Lecture 5

Introduction to Ship motions

This lecture introduces the response of the ship during her exposure the external environment starting from the governing equation to response evaluation and assessment. There are three main components that influence this response namely:

- Waves (the input to the system)
- Ship characteristics (the system)
- Ship motions (the output of the system)

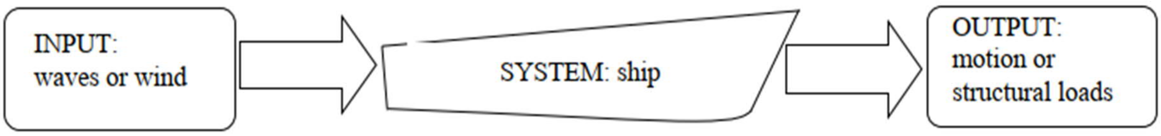


Figure 5-1 *The principle of ship response in wind and waves*

In the broadest sense, *seakeeping is the behavior of ships in waves*. Seakeeping is generally considered the ability of a vessel to withstand rough conditions at sea. It therefore involves the study of the motions of a ship or floating structure, when subjected to waves, and the resulting effects on humans, systems, and mission capability. With fast computers and sophisticated software readily available to designers, it is now possible for a vessel's seakeeping characteristics to be addressed much earlier in the design spiral. Regulatory bodies and operators are becoming increasingly aware of the importance of specifying seakeeping requirements which the vessel must meet. The seakeeping of a vessel depends on:

- General particulars. How does the length of a ship, for example, affect its performance in waves? For a given sea condition, ship length makes a big difference in behavior. Small ships suffer from large motions (relative to the ship). As the length reduces a ship begins to contour (travel along the surface of a wave as though it were a road). Thus, if a small ship is in big waves, it moves around a lot compared to its size. However, a big ship Will move less in the same set of waves, partly because it is longer relative to the waves, but also because it is more massive than the smaller ship. However, a long ship with a shallow draft runs a higher risk of keel emergence (the keel coming out of the water) when in rough seas.
- Hull form. It turns out that small detail changes (such as reducing the radius of curvature on the bilges) results in little Change in ship motions. However, changes to overall ship proportions (such as beam—to—length or beam—to—draft) can have important consequences with respect to the seakeeping qualities of a ship. Therefore, it is important for seakeeping considerations to be addressed early in the design process, before the final proportions and dimensions of the hull have been finalized. For example, reducing the draft of a ship (for a given length and beam) has the effect of reducing motions. However, it also increases the likelihood of keel emergence. Increasing the local beam at the bow results in greater wave excitation action the ship and, with flared bows, there is a

risk of flare slamming. However, a large forward waterplane can reduce overall motions and reduce the probability of keel emergence.

- Metacentric height. As you, hopefully, remember from Hydrostatics and Stability, the larger the metacentric height, the more initial stability a vessel has. However, from a seakeeping perspective, the trouble with too large a metacentric height is that it has too high a natural r011 frequency and this is associated with poor motion sickness indices (i.e. lots of people throwing up on your ship). However, if the metacentric height is too small the ship has a “lazy” r011 motion and there is the increased risk of capsize. As metacentric height depends strongly on the beam, reducing the beam results in a reduction in GM.

5.1 Ship motions

Ship rigid body motions (1 - 3) are linear displacements (translations) and the remaining (4 - 6) are rotations. All motions are measured relative to the ship itself, as shown in Figure 5-2.

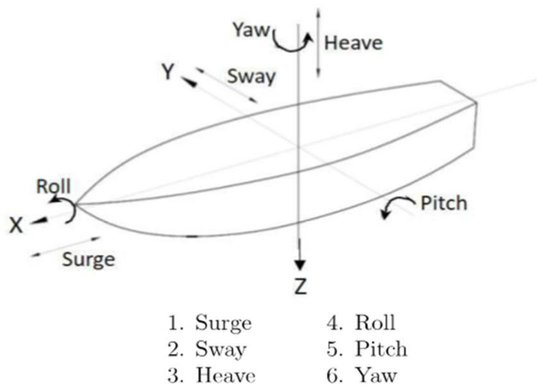


Figure 5-2 *Seakeeping notion of ship motions*

In specific, they are defined as follows:

- Surge describes the forward and back direction (forward is in the direction the bow is pointing and back is in the direction of the stern).
- Sway is the side—to—side direction, so a vessel moving in its starboard direction is traveling in the positive sway direction.
- Heave is the vertical direction and, by convention, positive heave is down (toward the water bottom). So, a vessel that is sinking into the water (increasing its draft) is moving in the positive heave direction.
- Roll is rotational motion about the surge axis. A vessel that has the starboard and port sides moving vertically but in opposite directions (Le. the starboard side is moving up while the port side is moving down) is rolling. Convention has a positive roll angle when the starboard side is down and the port side is up.
- Pitch is the rotational motion about the sway axis. When pitching, the bow and stem are moving vertically in opposite directions (i.e. when the bow is moving up and the stem is moving down). Pitch is positive when the bow is up relative to a level ship.

