sqrt{\frac{P}{EI}}$$

Berry's functions:



Olkoon $\lambda \equiv kL$,

$$\lambda \equiv kL$$

Puristettu sauva:

Compression:

$$\phi(\lambda) = \frac{6}{\lambda}\left(\frac{1}{\sin\lambda} - \frac{1}{\lambda}\right), \ \psi(\lambda) = \frac{3}{\lambda}\left(\frac{1}{\lambda} - \frac{1}{\tan\lambda}\right), \text{ ja } \chi(\lambda) = \frac{24}{\lambda^3}\left(\tan\frac{\lambda}{2} - \frac{\lambda}{2}\right),$$

Vedetty sauva:

Extension:

$$\phi(\lambda) = \frac{6}{\lambda}\left(-\frac{1}{\sinh\lambda} + \frac{1}{\lambda}\right), \ \psi(\lambda) = \frac{3}{\lambda}\left(-\frac{1}{\lambda} + \frac{1}{\tanh\lambda}\right), \text{ ja } \chi(\lambda) = \frac{24}{\lambda^3}\left(-\tanh\frac{\lambda}{2} + \frac{\lambda}{2}\right),$$

More concise notation:

$$A_{ij} = \frac{3\psi(\lambda)}{\mathrm{D}(\lambda)}\cdot\frac{4EI}{L}$$

$$B_{ij} = \frac{3\phi(\lambda)}{\mathrm{D}(\lambda)}\cdot\frac{2EI}{L}$$

$$C_{ij} = A_{ij} + B_{ij}$$

$$\mathrm{D}(\lambda) \equiv 4\psi^2(\lambda) - \phi^2(\lambda)$$

Examples:

$$A^0_{12}(\lambda) = \frac{1}{\psi(\lambda)}\underbrace{\frac{3EI}{L}}_{\equiv a^0_{12}} = \frac{1}{\psi(\lambda)}a^0_{12}(P=0)$$

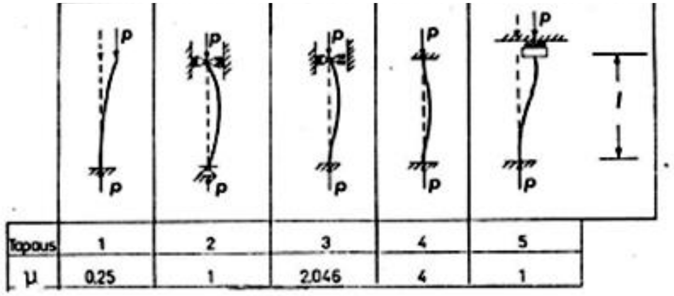
$\mathrm{M}(\lambda)$

Magnification factor depends on compressive/tensional load (Berry's stability functions)

$$A_{ij} = \frac{2\psi(\lambda)}{4\psi^2(\lambda)-\phi^2(\lambda)}\frac{6EI}{L} = \frac{3\psi(\lambda)}{4\psi^2(\lambda)-\phi^2(\lambda)}\overbrace{\frac{4EI}{L}}^{\equiv a_{12}} = \mathrm{M}(\lambda)\cdot a_{12}$$

Formulary

Eulerin peruskaavat nurjahdukselle: $P_{cr} = \mu \cdot \dfrac{\pi^2 EI}{l^2}$



Tapaus	1	2	3	4	5
μ	0.25	1	2.046	4	1

The stiffness coefficients (are symmetric)

Puristettu ja taivutettu sauva:

Kulmanmuutosmenetelmä

$$M_{ij} = A_{ij}\varphi_{ij} + B_{ij}\varphi_{ji} - C_{ij}\psi_{ij} + \overline{MK}_{ij}$$

$$M_{ij} = A^0_{ij}\varphi_{ij} - C^0_{ij}\psi_{ij} + \overline{MK}^0_{ij} \quad \text{(sauvan päässä } j \text{ on nivel)}$$

Tasajäykkä sauva :

