Sea Spectra Simplified

By Walter H. Michel[1]

A dissertation on the simple wave elements that make up the complex sea, this paper is intended to give the practicing naval architect a clearer view of how regular waves combine into an irregular pattern and how the consequent irregular behavior of a vessel at sea can be predicted on the basis of recent statistical formulations.

Prologue

MORE than 13 years have elapsed since St. Denis and Pierson introduced to this Society the exciting new theory of sea-wave behavior and its effect on ships ("On the Motions of Ships in Confused Seas," *Trans. SNAME*, vol. 61, 1953). Since that time, much effort has been expended in proving, refining, and applying this theory in research activities until today we are on the threshold of complete acceptability for valid engineering application to all types of seagoing vessels, in all ocean areas. Unfortunately, little has been done to bring this theory, and the techniques for applying it, to clear focus for the practicing naval architect, who is normally not engaged in such research activity and cannot otherwise devote sufficient time to "dig" the theory. However, he has a definite need to know and to apply its findings. This, then, is an attempt to explain the basic sea-spectrum theory in its simplest, elemental terms and to show how direct application of prevailing sea characteristics can be made to determine forces and motions of a body in that sea. Along with this, an attempt is made to eliminate the numbing effect of abstruse notation and format, by presenting the latest, most acceptable values for the sea spectrum in a simple and readily usable form.

Introduction

The sea is never regular. It does not take the form of a series of uniform waves, of constant height and length, proceeding in a steady and reliable sequence. Rather, a true sea is a random phenomenon, where waves are continually changing in height, length, and breadth.

While this fact has long been recognized, it has only been within the last 15 years that logical mathematical methods have been developed for properly characterizing the sea and for applying this information toward the engineering prediction of forces and motions of a body in that area.

The methods are fairly straightforward, if somewhat more lengthy than those classically employing the simple, regular wave. Although the theory is still in the throes of development and change, as more study and actual sea data are gathered, and, although there are still limitations to it (it does not as yet take good account of shallow water, or very steep waves, for example), it presents the most logical assessment of what the sea actually is and how it does what it does.

Even though this is now well recognized, much study and analysis of forces and motions are still made on the basis of the simpler, regular-wave theory. To the extent that one "knows what he's doing," it is still a useful concept and can give meaningful results. In fact, the wave-spectrum theory utilizes information on regular waves and their effects, as will be shown.

However, reliance on the old method of analyzing forces and motions produced by a few selected regular waves, to the exclusion of full consideration of the sea spectrum, has its pitfalls. In particular, a common but erroneous assumption is often made: that those properties indicating the intensity of the sea (designated "significant wave height" and "significant wave period") can be considered as the properties of an equivalent regular series of waves, the effects of which will accurately represent the effects of the sea. In so doing, one may make a gross overestimate or underestimate of the actual forces and motions that may be truly anticipated, depending on the sea state under consideration and the way in which the vessel would be affected by each of the individual wave components that make up the sea.

The following, then, is an explanation of what a sea comprises, and how a predicted or observed sea state can be analyzed to determine the forces and/or motions of a body in that sea.

Makeup of the Sea

Typically, the sea comprises a myriad of waves, of all different sizes, lengths, and directions, jumbled together as a result of wind-generated (usually) disturbances of different intensities, locations, and directions.

The theories so far developed, and in large part substantiated, explain how such seas build up, change shape, disperse, and so forth. Although we still have insufficient meteorologic and hydrographic data (not to mention the facilities for processing it) to make long-range