- Yaw is the rotational motion about the heave axis, or describes the turning motion of the ship. When the bow moves in the starboard direction, we consider that a positive yaw angle.

A ship operating in waves will depend on all 6 - DOF. However, *the most significant ones are those that have a restoring force associated with them*. For example, if the wave pushes the vessel to the side (sway motion effect), it may be inconvenient in terms of navigation, but the effect is just limited in time (there are no restoring forces). On the other hand, if a ship is pushed over so that her starboard deck drops while waves pass over return to original upright position is critical in terms of avoiding loosing stability leading to capsize.

As an example, Figure 5-3 illustrates the restoring force emerging from heave moment. It is proportional to the distance displaced since the disparity between displacement and buoyancy force is linear for different waterlines. Ships that have a large water plane area for their displacement will experience much greater heave restoring forces than ships with small water plane areas. So 'beamy' ships such as tugs and fishing vessels will suffer short period heave oscillations and high heave accelerations. Conversely, ships with small water plane areas will have much longer heave periods and experience lower heave accelerations. In general, the lower the motion acceleration, the more comfortable the ride and the less chance of damage to equipment and personnel. This concept is taken to extremes in the case of offshore floating platforms that have very small water plane area compared to their displacement.

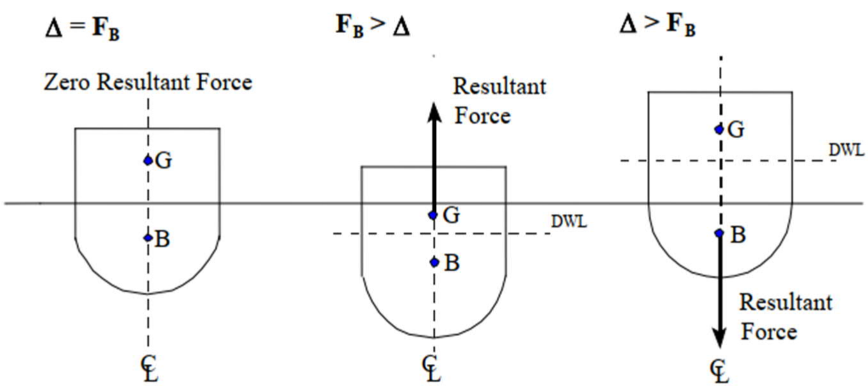


Figure 5-3 *Generation of the heave restoring force*

5.2 The ship encounter frequency

When ship dynamics are accounted for, the encounter frequency with the waves is used instead of the absolute wave frequency. This is because the ship is moving relative to the waves, and she will meet successive peaks and troughs in a shorter or longer time interval depending upon whether it is advancing into the waves or is travelling in their direction. Assuming the waves and ship are on a straight course, the frequency with which the ship will encounter a wave crest depends on the distance between the waves crests (λ — wavelength), the speed of the waves (c — wave celerity that depends on the wavelength), the speed of the ship (U), and the relative angle between the ship heading and the wave heading (μ), see Figure 5-4. The encounter period is thus the distance traveled (λ) divided by the speed the ship encounters the waves ($c - U\cos(\mu)$)- The encounter frequency is

