Lecture 3 - Wave mechanics

3.1 Wave formation

What are water waves? How do they form and how do we describe them? Looking out at the ocean, one often sees a seemingly infinite series of waves, transporting water from one place to the next. Though waves do cause the surface water to move, the idea that waves are travelling bodies of water is misleading. In naval architecture waves generated by local winds are termed sea and waves that travelled out of their area of generation are termed swell. Sea waves are characterized by relatively peaky crests and the crest length seldom exceeds some two or three times the wavelength. Swell waves are generally lower with more rounded tops. The crest length is typically six or seven times the wavelength. In a swell, the variation in height between successive waves is less than is the case for sea waves. Waves are created by anything that supplies energy to the water surface. Consequently, sources of wave systems are numerous. From our own experience we know that throwing a stone into a pool will generate a circular wave pattern. When examining the wave system creation sequence, it is important to be aware of the energy transfer that is constantly occurring in a wave. The energy of a wave is always being dissipated by the viscous friction forces associated with the viscosity of the sea. This energy dissipation increases with wave height.

For the wave to be maintained, the energy being dissipated must be replaced by the energy source of the wave - the wind. Hence, without the continued presence of the wind, the wave system will die. Deep water ocean waves are essentially energy passing through the water, causing it to move in a circular motion. When a wave encounters a surface object, the object appears to lurch forward and upward with the wave, but then falls down and back in an orbital rotation as the wave continues by, ending up in the same position as before the wave came by. If one imagines wave water itself following this same pattern, it is easier to understand ocean waves as simply the outward manifestation of kinetic energy propagating through seawater. On this basis it is fair to accept that the passing of air over the sea surface causes ripples or waves. Although there are many theories in literature explain the wave formation mechanism, there is no exact mechanism known by which the energy is transferred from wind to sea water. Therefore, the extent of disturbance depends on the wind strength, the time for which wind acts on the sea surface, and the portion of the water surface it acts upon it. Those features are termed wind strength, duration and fetch respectively. Wave characteristics become practically independent of fetch when the fetch is greater than about 500 km. *Practically this means that the water in waves doesn't travel much at all and within the context of ship dynamics in open seas serve the purpose of energy transmitters.*

This idea of waves being energy transmition medium (rather than water movement) makes sense in the open ocean. *On the coast, where waves are clearly seen crashing dramatically onto shore this phenomenon is a result of the wave's orbital motion being disturbed by the seafloor.* As a wave passes through water, not only does the surface water follow an orbital motion, but a column of water below it (down to half of the wave's wavelength) completes the same movement. The approach of the bottom in shallow areas causes the lower portion of the wave to slow down and compress, forcing the wave's crest higher in the air. Eventually this imbalance in the wave reaches a breaking point, and the crest comes crashing down as wave energy is dissipated into the surface.

(a) Ocean waves (b)The wave energy cycle

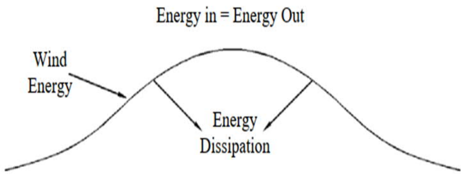


Figure 3-1 Principles of wave formation

Where does a wave's energy come from? There are a few types of ocean waves and they are generally classified by the energy source that creates them. Most common are surface waves, caused by wind blowing along the air-water interface, creating a disturbance that steadily builds as wind continues to blow and the wave crest rises. Surface waves occur constantly all over the globe and are the waves you see at the beach under normal conditions. Adverse weather or natural events often produce larger and potentially hazardous waves. Severe storms moving inland often create a storm surge, a long wave caused by high winds and a continued low-pressure area. Submarine earthquakes or landslides can displace a large amount of water very quickly, creating a series of very long waves called tsunamis. Storm surges and tsunamis do not create a typical crashing wave but rather a massive rise in sea level upon reaching shore, and they can be extremely destructive to coastal environments. Once a wave has been generated it will move away from the position at which it was generated until all its energy is spent.