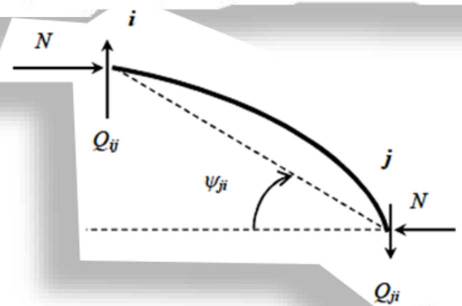
$$A_{ij} = A_{ji} = \frac{2\psi(kL)}{4\psi^2(kL) - \phi^2(kL)}\frac{6EI}{L}, \quad B_{ij} = B_{ji} = \frac{\phi(kL)}{4\psi^2(kL) - \phi^2(kL)}\frac{6EI}{L} \quad \text{ja} \quad C_{ij} = A_{ij} + B_{ij},$$

$$\overline{MK}_{ij} = -A_{ij}\overline{\alpha}^0_{ij} - B_{ji}\overline{\alpha}^0_{ji}, \qquad \overline{MK}_{ji} = -A_{ji}\overline{\alpha}^0_{ji} - B_{ij}\overline{\alpha}^0_{ij},$$

$$A^0_{ij} = C^0_{ij} = \frac{1}{\psi(kL)}\frac{3EI}{L}, \qquad \overline{MK}^0_{ij} = -A_{ij}\overline{\alpha}^0_{ij}$$

Leikkausvoima:

$$Q_{ij} = Q^0_{ij} - (M_{ij} + M_{ji})/L - N\psi_{ij} \quad (N \text{ positiivinen, kun sauva puristettu})$$



Berry's functions (stability function)

Berryn funktiot:

Olkoon $\lambda \equiv kL$,

Puristettu sauva: **Compression**

$$\phi(\lambda) = \frac{6}{\lambda}\left(\frac{1}{\sin\lambda} - \frac{1}{\lambda}\right), \quad \psi(\lambda) = \frac{3}{\lambda}\left(\frac{1}{\lambda} - \frac{1}{\tan\lambda}\right), \quad \text{ja} \quad \chi(\lambda) = \frac{24}{\lambda^3}\left(\tan\frac{\lambda}{2} - \frac{\lambda}{2}\right),$$

Vedetty sauva:

$$\phi(\lambda) = \frac{6}{\lambda}\left(-\frac{1}{\sinh\lambda} + \frac{1}{\lambda}\right), \quad \psi(\lambda) = \frac{3}{\lambda}\left(-\frac{1}{\lambda} + \frac{1}{\tanh\lambda}\right), \quad \text{ja} \quad \chi(\lambda) = \frac{24}{\lambda^3}\left(-\tanh\frac{\lambda}{2} + \frac{\lambda}{2}\right),$$

Extension

$$M_{ij} = A_{ij}\varphi_{ij} + B_{ij}\varphi_{ij} - C_{ij}\psi_{ij} + \overline{M}_{ij}$$