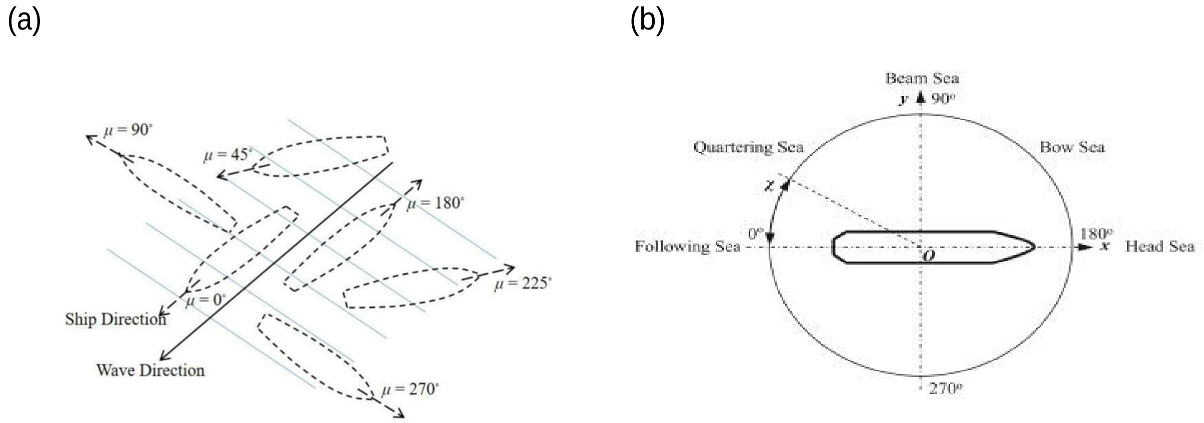


Figure 5-4 Ship seakeeping encounters idealization

$$\omega_e = \frac{2\pi}{T_E} = \frac{2\pi}{\lambda}(c - U\cos(\mu)) \tag{5-1}$$

Substituting in the relationships between wavelength, wave speed, and wave frequency, the encounter frequency can be written as

$$\omega_e = \omega - \frac{\omega^2 U}{g}\cos(\mu) \tag{5-2}$$

$$T_e = \frac{\lambda}{c - U \times \cos\mu} \tag{5-3}$$

The heading angle determines the "type" of seas the ship experiences. Heading angles are defined as follows :

- $\mu = 0^0$ – following seas
- $\mu = 180^0$ – head seas
- $\mu = 90^0$ – starboard beam seas
- $\mu = 270^0$- port beam seas
- $0 \leq \mu \leq 90^0$ – quartering waves on the ship starboard side
- $270^0 \leq \mu \leq 360^0$ – quartering waves on the ship port side
- $90^0 \leq \mu \leq 180^0$- bow waves on the starboard side
- $180^0 \leq \mu \leq 270^0$ – bow waves on the port side

5.3 Axis of reference

Another issue that concerns the description of ship motions is the point of reference (see Figure 5-5). This depends on what we are trying to describe. If we are giving the ship speed and heading we are measuring in reference to a point fixed on the earth (say x_E). If we want to describe the location of something on the ship we want to be able to use a reference frame that is fixed to the ship (x_B). For seakeeping motions (roll, heave and pitch), we want a frame that is co-located with the ship, but does not move with the ship in heave, roll, or pitch.

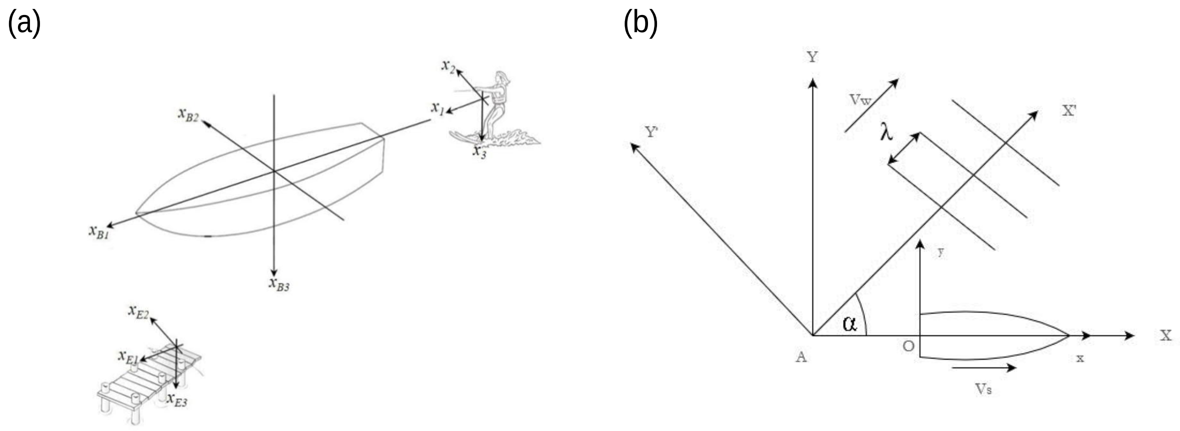


Figure 5-5 Illustration of the three reference frames (a) xB Body Fixed idelisation; xE Earth fixed idealization; and xi seakeeping notional frame, (b) axis transformation system.

Considering transformation of coordinates for a regular wave propagating at an angle α (from *A X' Y' Z'* to *A X Y Z*) as illustrated in Figure 5-5(b).