3.2 Brief overview of wave theories

Extensive explanation to the background of wave theories is given by (Karadeniz, Saka, and Togan 2013). This section outlines their basic taxonomy of key wave theories and qualitatively discusses the basic fluid mechanics assumptions used for the basic wave terms outlined in section 0 of this chapter. Linear water surface waves are obtained from inviscid, incompressible and irrotational flow under certain boundary conditions by using the principles of potential flow hydrodynamics. To obtain wave equations used in linear Airy wave theory the continuity of water is used. Thus, mass and momentum are defined respectively by the Laplace and Bernoulli equations:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (3\text{-}1)$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + \frac{p}{\rho} + gz = 0 \quad (3\text{-}2)$$

The boundary conditions for the above equations are defined as kinematic and dynamic. They relate to the motions of the water particles and the forces acting on the water particles respectively. A linear wave is assumed to be periodic and infinitely long in the longitudinal direction. Variations in the transverse direction are ignored and the system boundaries limit in way of the water surface and the seabed (see Figure 3-2). The kinematic boundary condition at the water surface is based on that the particles may not leave the water surface. This means that the velocity of the water particle normal to the surface is equal to the speed of the surface in that direction. Thus, at the sea surface (z=0) we assume that:

$$u_z = \frac{\partial \zeta}{dt} \rightarrow \frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{dt} \tag{3-3}$$

At the bottom (z=-d), the water particles may not penetrate the seabed and accordingly:

$$u_z = 0 \rightarrow \frac{\partial \phi}{\partial z} = 0 \tag{3-4}$$

The dynamic boundary condition at the surface assumes that the pressure is always the atmospheric pressure (taken as the reference pressure=0), and the wave is only subjected to gravity force.

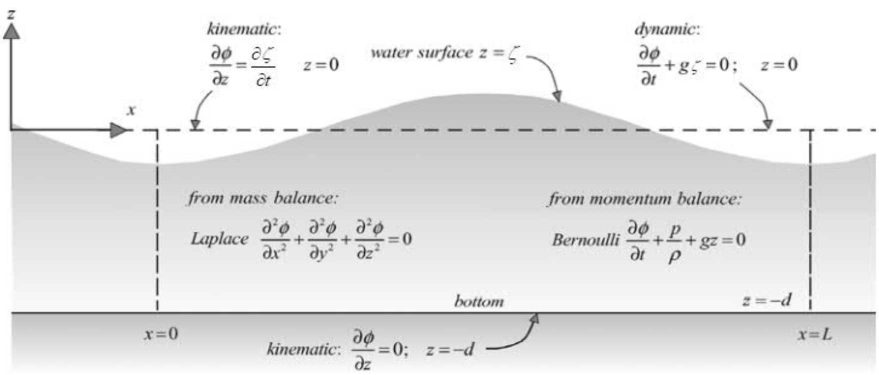


Figure 3-2 Linear wave theory idealization

Generally, flow disturbances propagate in various directions interacting non-linearly over water of probable non-uniform density and varying or deformable topography. A well-known nonlinear theory is Stokes theory. It is considered most suitable for waves which are not very long relative to the water depth. Stokes theory assumes that all the variations in the longitudinal direction can be represented by Fourier series and that the coefficients in these series can be written as perturbation expansion in terms of a parameter increases by the wave height. By substituting the higher order perturbation expansions in the governing equations (the mass and the momentum balance equation) and manipulation of the series yields to solving the velocity potential. Stokes wave theories are most suitable for deep and intermediate water depth. Even higher order terms in the Stokes theory for steeper waves produce unrealistic results. For shallower water, a finite-amplitude wave theory is required. Cnoidal wave theory and, in very shallow water, solitary wave theory are the analytical wave theories most commonly used. Solutions in the cnoidal wave theory are obtained in terms of elliptical integrals of the first kind. The solitary wave theory is a special case combining mathematical principles of cnoidal and linear wave theories. As the relative depth decreases the cnoidal wave becomes the solitary wave, which has a crest that is completely above the still water level and has no trough.