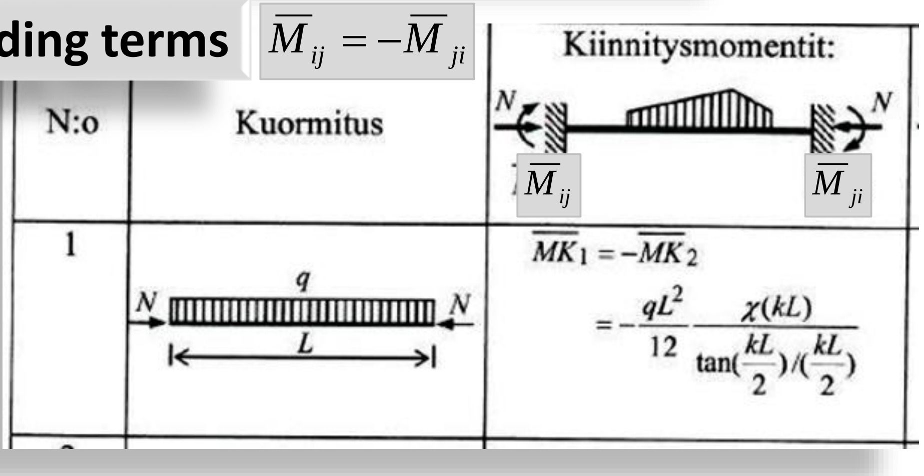
EI constant

$$A_{12} = \frac{2\psi(kL)}{4\psi^2(kL) - \phi^2(kL)}\frac{6EI}{L} = A_{21},$$

$$B_{12} = \frac{\phi(kL)}{4\psi^2(kL) - \phi^2(kL)}\frac{6EI}{L} = B_{21},$$

$$C_{12} = A_{12} + B_{12}, \quad C_{21} = A_{21} + B_{21}.$$

Loading terms $\overline{M}_{ij} = -\overline{M}_{ji}$

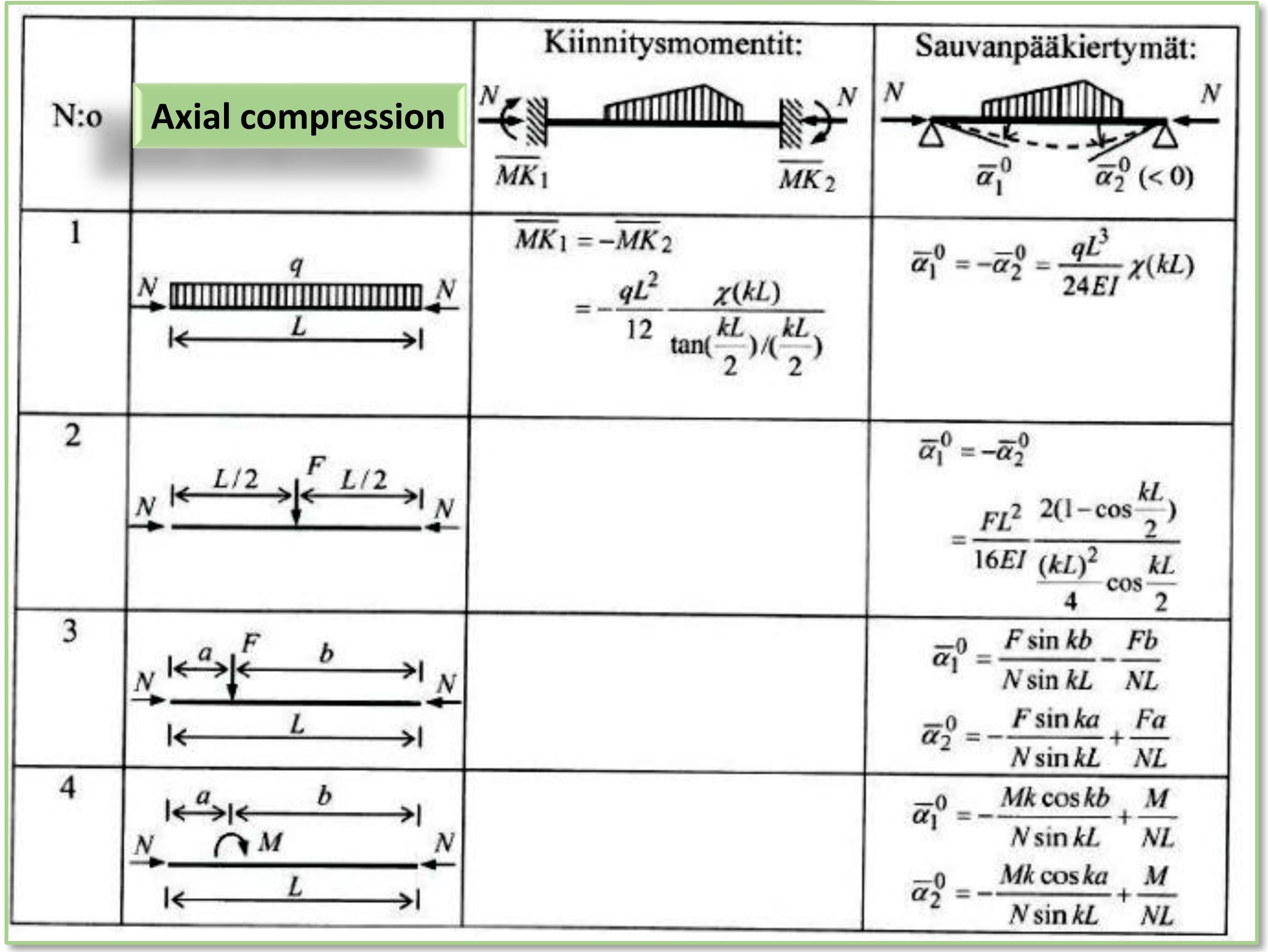


N:o	Kuormitus	Kiinnitysmomentit:
1		$\overline{MK}_1 = -\overline{MK}_2 = -\dfrac{qL^2}{12}\dfrac{\chi(kL)}{\tan(\frac{kL}{2})/(\frac{kL}{2})}$