$$X' = X \cos \alpha + Y \sin \alpha \tag{5-4}$$

Then the transformation to the ship's-fixed coordinate system (oxy) coordinate system:

$$X = x + V_s t,\ \text{Y=y} \tag{5-5}$$

5.4 Review of rigid body dynamics

The behavior of ships in rough weather is similar to the classical oscillatory response of damped spring-mass system. With this in mind and with the ultimate aim to set sufficient background for the mathematical derivation of the dynamics of seakeeping this section reviews principles of rigid body dynamics. Let us consider the general form of the typical single degree of freedom (1-DOF) damped spring mass system with force excitation varying in time shown in Figure 5-6. If the mass is displaced in either direction, the spring will either be compressed or placed in tension. This will generate a force that will try to return the mass to its original location - a restoring force. Provided the spring remains within its linear operating region, the size of the force will be proportional to the amount of displacement - a linear force. If the mass is let go, the linear restoring force will act to bring the mass back to its original location. However, because of inertia effects, the mass will overshoot its original position and be displaced to the other side. At this point the spring creates another linear restoring force in the opposite direction, again acting to restore the mass to its central position. This motion is repeated until the effects of the damper dissipate the energy stored by the system oscillations. The important point to note is that no matter which side the mass moves, the mass always experiences a linear restoring force ; i.e. it exhibits simple harmonic motion (SHM) according to Newton's law.The main assumption of this formulation is that the mass is constant and the body is rigid. Accordingly, the classic equation:

$$\sum \vec{F} = m\ddot{x} \tag{5-6}$$

becomes:

$$-kx - c\dot{x} + F = m\ddot{x} \rightarrow m\ddot{x} + c\dot{x} + kx = F(t) \tag{5-7}$$

where m is the mass of the body, c is the damping coefficient and k is the spring stiffness.

For a ship, the stiffness term is mainly attributed to buoyancy. To realize the significance of this term just imagine the ideal case for which a ship undergoes pure heave motion. If you push the ship down in the water, based on Archimedes principle there will be an extra buoyant force acting up on the ship in excess of the ship's displacement. If you then release the downward force on the ship, it will move up. Likewise, lifting a ship out of the water will result in lower buoyancy force than the ship's displacement. So when released the ship will move down. Thus, *buoyancy is a restoring force, i.e. the spring in the system*.

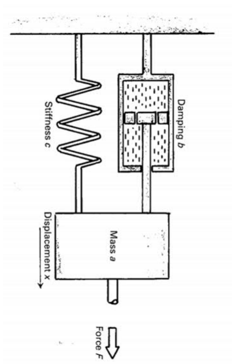


Figure 5-6 Typical spring-mass system with damper (1-DOF idealization)

5.4.1 Free undamped vibration of 1- DOF system

If we assume the system is conservative and the vibration is free and the equation of motion reduces to:

$$m\ddot{x} + kx = 0 \tag{5-8}$$

Rigid body dynamic response can be extracted by assuming the sinusoidal solution $x = e^{\lambda t}$ leading to:

$$\lambda^2 m + k = 0,\ \lambda = \pm\sqrt{\frac{k}{m}} = \pm j\omega_n \tag{5-9}$$

where $\omega_n = \sqrt{k/m}$ is known as the natural frequency of the system and the response is defined as:

$$x = A_1 e^{j\omega_n t} + B_1 e^{-j\omega_n t} = A\,sin(\omega t) + B\,cos(\omega t) = X\,sin(\omega t + \phi) \tag{5-10}$$

Where amplitude $X = \sqrt{A^2 + B^2}$ and the phase $\phi = \tan^{-1}(B/A)$.

If we assume the initial conditions:

$$x(t = 0) = x_0, \dot{x}(t = 0) = v_0 \quad (5\text{-}11)$$

this leads to:

$$X = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n}$$
$$\phi = tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) \quad (5\text{-}12)$$

Therefore the final solution of this system is defined as:

$$x(t) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} sin(\omega t + \phi) \quad (5\text{-}13)$$

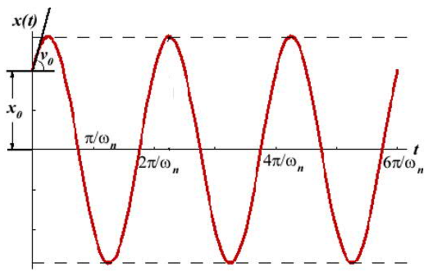


Figure 5-7 Free undamped vibration response

5.4.2 Free damped vibration of single DOF system

In reality, the amplitude of oscillation of the spring, mass, damper system will reduce with time due to damping effects. The damper works by dissipating the energy of the system to zero. Changing the viscosity of the fluid in the damper the level of damping can be altered. A low level of damping will allow several oscillations before the system comes to rest. If we add damping to the system Newton's equation of motion under free vibration conditions reduces to:

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (5\text{-}14)$$

By using the same trial solution we applied in equation (5.9) :

$$m\lambda^2 + c\lambda + k = 0, \quad \lambda_{1,2} = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \quad (5\text{-}15)$$

There are three solutions to the above differential equation that link to three different types of motions:

- if $\lambda_{1,2}$ are real ($c^2 - 4mk > 0$) corresponding to overdamped case.
- if $\lambda_{1,2}$ are imaginary ($c^2 - 4mk < 0$) corresponding to underdamped case.
- if $\lambda_1 = \lambda_2$are real ($c^2 - 4mk = 0$) leading to $c_{cr} = \sqrt{4mk} = 2m\omega_n$ that corresponds to critically damped case (i.e. case where the system is allowed to overshoot and then come back to rest).