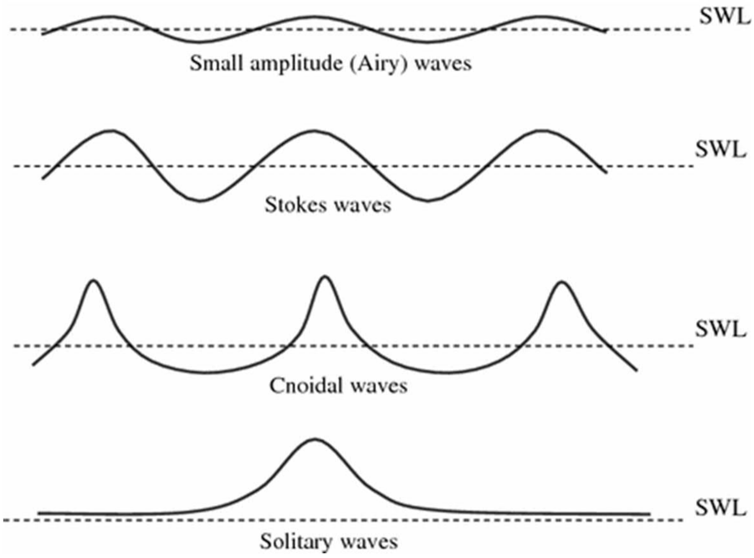


Figure 3-3 Different forms of wave surface elevation (SWL : idealization of Still Water Line)

3.3 The regular wave

Evaluation of the properties of random waves is almost impossible. However, if we consider the randomly changing waves as a stochastic process, then it is possible to evaluate the statistical properties of irregular waves as an aggregate of regular waves. Regular waves are shaped like a sine wave moving along water surface. This type of waves is periodic, meaning it has consistent frequency or period of occurrence. Figure 3-4 gives the notation we will use to describe the characteristics of a regular wave and shows a wave with the components labeled. Our wave is progressive, meaning it moves horizontally over the water surface (a wave that merely oscillates up and down is considered a standing wave). the shape of each wave passing by looks the same and the whole wave train can be viewed as an advancing rigid corrugated sheet.

If we were to consider only a single point in space and describe the water surface elevation at that point as the waves move past then

$$\zeta(t) = \zeta_0 sin(\omega t - \varepsilon) \qquad (3\text{-}5)$$

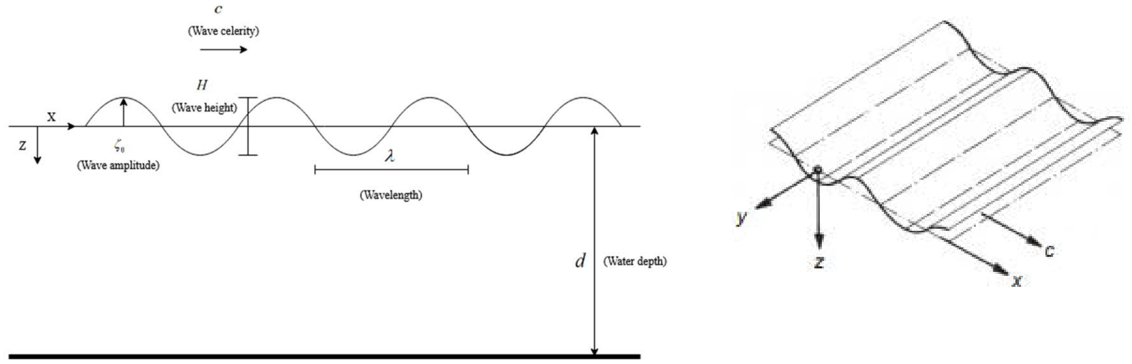
In this expression t is the variable for time, ζ_0 is the wave amplitude, ε is the phase angle (the degrees the shape is different from a perfect sine wave) and ω is the wave frequency (in radians / second) ; i.e. a measure of the oscillations that pass this point in one second. If we instead consider the entire wave train in space, but only for a single moment in time, the mathematical expression becomes:

$$\zeta(x) = \zeta_0 sin(kx) \qquad (3\text{-}6)$$

where x is the variable for position and k is the wave number representing the frequency as a function of wavelength (i.e. the number of cycles that occur over a unit of length). The expression for wave number is :

$$k = \frac{2\pi}{\lambda} \tag{3-7}$$

(a) 2D surface elevation (b) 3D surface formation



ζ : instantaneous depression of water surface below mean level (y = 0)
ζ_a : wave amplitude from mean level ($y = 0$) to a crest or trough
H : wave height (always twice the wave amplitude)
λ : wave length (distance from one crest – or trough – to the next)
c : wave celerity
T : wave period (time interval between successive crests or troughs passing a fixed point
α : the instantaneous wave slope (gradient of the surface profile)
α_0 : maximum wave slope or wave slope amplitudes
H/λ : wave steepness