EI is constant

Loading terms: Fixed-End-Moments

$\overline{M}_{12} \equiv MK_1$

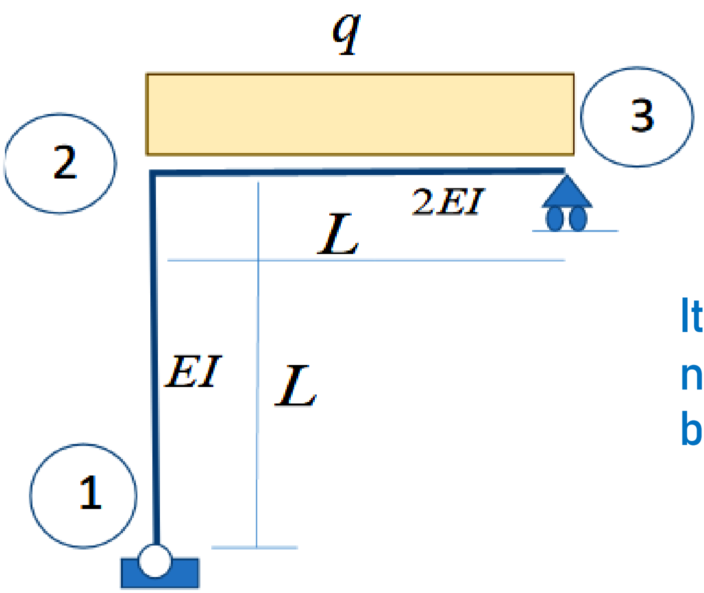


N:o	**Axial compression**	Kiinnitysmomentit: N N $\overline{MK}_1$ $\overline{MK}_2$	Sauvanpääkiertymät: N N $\bar{\alpha}_1^0$ $\bar{\alpha}_2^0\ (<0)$
1	N q N L	$\overline{MK}_1 = -\overline{MK}_2$ $= -\frac{qL^2}{12}\frac{\chi(kL)}{\tan(\frac{kL}{2})/(\frac{kL}{2})}$	$\bar{\alpha}_1^0 = -\bar{\alpha}_2^0 = \frac{qL^3}{24EI}\chi(kL)$
2	N $L/2$ F $L/2$ N		$\bar{\alpha}_1^0 = -\bar{\alpha}_2^0$ $= \frac{FL^2}{16EI}\frac{2(1-\cos\frac{kL}{2})}{\frac{(kL)^2}{4}\cos\frac{kL}{2}}$
3	N a F b N L		$\bar{\alpha}_1^0 = \frac{F\sin kb}{N\sin kL} - \frac{Fb}{NL}$ $\bar{\alpha}_2^0 = -\frac{F\sin ka}{N\sin kL} + \frac{Fa}{NL}$
4	a b N M N L		$\bar{\alpha}_1^0 = -\frac{Mk\cos kb}{N\sin kL} + \frac{M}{NL}$ $\bar{\alpha}_2^0 = -\frac{Mk\cos ka}{N\sin kL} + \frac{M}{NL}$

Geometrically non-linear analysis of frames by the Slope-deflection method

Moderate rotations and loads close to critical load but not over

Q: DETERMINE THE BENDING MOMENT AT RIGID JOINT #2



$$\varphi_{21} = \varphi_{23} \Rightarrow \frac{L}{3EI}\Psi(kL)M_{21} + \psi_{21} = \frac{L}{6EI}M_{23} + \frac{qL^3}{48EI}$$

$$(1+2\Psi(kL))M_2 + \frac{6EI}{L}\psi_{21} = \frac{qL^2}{8}$$

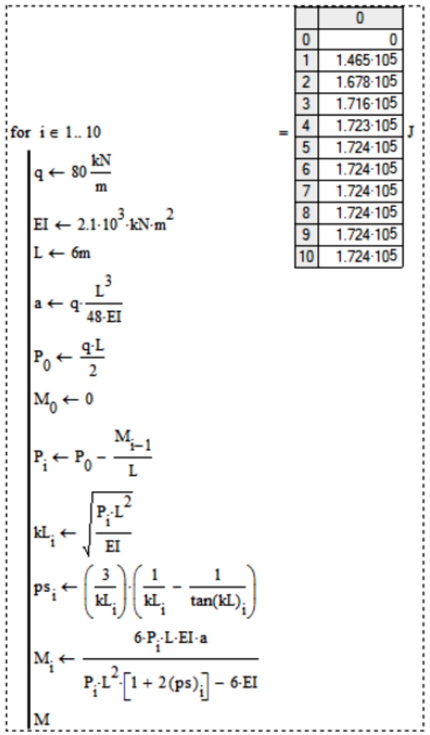
$$Q_{21} = 0 \Rightarrow -\frac{M_2}{L} - P\psi_{21} = 0 \Rightarrow \psi_{21} = -\frac{M_2}{PL}$$

$$(1+2\Psi(kL))M_2 - \frac{3M_2}{k^2L^2} = \frac{qL^2}{8}$$

Iterations are needed to solve the bending moment:

$$\Rightarrow \begin{cases} M_2 = \dfrac{qL^2}{8}\dfrac{PL^2}{PL^2(1+2\Psi(kL)) - 6EI} \\ N_{21} + Q_{32} = 0 \end{cases}$$