Another approach used to solve Eq (5.8) is the damping ratio (ζ) which the ratio of the damping coefficient of the system to the critical damping coefficient:

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}} \rightarrow c = 2m\omega_n\zeta \tag{5-16}$$

If we use this notation the roots of Equation (5.15) become:

$$\lambda_{1,2} = \omega_n[-\zeta \pm \sqrt{\zeta^2 - 1}] \tag{5-17}$$

The three types of motions can then be defined by the damping ratio as:

- $\zeta > 1$ (for overdamped case);
- $\zeta < 1$ (for underdamped case) and
- $\zeta = 1$ for the critically damped case.

The response of the system in terms of these two roots is defined as:

$$x(t) = a_1e^{\lambda_1 t} + a_2e^{\lambda_2 t} \tag{5-18}$$

For the underdamped case where the damping ratio range is $0 < \zeta < 1$ this leads to:

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_d j \tag{5-19}$$

$$\omega_d = \sqrt{1 - \zeta^2} \tag{5-20}$$

$$x(t) = Ae^{-\zeta\omega_n t}sin(\omega_d t + \phi) = \sqrt{\frac{(v_0 + x_0\zeta\omega_n)^2 + (x_0\omega_d)^2}{\omega_d^2}}e^{-\zeta\omega_n t}sin(\omega_d t + tan^{-1}(\frac{x_0\omega_d}{v_0 + x_0\zeta\omega_n})) \tag{5-21}$$

If we follow the same procedure, the solution of the overdamped case is given by:

$$x(t) = a_3e^{(-\zeta\omega_n+\omega_d)t} + a_4e^{(-\zeta\omega_n-\omega_d)t} \tag{5-22}$$

Noted that the response of the overdamped solution is not oscillatory, which is considered a preferable case however it is difficult to achieve.

For the critically damped case, the solution is given by:

$$x(t) = [x_0 + (v_0 + \omega_n x_0)t]e^{-\omega_n t} \tag{5-23}$$

This response is also non-oscillatory, but it provides the fastest solution that return to zero after time.

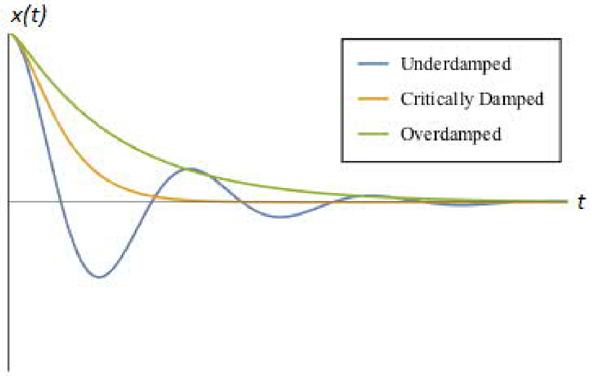


Figure 5-8 The three cases of damped free vibration response

A practical way to assess damping that is broadly applicable in the area of ship hydrodynamics is the damping decay. This can be mathematically expressed using the log decrement δ that is the natural logarithm of the ratio of two successive amplitudes. The natural logarithm of the ratio of the first two successive amplitudes X_1 and X_2 is defined based on the underdamped solution as follows:

$$\delta = ln\frac{X_1}{X_2} = ln\frac{Ae^{-\zeta\omega_n t_1}}{Ae^{-\zeta\omega_n(t_1+T_d)}} = ln\, e^{\zeta\omega_n T_d} = \zeta\omega_n T_d \tag{5-24}$$

$$\because T_d = 2\pi/\omega_d \rightarrow \therefore \delta = \frac{2\pi\zeta\omega_n}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{5-25}$$

Since the damping ratio is very small in that case, the log decrement can be approximated by:

$$\delta = 2\pi\zeta \tag{5-26}$$

5.4.3 Forced vibration of 1- DOF system

To enable the spring, mass damper system to remain oscillating, it is necessary to inject energy into the system. This energy is required to overcome the energy being dissipated by the damper. The force injects energy to coincide with the movement of the mass, otherwise it is likely to inhibit oscillation rather than encourage it. To maintain system oscillation, a cyclical force is required that is at the same frequency as the SHM of the system. When this occurs, the system is at resonance and maximum amplitude oscillations will occur. If the forcing function is applied at any other frequency, the amplitude of oscillation is diminished. To illustrate this principle let us consider adding harmonic excitation to the vibration system where $F(t)$ varies in sinusoidal manner instead of being arbitrary function in time:

$$m\ddot{x} + c\dot{x} + kx = F(t) = F_0\, cos(\omega t) \tag{5-27}$$

$$\rightarrow \ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = f_0 \cos(\omega t) \tag{5-28}$$

$$\text{where } f_0 = F_0 / m \tag{5-29}$$

This is a differential equation of the 2nd order. Accordingly, it is prone to a general and particular solution which when combined together they may give the response function of the system. The general solution is given when the left-hand side of the equation is equal to zero:

$$\ddot{x}_g(t) + 2\zeta\omega_n \dot{x}_g(t) + \omega_n^2 x_g(t) = 0 \tag{5-30}$$

leading to :

$$x_g(t) = Ae^{-\zeta\omega_n t} sin(\omega_d t + \phi), \omega_d = \omega_n\sqrt{1-\zeta^2} \tag{5-31}$$

Terms $A$and ϕ can be derived after deriving the full solution.

The particular solution is defined as follows:

$$\ddot{x}_p(t) + 2\zeta\omega_n \dot{x}_p(t) + \omega_n^2 x_p(t) = f_0 \cos(\omega t) \tag{5-32}$$

There are two possible trial solutions to the particular solution namely

$$x_p(t) = A_s \cos(\omega t) + B_s \sin(\omega t) \text{ or } x_p(t) = X \cos(\omega t - \theta) \tag{5-33}$$

Where

$$X^2 = A_s^2 + B_s^2, \quad \theta = \tan^{-1}(B_s / A_s) \tag{5-34}$$

Substituting the trial solution in the equation of motion leads to:

$$\begin{aligned}(-A_s\omega^2 + 2B_s\zeta\omega_n\omega + A_s\omega_n^2 - f_0)\cdot\cos(\omega t) + \\ (-B_s\omega^2 - 2A_s\zeta\omega_n\omega + Bs\omega_n^2)\cdot\sin(\omega t) = 0\end{aligned} \tag{5-35}$$

For this equation to be zero at any time t, the two coefficients multiplied by $sin(\omega t)$ and $sin(\omega t)$ must be zero:

$$-A_s\omega^2 + 2B_s\zeta\omega_n\omega + A_s\omega_n^2 - f_0 = 0 \tag{5-36}$$

$$-B_s\omega^2 - 2A_s\zeta\omega_n\omega + Bs\omega_n^2 = 0 \tag{5-37}$$

Solving these two equations we can find the two unknowns:

$$A_s = \frac{(\omega_n^2-\omega^2)f_0}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2} \quad \text{and} \quad B_s = \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2} \tag{5-38}$$

The particular solution after solving the unknowns becomes:

$$x_p(t) = A_s cos(\omega t) + B_s sin(\omega t) = \frac{(\omega_n^2-\omega^2)f_0}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2} cos(\omega t) + \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2} sin(\omega t) \tag{5-39}$$

Or

$$x_p(t) = X cos(\omega t - \theta) = \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2}} cos(\omega t - arctan[\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}]) \tag{5-40}$$

Eventually, the full solution is the summation of the general solution and the particular solution:

$$x(t) = x_p(t) + x_g(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + X\cos(\omega t - \theta) \tag{5-41}$$

If we solve A and ϕ using the initial conditions $x(0) = x_0$, and $\dot{x}(0) = v_0$

$$\phi = arctan[\frac{\omega_d(x_0 - X\cos\theta)}{v_0 + (x_0 - X\cos\theta)\zeta\omega_n - \omega X\sin\theta}] \tag{5-42}$$

$$A = \frac{x_0 - X\cos\theta}{\sin\phi} \tag{5-43}$$

The first term in the full solution is the transient solution, which tends to zero as the time goes to infinity, while the second term is the steady oscillatory solution (see Figure 5-9). The second term is of more importance as it is the steady solution. In many cases, we neglect the transient solution. The full solution then reduces to:

$$x_p(t) = X\cos(\omega t - \theta) \tag{5-44}$$

$$X = \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}} \text{ and } \theta = tan^{-1}[\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}] \tag{5-45}$$

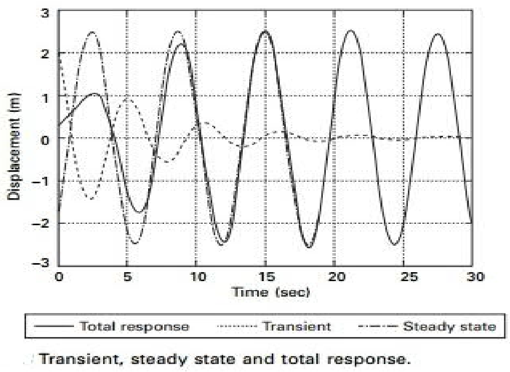


Figure 5-9 Harmonic excitation damped vibration

If we rewrite these equations as a function of the frequency ratio $\omega^* = \omega / \omega_n$ we get the expression:

$$\frac{Xk}{F_0} = \frac{X\omega_n^2}{f_0} = \frac{1}{\sqrt{(1-\omega^{*2})^2+(2\zeta\omega^*)^2}} \text{ and } \theta = tan^{-1}(\frac{2\zeta\omega^*}{1-\omega^{*2}}) \tag{5-46}$$