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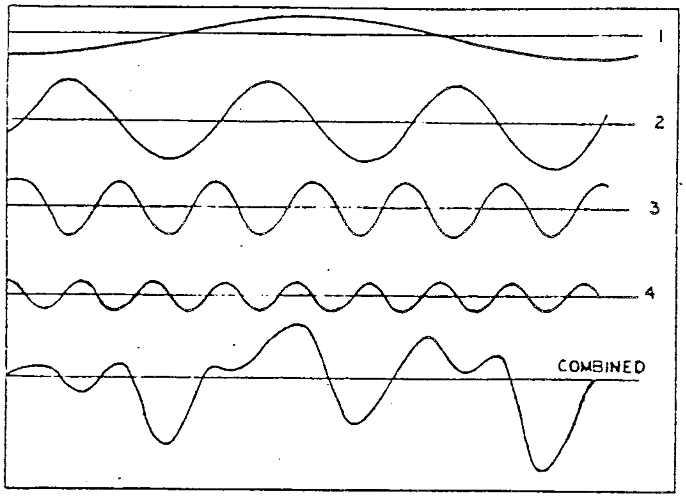

Fig. 1 Wave pattern combining four regular waves

and accurate programmed predictions of exactly when and where storms of what intensities will arise, we do have fairly good information on whether and how frequently they will arise in a general area. Further, once a sea has been generated, we have a good idea of its characteristics, how it will behave, and how it will affect bodies in its path—which this paper describes.

We shall consider primarily the basic two-dimensional seaway, that which is classed as an "irregular sea," as generated by a broad-scoped wind. The wave crests are continuous in a breadthwise direction, and all waves move in the same forward direction. Ultimately, of course, as an actual sea moves out of the storm area it spreads sideways (losing height as it does) and thins out progressively as the longer waves in the sea tend to outrun the others. If, in the course of its travels, it also meets waves from other disturbances coming from different directions, as is usually the case, the short-crested "confused sea" results. Whereas this three-dimensional "confused sea" is more prevalent in nature and is of importance in such instances as evaluating a long-range history of the motions of a ship, the two-dimensional "irregular sea" is considered to have maximal effect on a body situated within it, particularly at or near the area where the sea has been generated.

As to the actual makeup of the seaway (in two dimensions now) the mathematician may describe it as:

"An infinite number of unidirectional sinusoidal waves, with continuous variation in frequency; with each wave of an infinitesimal height and random in phase."

For more immediate understanding and visualization of the seaway, we may more simply describe it as:

"A collection of a great number of simple, regular waves of different lengths, all of small height and all mixed together with no apparent relation to each other except that they are all there and are all traveling in the same direction."

The result is an irregular sea, with no set pattern to the wave height, length, or period. For illustration, we may consider the result of combining a small number of regular waves of different lengths and heights, Fig. 1.

It can be seen how irregular the resulting wave is formed by only four basic regular waves; and it requires very little imagination to foresee that with an infinite number of simple waves, all of different lengths (or periods), the summed-up resultant wave would be completely irregular and impossible to predict in shape. In fact, the most distinctive feature of the irregular sea is that it never repeats its pattern from one interval to any other. Thus, we cannot characterize or define an irregular sea by its pattern or shape.

There is, however, one way we can define the sea in simple terms. Its total energy must necessarily be made up of the sum of the energies of all the small, regular waves that make up the sea—no more and no less.

Note that the energy of a simple, sinusoidal wave is readily shown to be $\rho g H^2/8$, for each square foot of water surface (ρg is the unit weight of water, H is wave height from crest to trough). Then the total energy in each and every square foot of the seaway is

$$\text{Energy} = \frac{\rho g}{8}(h_1^2 + h_2^2 + h_3^2 + \ldots\ldots)$$

or simply a constant times the sum of the squares of the heights of all the simple, small waves that exist in that seaway.

Thus, the intensity of the sea is characterized by its total energy; and, what is most important, we can show the individual contribution made by each of its component waves. In other words, with each component wave of different length or period (or, more conveniently, of different frequency), we can show how the total energy of the sea is distributed according to the frequencies of the various wave components.