Express Q_{32} in terms of end-moments



for $i \in 1..10$

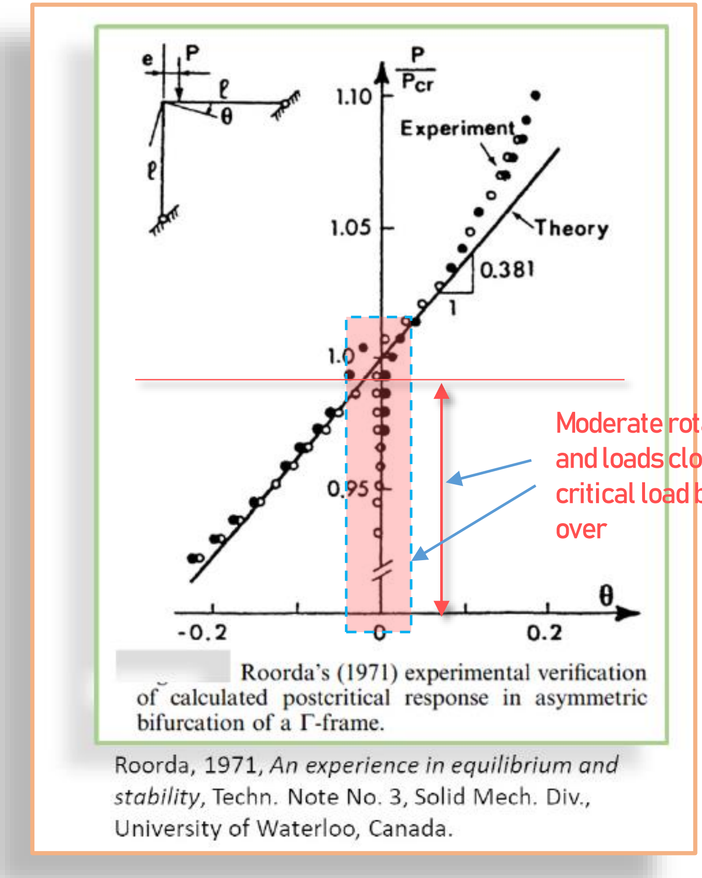
- $q \leftarrow 80\frac{kN}{m}$
- $EI \leftarrow 2.1 \cdot 10^3 \cdot kN \cdot m^2$
- $L \leftarrow 6m$
- $a \leftarrow q \cdot \frac{L^3}{48 \cdot EI}$
- $P_0 \leftarrow \frac{q \cdot L}{2}$
- $M_0 \leftarrow 0$
- $P_i \leftarrow P_0 - \frac{M_{i-1}}{L}$
- $kL_i \leftarrow \sqrt{\frac{P_i \cdot L^2}{EI}}$
- $ps_i \leftarrow \left(\frac{3}{kL_i}\right)\cdot\left(\frac{1}{kL_i} - \frac{1}{\tan(kL)_i}\right)$
- $M_i \leftarrow \frac{6 \cdot P_i \cdot L \cdot EI \cdot a}{P_i \cdot L^2 \cdot \left[1 + 2(ps)_i\right] - 6 \cdot EI}$
- M

= J

	0
0	0
1	1.465·105
2	1.678·105
3	1.716·105
4	1.723·105
5	1.724·105
6	1.724·105
7	1.724·105
8	1.724·105
9	1.724·105
10	1.724·105