The term in the left-hand side is known as the amplitude ratio. Figure 5-10, shows the amplitude ratio values and the phase values for different damping ratio values. When the system is undamped, the amplitude ratio, when the frequency of vibration approaches the natural frequency, gets to extremely significant value, and such case is known by resonance.

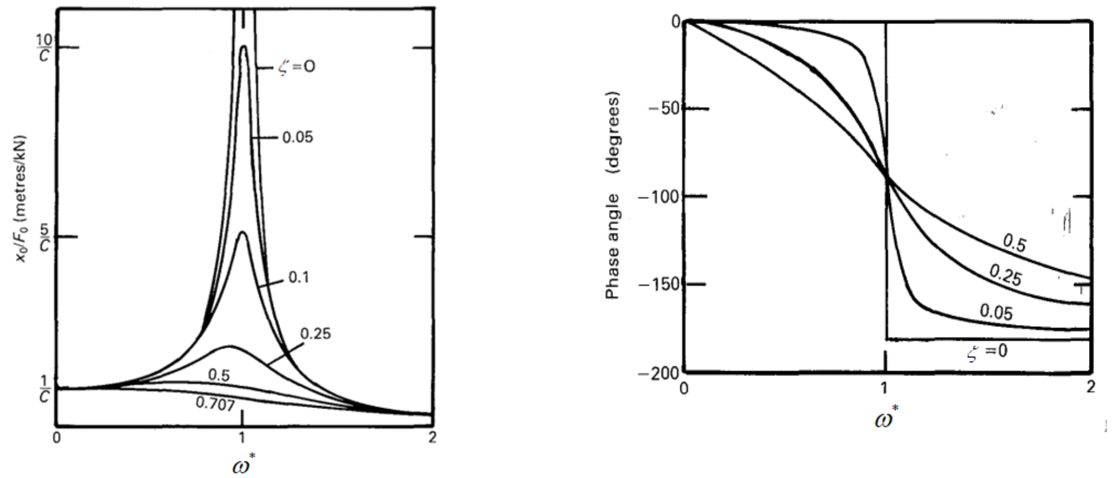


Figure 5-10 Amplitude ratio and phase angle representation

5.4.4 Quasi static, dynamic and resonant responses

If we apply a Fourier integral on the excitation force of Newton's equation of motion (see Equation 5.22) then external loading is defined as:

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_P(\omega)e^{i\omega t}d\omega \tag{5-47}$$

and the response becomes:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_x(\omega)e^{i\omega t}d\omega \tag{5-48}$$

Substituting the response function and its derivatives into Equation 5.22 leads to

$$\begin{aligned}&-m\int_{-\infty}^{\infty} A_x(\omega)\omega^2 e^{i\omega t}d\omega + c\int_{-\infty}^{\infty} A_x(\omega)i\omega e^{i\omega t}d\omega + k\int_{-\infty}^{\infty} A_x(\omega)e^{i\omega t}d\omega \\ &= \int_{-\infty}^{\infty} A_P(\omega)e^{i\omega t}d\omega\end{aligned} \tag{5-49}$$

$$\begin{aligned}&\int_{-\infty}^{\infty}[(-m\omega^2 + ci\omega + k)A_x(\omega) - A_p(\omega)]e^{i\omega t} = 0 \\ &\rightarrow [(-m\omega^2 + ci\omega + k)A_x(\omega) - A_p(\omega)]e^{i\omega t} = 0\end{aligned} \tag{5-50}$$

For

$$A_x(\omega) = \frac{A_P(\omega)}{-m\omega^2 + ci\omega + k} \tag{5-51}$$

Equation (5.47) can be written in terms of the frequency ratio:

$$A_x(\omega^*) = \frac{A_P^n(\omega^*)}{-m\omega^{*2}+\delta i\omega^*+1} \quad (5\text{-}52)$$

Where

$$A_P^n(\omega^*) = \frac{A_P(\omega)}{m\omega_n^2}, \quad \text{and} \quad \delta = \frac{c}{m\omega_n} \quad (5\text{-}53)$$

