Appendix 1

Review of Probability and Statistics for Marine Applications.

We have shown that the irregular time histories of waves can be characterized in terms of energy spectra and various statistical quantities. Seakeeping studies, however, often demand a more intimate knowledge of waves. In particular, we need to be able to answer questions like "What is the likelihood of a particular wave height being exceeded?" We can use wave energy spectra and probability distributions to answer this type of question.

Probability Density Function (PDF)

The probability density function is defined such that the area enclosed by the PDF curve over a bin is equal to the probability of the measurement falling within that bin. So, the probability of the x-axis value falling between a and b is equal to the area under the curve from a to b. Figure 3.17 shows the area from a to b for a normal probability distribution curve. The probability the water elevation falls between these two limits is equal to the shaded area on the plot. The area under the entire probability density function equals one, since there is 100% probability that any measurement falls within the set of collected measurements. Water elevation typically follows a Gaussian or normal distribution. This is the typical "bell" curve, see Figure 3.18. The empirical rule states that there is about a 68% probability a measurement will fall between $\pm\sigma$ (one standard deviation), there is about a 95% probability a measurement will fall between $\pm 2\sigma$, and a 99% probability any measurement will fall between $\pm 3\sigma$. While *water elevation* typically follows a Gaussian distribution, *wave heights* (and amplitudes) follow a Rayleigh distribution for narrow-banded wave spectra.

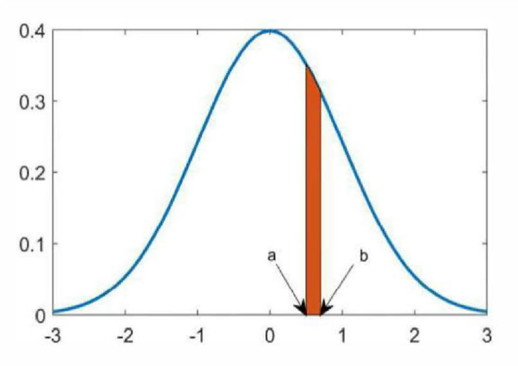

Figure A-1 The probability of the wave elevation falling between a and b equals the area under the pdf curve between these two values

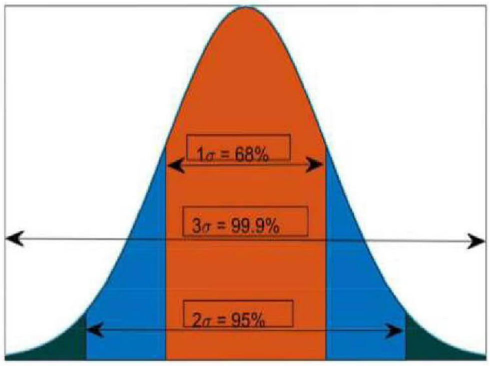

Figure A-2 Gaussian or Normal Probability Distribution

The probability for the wave amplitudes depends on the variance of the water elevation. Figure A-3 shows a typical Rayleigh distribution. The probability a wave amplitude would fall between two heights is equal to the area under the curve between those two points. The Rayleigh probability distribution equals

$$f = \frac{\zeta_a}{m_0} e^{-\frac{\zeta_a^2}{2m_0}} \qquad \text{(A-1)}$$

where ζ_a is the wave amplitude and m_0 is the variance from the water elevation time history or area under the wave energy spectrum curve.

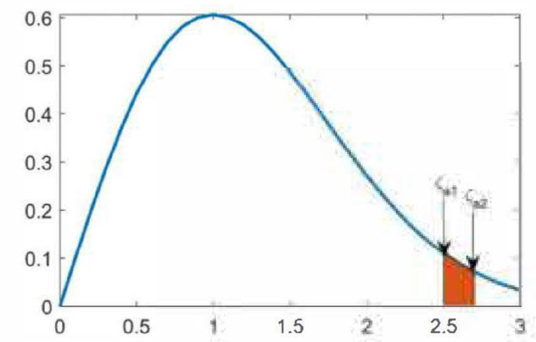

Figure A-3 Rayleigh Probability Distribution

Considering the wave amplitudes (Rayleigh probability distribution), the probability that an amplitude ζ_a will exceed a specific amplitude, ζ_{A1} is

$$P\left(\zeta_a > \zeta_{A1}\right) = e^{-\frac{\zeta_{A1}^2}{2m_0}} \qquad \text{(A-2)}$$

The probability that the wave amplitude will fall *between* amplitudes ζ_{A1} and ζ_{A2} is

$$P\left(\zeta_{A1} < \zeta_a < \zeta_{A2}\right) = e^{-\frac{\zeta_{A2}^2}{2m_0}} - e^{-\frac{\zeta_{A1}^2}{2m_0}} \qquad \text{(A-3)}$$

i.e. the probability of exceeding ζ_{A2} minus the probability of exceeding ζ_{A1}.