This distribution is what we call the energy spectrum of the sea, or more simply the "wave spectrum." For simple illustration, Fig. 2 shows a crude spectrum made up of the same waves used in Fig. 1.[2]

The ordinate of the "curve" is expressed as energy-seconds and may be regarded as an abstract term conveniently selected so that the area under the spectrum curve represents the entire energy of the system when plotted on a frequency base that has the dimension of 1/seconds. Note that while we have centered the energy of each component wave at its designated frequency, we have given it a small "bandwidth," so that the energy-seconds ordinates have finite values and so that the curve has a semblance of continuity over a wide range of frequency.

[2] A little reminder: the deep-water characteristics of regular waves are as follows:

wave period, sec $= T$
wave length, L, ft $= gT^2/2\pi$
wave speed, C, fps $= L/T = (gL/2\pi)^{1/2} = gT/2\pi$
cyclic frequency, f, cps $= 1/T$
circular frequency, ω rad/sec $= 2\pi/T$

And continuity it must have, according to observed behavior of the actual sea. That is, we must consider that the sea contains a great number of waves varying slightly in frequency, one from the next. Otherwise, if we only had four different waves, as per this elemental example, or even ten or twenty, sooner or later we would see the wave pattern of the sea repeating itself exactly. Furthermore, as the sea proceeded into new areas, it would separate into groups of regular waves, followed [illegible] [illegible] run those of longer frequency, shorter length). These conditions certainly are not representative of those that actually exist since the sea disperses or "dissipates" in a continuous and gradual manner.

So we need to consider that a sea is composed of a very great number of different frequency waves; and, for a given amount of total energy of the sea, we can see that the greater the number of waves considered, the less energy (or height) each of these component waves possesses. Ultimately, then, the most factual energy spectrum of the sea is a smooth, continuous curve made up of the contribution of an infinite number of regular waves, all of different period and exceeding small height, as in Fig. 3.

Add to this the fact that we do not know, and cannot predict, what relative position one component wave has to the next (i.e., what the phase relationship of one sine wave has to another), and we can never tell just when several waves will group together to form a high sea wave, or when they will tend to cancel out, or whatever, in any systematic sequence. Thus, we have "randomness," and we now see the logic of the mathematician's definition of the seaway.

Note that instead of using energy-seconds as the ordinate of the curve, resulting in energy as the area, we may conveniently substitute square feet-seconds for the ordinate and square feet for the area as a direct indication of component wave height variations since energy and height^2 are directly proportional.

We shall see later just what values may be ascribed to the spectrum curve; that is, what mathematical function can be used to define both shape and area under the curve for given conditions of wind force or sea state. For now, we note that the spectrum builds from the high frequency end. That is, for a given wind speed, the first waves generated are those that are of short length; and then, as the wind continues to blow, longer and longer waves are generated until finally the condition known as "the fully developed sea" is reached, where the system is stable and no further effect is produced, regardless of how much longer the wind blows and over how much more area, Fig. 4.

Thus, as more and more energy is put into the seaway, its spectrum changes. As it grows, its total scope includes more low-frequency (longer) waves, and its maximum value shifts toward the low-frequency end, as well. This is true also for fully developed seas as a function of increasing wind speed.

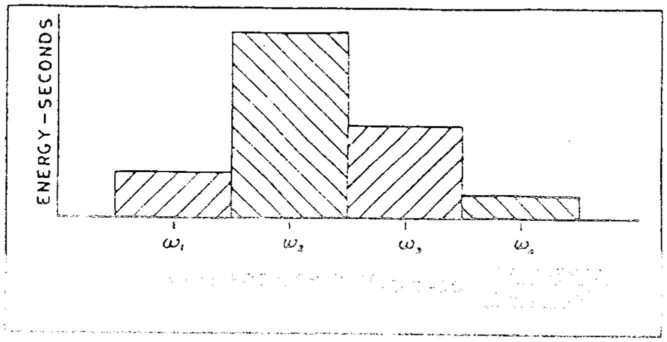

Fig. 2 Spectrum of four waves

Heights of Sea Waves

Now that the form and content of the spectrum become clear, the question remains how to determine the actual physical wave heights that are produced within the seaway itself. As stated previously, we cannot predict the actual pattern of the sea surface insofar as which wave follows which. However, we can predict by statistical methods how often waves of various heights will occur over any given period of time for a sea of a given amount of energy.