Figure 3-4 Regular wave idealization

Waves exist and change in both time and space. So you may stay with a wave and move through space in time or you can stay at one location and see the wave move past in time. Accordingly, the equation for the water elevation must account for the point at which we are measuring (i.e. where are we standing?) and the time the measurement is made (i.e. what time are we looking?). Therefore,

$$\zeta(t) = \zeta_0 sin(kx - \omega t - \varepsilon) \tag{3-8}$$

For regular waves in deep water there is a fixed relationship between wave frequency, length and speed. For a high frequency wave there is only a short time between peaks and therefore the wave length is very short. For a low frequency wave there is a long time between peaks and the wave length is long. For deep waters these relationships are:

$$T = \frac{2\pi}{\omega} \tag{3-9}$$

$$\lambda = \frac{2\pi g}{\omega^2} = \frac{gT^2}{2\pi} \quad (3\text{-}10)$$

In shallow water the wave length (*λ*) depends only on the water depth (*d*). Thus:

$$\lambda = 2\pi d \quad (3\text{-}11)$$

The general relationship between wave frequency, length and depth is given by the so called ‘dispersion equation’:

$$\omega^2 = gk \times tanh(kd) = gk\frac{sinh(kd)}{cosh(kd)} \quad (3\text{-}12)$$

where $kd = \frac{2\pi d}{\lambda}$ and therefore when the water depth is very large relative to the wave length, kd is large and vice versa in shallow waters when depth is small relative to the wave length kd becomes small. For these situations we can simplify the hyperbolic expressions as:

Function	Large *kd*	Small *kd*
Sinh(kd)	$e^{kd}/2$	1
Cosh(kd)	$e^{kd}/2$	*kd*
Tanh(kd)	1	*kd*

Using these simplifications of the hyperbolic functions we can show that in deep water the wave frequency depends only on the wave length,

$$\omega = \sqrt{gk} \quad (3\text{-}13)$$

while in shallow water the wave frequency also depends on water depth

$$\omega = \sqrt{gk^2d} \quad (3\text{-}14)$$

The wave celerity, i.e. the speed of the wave travelling over the water surface is given by:

$$c = \sqrt{\frac{g}{k}\tanh(kd)} \quad (3\text{-}15)$$

As with wave length, the equation takes different forms in deep and shallow water conditions namely:

$$c = \sqrt{\frac{g}{k}} = \frac{g}{\omega} \quad (3\text{-}16)$$

$$c = \sqrt{gd} \quad (3\text{-}17)$$

This relationship helps explain why tsunamis are difficult to observe out in the ocean yet develop into towering waves as they approach the shore. In deep water the tsunami wave has a very long wave length and is travelling extremely fast. However, as the wave approaches the shore the wave speed (and length) becomes determined by the water depth and the wave has to slow down. Thus, the energy in the wave that was stored in speed (kinetic energy) is transformed into energy stored in wave amplitude (potential energy). While wave celerity can give the velocity of the crest of a wave moving over the water surface the wave group velocity gives the velocity of energy associated with the wave. The concept can be demonstrated in a wave tank. When the wave maker sends the first regular wave train down the tank you may witness the height of the wave decreasing as it travels. Eventually that first wave disappears. In deep water conditions this is because the energy in the wave (which is seen in the wave height) is travelling as fast as the wave crest. *The energy associated with a train of regular waves includes contributions from both potential and kinetic energy* (see Figure 3-5). If we assume a portion of regular wave idealisation with height ζ, the center of gravity of this is located in the middle i.e. in way of ζ /2 and has a mass *ρgδx*. The potential energy (*P.E.*) of this portion is defined as:

$$P.E. = mgh = (\rho g \delta x)\frac{\zeta}{2} = \frac{\rho g \zeta^2 \delta x}{2} \quad (3\text{-}18)$$