Significant Wave Height and Related Statistics

The significant wave height is the mean of the highest 1/3rd of the heights recorded in a wave time history. It closely correlates with the average wave height estimated visually by an experienced observer. It is expected that the experienced sailor's estimates of "average" wave heights might be similar to the significant wave height. The Rayleigh formula for the mean value of the highest 1/nth of all observations is

$$\zeta_{\frac{1}{n}} = \sqrt{-2m_0 \ln\frac{1}{n}} \quad \text{(A-4)}$$

So, for n = 1, the mean of all amplitudes, $\bar{\zeta}_a = 1.25\sigma_0$ where σ_0 is the standard deviation from the water surface elevation$\left(\sigma_0 = \sqrt{m_0}\right)$. For the significant wave amplitude $\bar{\zeta}_{1/3} = 2.00\sigma_0$. Significant wave height, $\bar{H}_{1/3}$ = 4.00$\sqrt{m_0}$. This is the same as saying that the significant wave height is equal to twice the significant wave amplitude. These results are widely assumed to apply to all wave records. However, this is only strictly true if the Rayleigh formula applies. Table below shows the values that can be multiplied by the water elevation standard deviation (σ_0) to determine the average of the $1/n^{th}$ highest amplitudes.

n	$\frac{\zeta_{1/n}}{\sigma_o}$	n	$\frac{\zeta_{1/n}}{\sigma_o}$
1	1.25	10	2.54
2	1.77	100	3.34
3	2.00	1000	3.72

Probability of Exceedance

So, what is the procedure for predicting the probability of the wave amplitude exceeding a particular value ζ_B in a specific sea state?

1. First we build an ITTC wave energy spectrum for our sea state (using the mean significant wave height and most probable modal period). We will need to convert the modal period, T_0, into modal frequency, $\omega_0 = 2\pi / T_0$.

$$S_\zeta(\omega) = \frac{1.25}{4}\left(\frac{\omega_0}{\omega}\right)^4 \frac{\bar{H}_{1/3}^2}{\omega} e^{-1.25\left(\frac{\omega_0}{\omega}\right)^4} \quad \text{(A-5)}$$

2. We can find the variance, m0 from this wave energy spectrum.

$$m_0 = \int_0^\infty S_\zeta(\omega)d\omega \quad \text{(A-6)}$$

3. Then, we use the variance and the value of interest to calculate the probability of exceedance

$$P(\zeta_a > \zeta_B) = e^{-\frac{\zeta_B^2}{2m_0}} \quad \text{(A-7)}$$

<u>Example</u>

Consider the ocean spectrum for a Brettschneider sea state 6. For the time history recorded (a total of 23.5 minutes), the variance of the water elevation was 16.81f^2. Find the significant wave height and the probability of a wave height exceeding 25 ft. Find the probability of exceeding the significant wave height. How did we get m_0 (variance of the water elevation)? It was found either by taking the variance of the time history (as in this problem) or by finding the area under the wave energy spectrum (as explained in the procedure above). Once we have it, we can find the significant wave height directly

$$\bar{H}_{1/3} = 4.00\sqrt{m_0} = 4.00\sqrt{16.81} = 16.4\text{ft} \quad \text{(A-8)}$$

To find the probability of exceedance we plug this into the equation 3.12. This equation requires us to use the wave amplitude. Since we want the probability of exceedance for a wave height of 25 ft, the corresponding wave amplitude is 25/2 = 12.5 ft.

$$P(\zeta_a > 12.5) = e^{-\frac{12.5^2}{2(16.81)}} = 0.0096 = 0.96\% \quad \text{(A-9)}$$

So, there is a 0.96% probability that we will encounter a wave height over 25 ft. The probability we will exceed the significant wave height of 16.4 ft (amplitude of 8.2 ft) is

$$P(\zeta_a > 8.2) = e^{-\frac{8.2^2}{2(16.81)}} = 0.1353 = 13.53\% \quad \text{(A-10)}$$