The most acceptable formulation to date for deriving statistics of wave-height distribution is one which has been corroborated by actual wave measurements that have shown a very consistent pattern over many years of investigation. To illustrate how such a distribution is determined, the heights of all the waves in a given record are measured and the percentage of occurrence calculated, that is, the *number* of waves under two feet high, from two to four feet high, four to six, and so forth, are each divided by the *total number* of waves in the record. These percentages are then plotted against the wave heights themselves, resulting in Fig. 5.

It was found that one simple form of curve fits most sea-wave histogram records very closely. This is known as the Rayleigh distribution and is written

$$p(H_i) = \frac{2H_i}{\overline{H^2}} e^{-H_i^2/\overline{H^2}}$$

This may be expressed as "the percentage of times that a wave of height, H_i feet ($\pm 1/2$ ft), will occur in all the waves of that series."

Let us identify the term $\overline{H^2}$. This is the average of all the squared values of the wave heights in the record, or expressed mathematically

$$\overline{H^2} = \frac{1}{N} \sum_{i=1}^{i=N} H_i^2$$

where N is the total number of waves in the record.

Intuitively, we can see that this value, being the average over the entire area of the sea (for the actual sea waves which are made up by the collective action of the small regular waves in the spectrum), should be very close in representing the average energy of the sea, that is