If we multiply this term with the complex conjugate, we get the spectral density for that system:

$$S_x(\omega) = \lim_{T\to\infty} \frac{1}{\pi T} |A_x(\omega)|^2 = \lim_{T\to\infty} \frac{1}{\pi T} \left| \frac{A_P^n(\omega^*)}{-m\omega^{*2} + \delta i\omega^* + 1} \right|^2 = \frac{S_P^n}{(1-\omega^{*2})^2 + \delta^2\omega^{*2}} \quad (5\text{-}54)$$

At sub-critical case (also known as quasi-static response) the system can reach high values of spectral density at small frequencies relative to the natural frequency and the stiffness has the major effect on the system:

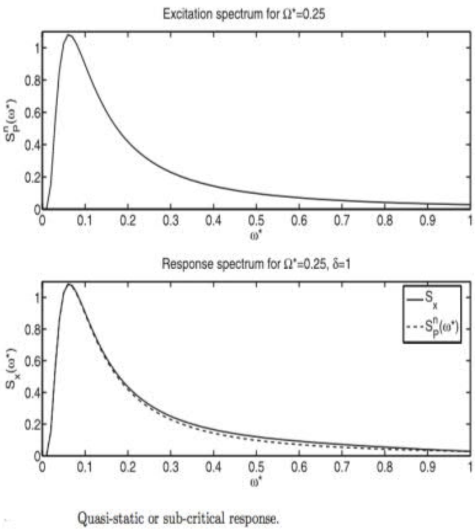
$$S_x(\omega^*) = \frac{S_P^n(\omega^*)}{(1-\omega^{*2})^2 + \delta^2\omega^{*2}} \to S_x \approx S_P^n \quad (5\text{-}55)$$

At super-critical (also known as dynamic response) the highest values of spectral density lie in only high values of frequencies with respect to the natural frequency and damping plays an important role:

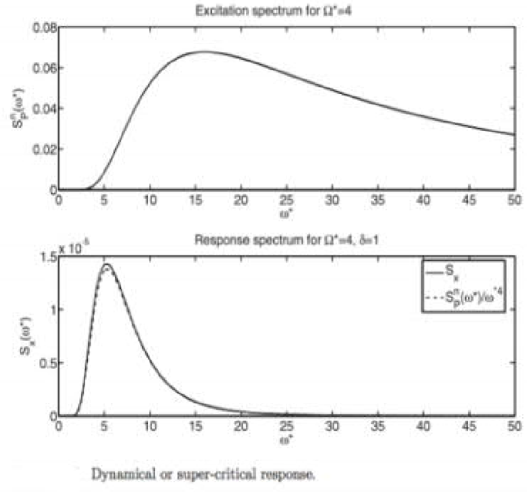
$$S_x(\omega^*) = \frac{S_P^n(\omega^*)}{(1-\omega^{*2})^2+\delta^2\omega^{*2}} \to S_x \approx \frac{S_P^n}{\omega^{*4}} \quad (5\text{-}56)$$

At resonance condition when there is very low damping the frequency ratio ω^*approaches unity. The denominator approaches zero, and the spectral density approaches extremely large value:

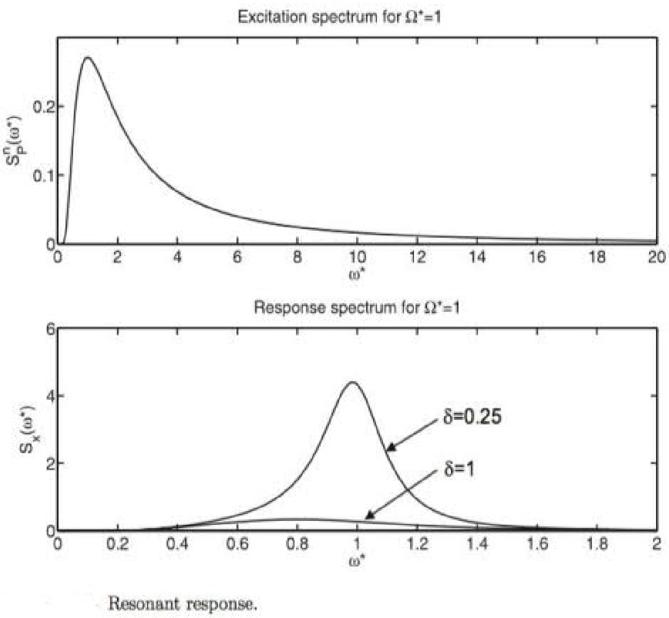
$$S_x(\omega^*) = \frac{S_P^n(\omega^*)}{(1-\omega^{*2})^2+\delta^2\omega^{*2}\to\approx 0} \to S_x >>>> \quad (5\text{-}57)$$



(a)



(b)



(c)

Figure 5-11 (a) Quasi-static response (b) Dynamic response (c) Resonant response