Geometrically non-linear analysis of frames by the Slope-deflection method

Moderate rotations and loads close to critical load but not over

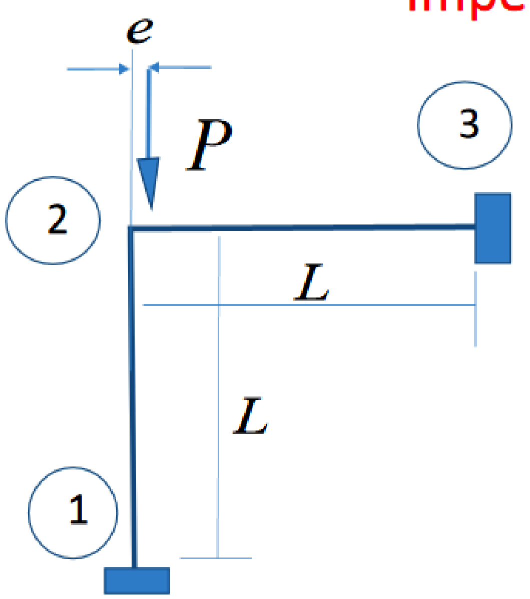


Roorda's (1971) experimental verification of calculated postcritical response in asymmetric bifurcation of a Γ-frame.

Roorda, 1971, *An experience in equilibrium and stability*, Techn. Note No. 3, Solid Mech. Div., University of Waterloo, Canada.

Moderate rotations and loads close to critical load but not over

Imperfection, eccentricity



$$M_{21}+M_{23}-Pe=0 \Rightarrow \varphi_2=\frac{Pe}{(A_{21}+a_{23})}$$

$$\begin{cases} M_{21}=A_{21}\varphi_2=Pe\dfrac{A_{21}}{(A_{21}+a_{23})} \\ M_{23}=a_{23}\varphi_2=Pe\dfrac{a_{23}}{(A_{21}+a_{23})} \\ M_{32}=b_{32}\varphi_2=\dfrac{M_{23}}{2} \end{cases}$$

$$Q_{23}=-\frac{M_{23}+M_{32}}{L}=-\frac{3}{2}\frac{M_{23}}{L}=-Pe\frac{3a_{23}}{2(A\ +a\ )}$$

$$P=P+Q_{23}=P(1-e\frac{3a_{23}}{2(A_{21}+a_{23})})=P(1-e\frac{1}{\dfrac{\Psi(kL)}{4\Psi^2(kL)-\Phi^2(kL)}+\dfrac{2}{3}})$$

Should be N_{21}

(the normal stress resultant in column 12)

If now the eccentritcity *e* is negative, the value of the compressive load *P* is increasing all the time, and no convergence will be reached. If positive, the convergence is reached.