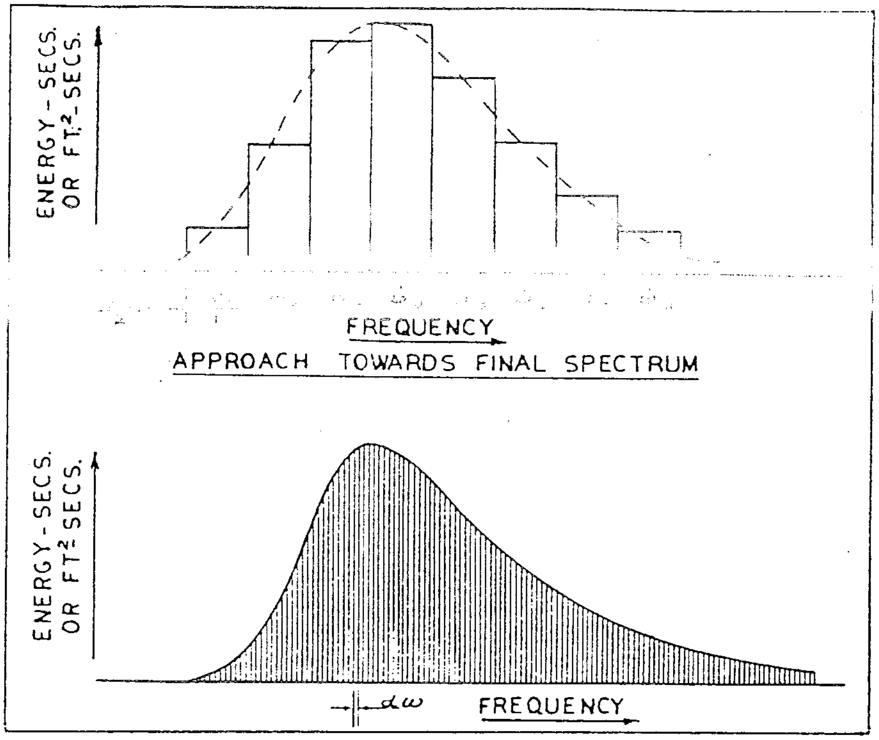

Fig. 3 Final spectrum

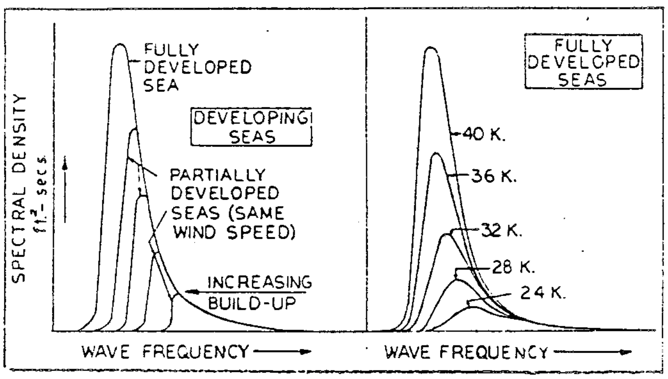

Fig. 4 Growth of spectrum

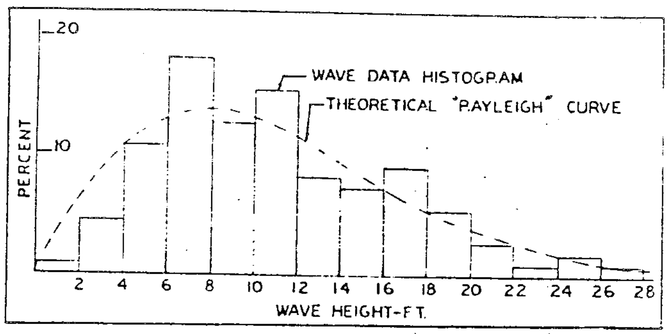

Fig. 5 Histogram of wave height measurements

$$\frac{\rho g}{8}\overline{H^2} \approx \frac{\rho g}{8}(h_1^2 + h_2^2 + h_3^2 + \ldots) = \text{Energy}$$

Thus, once we know the area under the spectrum curve, we can relate it directly to the Rayleigh distribution formula and determine from this all sorts of useful probabilities of occurrence of different wave heights. For example

$$P(H > H_i) = 1 - \int_0^{H_i} \frac{2H_i}{\overline{H^2}} e^{-H_i^2/\overline{H^2}} dH = e^{-H_i^2/\overline{H^2}}$$

gives "the probability that the wave height will be greater than H_i," or, in other words, out of a number of waves, N, there will be $Ne^{-H_i^2/H_2}$ waves that will be higher than H_i.

We can go from this into determining what the average wave height will be, or the average height of the one-third highest wave, or the one-tenth highest wave, and so forth. For instance:

Average wave height, $H_0 = 0.89(\overline{H^2})^{1/2}$

Average height of one-third highest waves, $H_{1/3} = 1.41(\overline{H^2})^{1/2}$

Average height of one-tenth highest waves, $H_{1/10} = 1.80(\overline{H^2})^{1/2}$

A more comprehensive tabulation of these various statistical probabilities, all derived from the Rayleigh distribution, are given in Appendix 1.

Let us note here the particular importance of "the average height of the one-third highest waves." This is identical to the value assigned to the "significant height," which stems from the fact that psychological (and physiologically) an observer tends to neglect the small waves and only notices the larger ones when evaluating the wave conditions he is experiencing. The *significant height* remains a most significant index since the practice of reporting sea conditions on this basis is widespread, among oceanographers and seafarers alike.

Let us dwell briefly on the validity of the Rayleigh distribution in the sense of whether and how it fits in the overall statistical theory which has been developed for sea waves. In the first place, the laws of statistics indicate that the sea surface should follow the well-known Gaussian or "normal" distribution (the good old "bell-shaped" curve), that is, the probability that the water surface at a given location has a certain elevation could be determined using the normal distribution.

However, this in itself does not do us much good since we are primarily interested in the frequency and value of the maximum (or minimum) elevations, that is, crests and troughs. A more useful formulation would then be the envelope curve of the maximum surface elevations, and happily this is what the Rayleigh distribution works out to be.