If we integrate this energy over the entire wavelength, we get the potential energy of the wave per unit width as:

$$E_{PE} = \frac{\rho g \lambda \zeta_0^2}{4} \quad (3\text{-}19)$$

The wave has also kinetic energy (*K.E.*) and if the total velocity of the segment is q then the kinetic energy of the segment is:

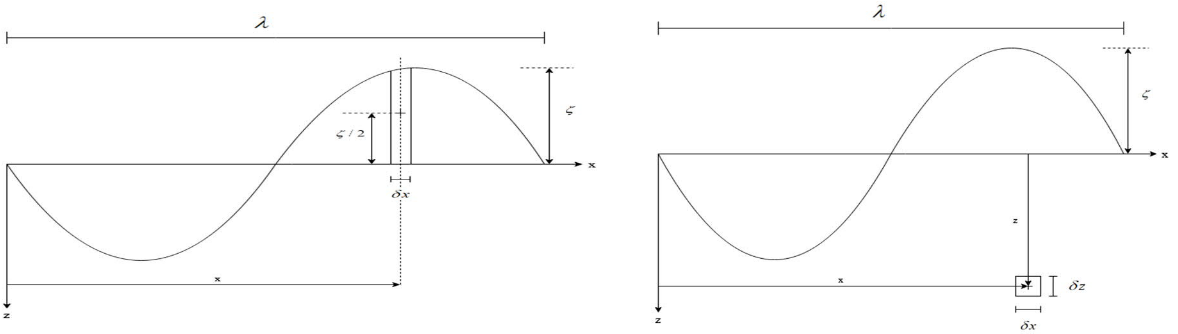
$$K.E. = \frac{1}{2}mq^2 = \frac{\rho q \delta x \delta z}{2} \quad (3\text{-}20)$$

Consequently, integrating this over the full wave length and by utilizing the relationship between the speed and wavelength gives the total *K.E.* per unit width as:

$$E_{KE} = \frac{\rho g \lambda \zeta_0^2}{4} \quad (3\text{-}21)$$

Thus the total energy becomes:

$$E_{TOT} = E_{PE} + E_{KE} = \frac{\rho g \zeta_0^2}{2} \quad (3\text{-}22)$$



(a) *K.E.* idelisation (b) *P.E.* idelisation

Figure 3-5 Wave energy integration

Deep water idealization implies that water particles involved in the wave motion do not detect the bottom. For deep water waves, the water particles move in circular motion. This means that the particles are not travelling with the wave but the wave passes along while the particles stay in the same spot. You have experienced this in the ocean if as waves pass you by. Although the waves move you up and down there is very little sideways motion. The particles near the surface of the water make large circular motions. However, as you go deeper in the water the particle motions decrease in amplitude (see Figure 3-6). Eventually the circular motion becomes so small that the water particles do not move as the wave passes by. You have experienced something like this if you have tried to avoid a wave by diving deep as it travels past.

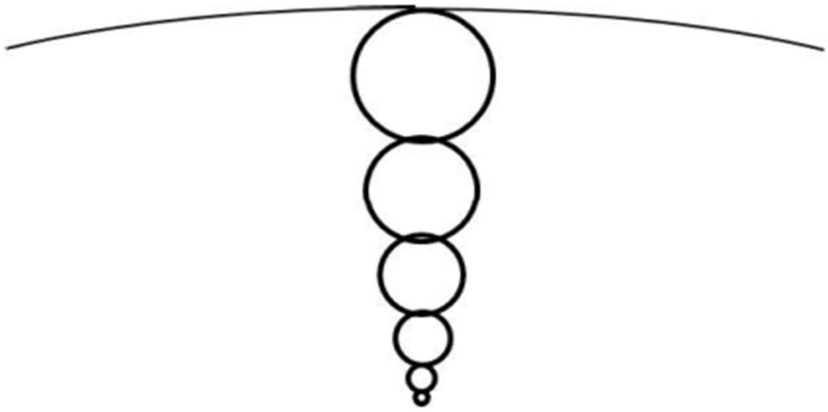


Figure 3-6 Decreasing water particle motion as a function of depth

3.4 References

Karadeniz, Halil, Mehmet Polat Saka, and Vedat Togan. 2013. "Water Wave Theories and Wave Loads." In *Stochastic Analysis of Offshore Steel Structures*, 177-252. Springer.