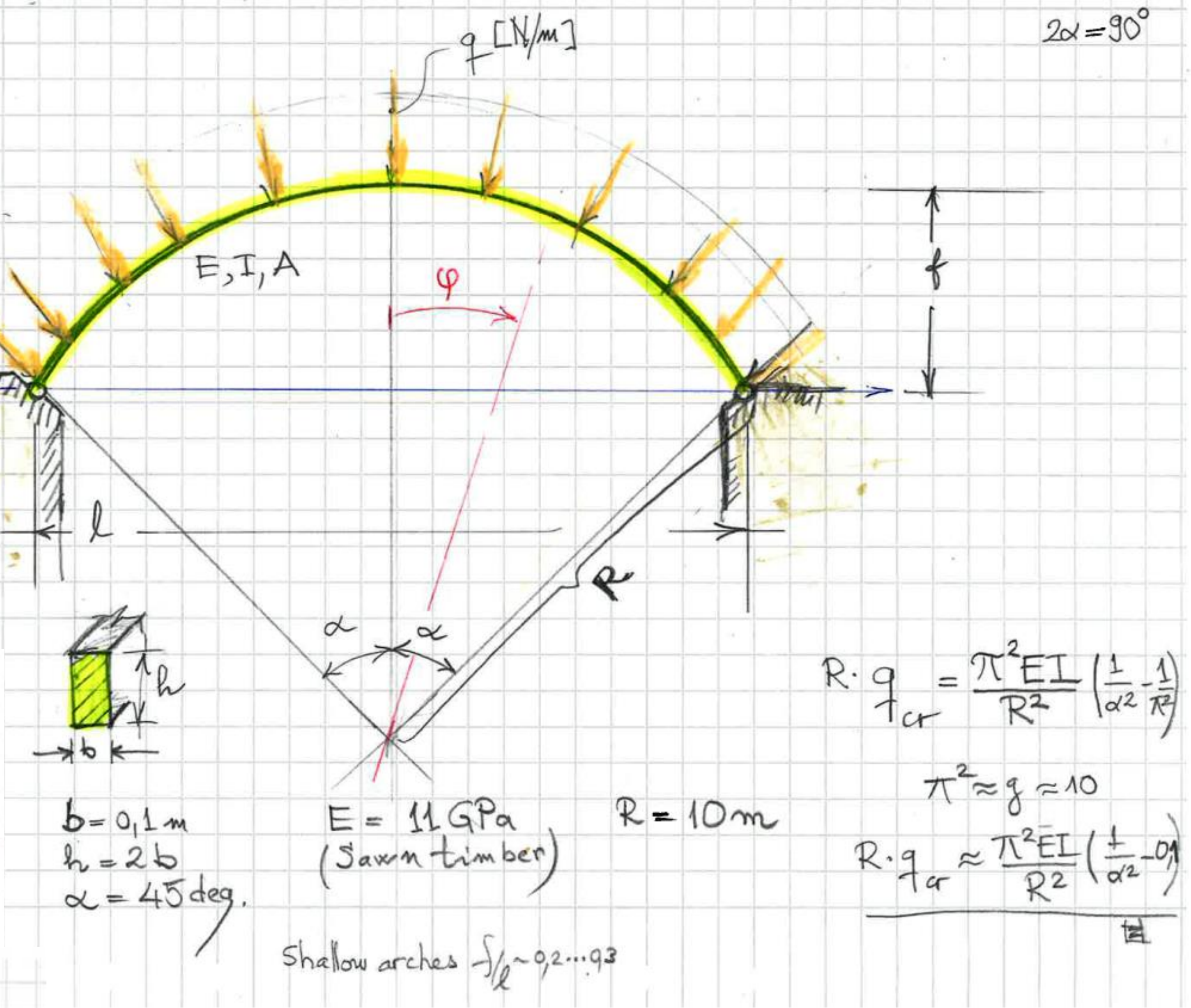
Linear and non-linear buckling analysis

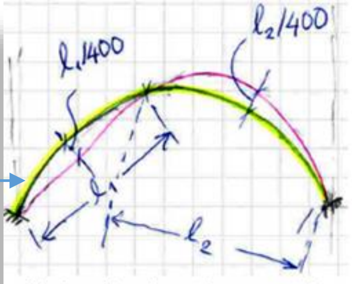
Free Exercise - 20 extra-points for HW

1. Perform linear buckling analysis for the perfect geometry and find the critical load and the respective buckling mode
2. Find the second buckling load and the buckling mode
3. Analysis the shape imperfection effect on the buckling load (GNA)

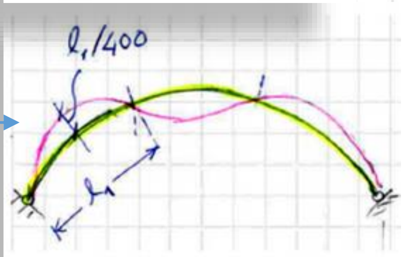
For that do:

- Take the first buckling mode and then the second one (or their combination) multiplied by L/400 (L distance between mode nodes, as in Figs. on right) as a shape imperfection to add for the perfect geometry.
- Determine the load-displacement curve at some characteristic points
- What is the limit load? How much the buckling load of the perfect arch is reduced?





b) loading is antisymmetric



a) loading is symmetric

Example of initial shape imperfections in an arch (Standards: design of wood structures - EN 1995-1-1)

Assume that stresses remains in the elastic range.

1) One way to think **how form the increment of total potential energy** is through a real loading sequence where the load increases quasi-statically and monotonically from zero to the buckling load $P_E^+ = P_E + \varepsilon$ where it buckles where ε being infinitesimally small > 0. The primary non-buckled configuration (primary equilibrium) corresponds to $P_E^- = P_E - \varepsilon$. Now one can form the increment of the total potential energy between these two real states and takes the limit when $\varepsilon \to 0$ to say that we are at the bifurcation or limit-point where now the critical load being P_E.