There is a theoretical reservation. The Rayleigh distribution is mathematically indicated to apply accurately only to a narrow spectrum (one which is highly peaked in shape with most of the energy contained in a narrow range of frequency) or to a narrow band of a general spectrum. It presumably loses validity when applied to an entire broad spectrum or a multipeaked spectrum unless certain corrective factors are applied.

Furthermore, mathematical laws are obeyed only if we consider the amplitude of the wave, measured either above or below the still-water level. Therefore, theory says we cannot precisely deduce wave heights (measured from crest to succeeding trough) from a given distribution, nor can we derive the proper distribution from measurements of wave height simply because the sea does not give us individual waves having crests the same distance above the still-water level as succeeding troughs below. In other words, the probability of a crest being a certain height above still-water level is not associated with an equal probability that the succeeding trough will be the same distance below.

Where does all this leave us? Repeated tests and observation indicate that the uncorrected Rayleigh distribution still gives excellent correlation, regardless of spectrum shape, and does so in terms of observed wave heights as produced by the sea. At least, it does so with sufficient accuracy for engineering application, and no other distribution yet has been shown to give as consistently good results.

Applications to Ship Behavior

With the knowledge that the Rayleigh distribution holds for the wave spectrum, regardless of its shape, and that we can thereby determine wave heights in the seaway and their probability of occurrence, can we take the bold, giant step forward and apply this same analysis toward other things that occur in that sea? Can we, for example, derive a "heave" spectrum for a ship in that sea whereby, instead of plotting the square of the height of each of the component regular waves, we plot the square of the heave that a regular wave of that height and frequency would produce? Can we do it for other ship motions and for wave forces on a body; and do those same statistical factors apply to the resulting spectrum for determining the magnitudes of the motions or forces and their probabilities of occurrence in that seaway?

Fortuitously, the answer is a firm yes,[3] as has been substantiated by a goodly number of performance tests and analyses. And this is the real import of the theory and the developed techniques, for the wave data alone would have little significance if we could not reliably determine the forces and motions of the bodies in those waves.

We need to devise the spectrum for the particular motion or force on the body. For this we need

1 The height characteristics of the component waves of different frequencies that occur in the sea. These are, of course, given by the sea spectrum in terms of square feet-seconds.

2 The unit response of the vessel for each of the component waves of different frequency. For example, if we are investigating motions such as pitching, we need to determine the maximum pitch angle (up or down) for a wave one foot high, and we must do this for a sufficient number of regular waves of selected frequencies (different length) corresponding to the range of frequencies given by the sea spectrum. If we are looking for wave forces on the body, we determine the maximum force (plus or minus) per foot of wave height for a similar range. The ordinate of the unit response is then pitch angle/foot of height, or force/foot of height, or whatever it is we are investigating.

By multiplying the spectral wave height by the square of the unit response (at the same frequency), we get ordinates of (pitch angle)^2-sec., or force^2-sec., and so forth, and, plotting these for the full range of frequencies involved, we get the particular motion or force spectrum we desire.

We then can proceed as for the wave spectrum itself by getting the area under this new spectrum curve and applying the same Rayleigh constants to get one-third highest motion, one-tenth, and so forth, as we may desire.

Let us illustrate this technique of deriving force and motion spectra from a characteristic wave-height spectrum, Fig. 6. Suppose we first wish to find the peak vertical accelerations of a small floating object, such as a raft, that follows the wave surface closely. The vertical acceleration of the water particles on the surface (and thus the same for the raft) is the second derivative of the regular sine wave equation, or

$$\ddot{y} = \alpha = -\frac{h\omega^2}{2}\cos\omega t \text{ (ft/sec}^2\text{)}$$

[3] Surprisingly, it is true even for nonlinear systems such as ship's roll motions, and slamming.