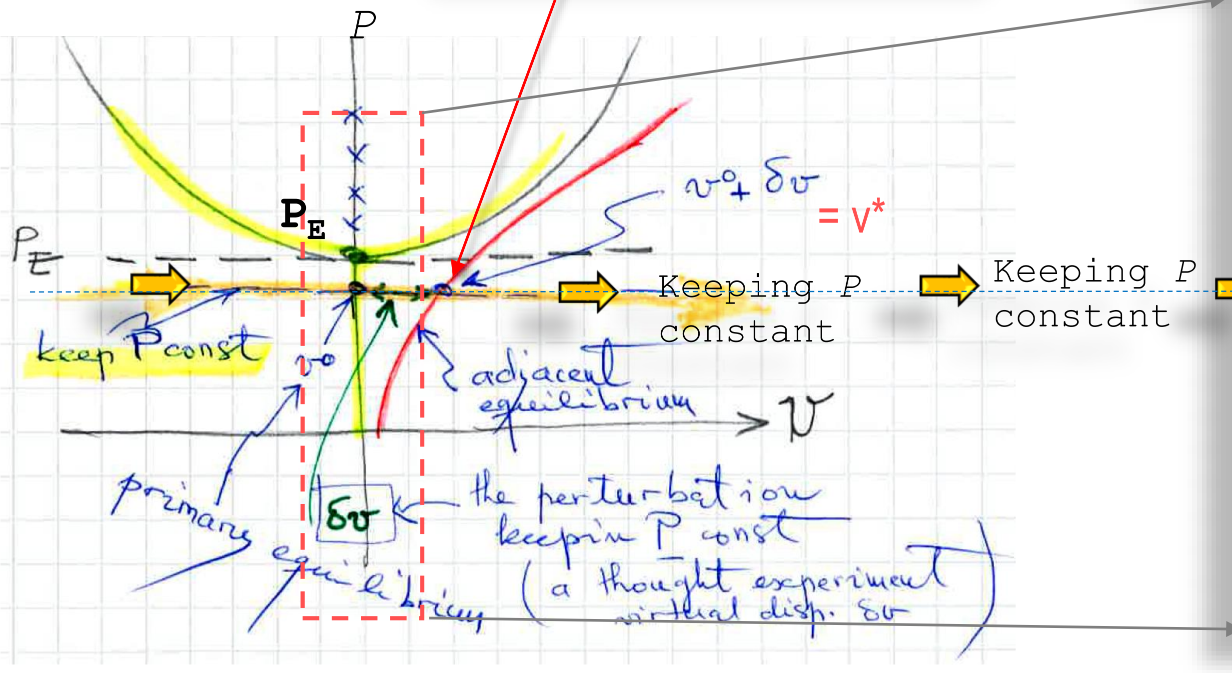
$$\boxed{\Delta\Pi = \Pi^* - \Pi^0} \Longrightarrow \delta(\Delta\Pi) = 0 \Longrightarrow$$

Equations (of loss) of stability

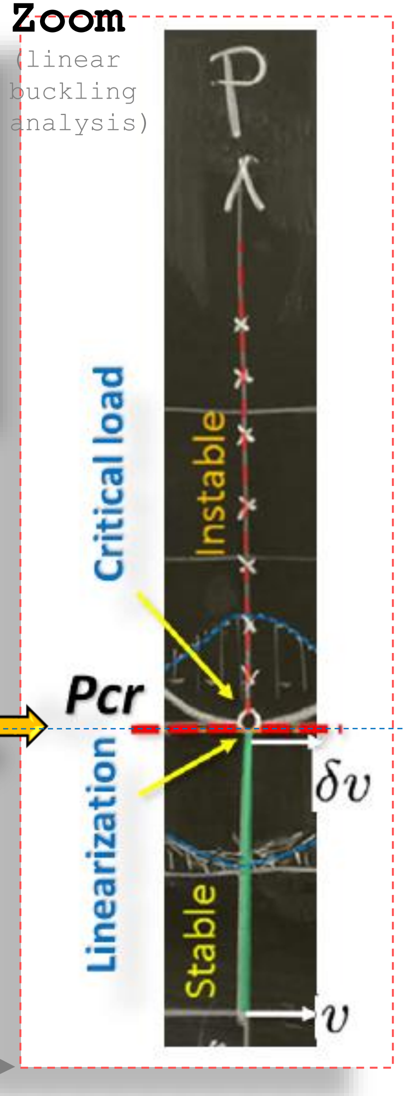
This is a <u>thought</u> <u>experiment</u>

N.B. The perturbed configuration [.]* can be (also) **<u>thought</u>** achieved keeping the load constant and for instance, giving the primary equilibrium configuration v_0 a tiny kinematical (virtual) <u>perturbation</u> to a an <u>adjacent</u> equilibrium configuration v*

$$\boxed{\Delta\Pi = \Pi^* - \Pi^0}$$



Zoom (linear buckling analysis)



2) the other <u>more classical</u> way how form the increment of total potential energy is by a *thought experiment where we give an infinitesimal virtual perturbation to the primary equilibrium configuration to an adjacent neighbor equilibrium configuration while keeping all the loads unchanged. Then we write the increment od total potential energy between these to states of equilibrium.*