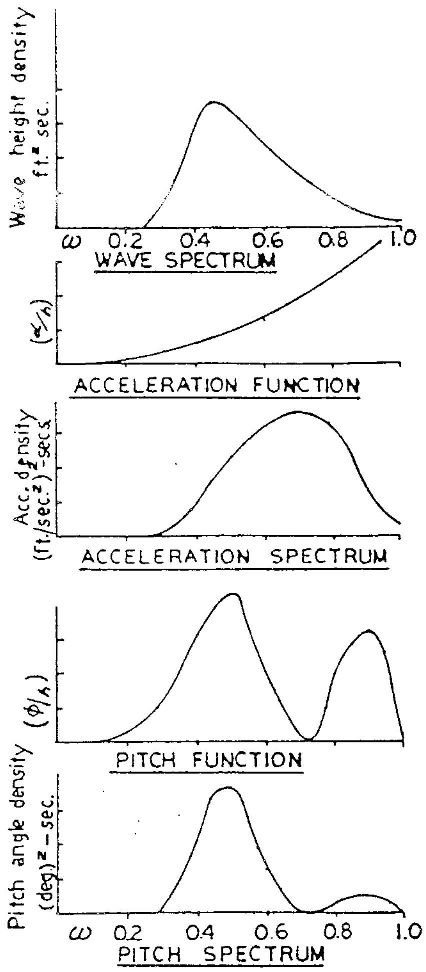

Fig. 6

where h is the height of the component regular wave of cyclic frequency, ω.

Then, the unit of maximum acceleration per foot of wave height is:

$$\alpha/h = \frac{\omega^2}{2}$$

From the wave-height spectrum, we note the value of the component wave-height "density" (in terms of square feet-seconds) at each frequency.

Multiplying this by the square of the unit acceleration, $(\alpha/h)^2$, gives us the value of the acceleration "density" at the same corresponding freuqency. Plotting these new values on the frequency base, we get the acceleration spectrum.

We also may have double-peaked spectra in certain applications. For instance, with two rafts rigidly connected some distance apart and the whole configuration heading into a sea, we can investigate the pitch angle. The equation for pitch single amplitude would be

$$\phi = \frac{h}{x} \sin \frac{x\omega^2}{2g} \sin \omega t$$

If we take x, the separation between rafts, equal to about 380 ft and put pitch angle in degrees, we get for the maximum pitch per foot of wave height

$$\phi/h = 0.15 \sin (6\omega^2)$$

and the pitch spectrum is found to be double peaked.

See Appendix 2 for several other interesting examples.

You can see from the figures used in these examples the importance of the shape of the wave-height spectrum, particularly in regard to the value of the maximum "density" and the frequency at which it occurs. Any appreciable change in "peaking" of the wave spectrum, whether it be a change in the value of the density ordinate or a shift in frequency (even though the area remained the same), would change the shape of the derived spectrum and, more importantly, would change its area and thus its maximum values. Thus, it frequently may be necessary to investigate conditions using more than one specified wave-spectrum form. We have noted that the shape can be different under different wind conditions, and it may be that, for some ship characteristics, a fully developed sea with relatively low wind speed may be critical, whereas for other characteristics a young sea with high wind speed may be the worst (even though both seas may produce the same wave heights).

You will note that in this spectral analysis technique we resort to the familiar regular wave concept to get the ship response functions, but we do not stop there. We must apply these characteristics to more realistic sea conditions to get more realistic predictions of forces and motions.

There is nothing particularly different in the wa[y] response functions in regular waves are determined now from in the past. Regular analysis may be used for the simpler motion or force considerations or for preliminary evaluation studies, after which we may resort to model tests in regular waves for more definite and accurate values.

In this latter regard, one may wonder why, with new model tank techniques for reproducing irregular sea waves to controlled spectra, we could not dispense with the intermediate step of determining regular-wave ship responses and the ensuing analyses. Why not go directly to testing the model in the irregular sea and get the results forthwith? As a matter of fact we can and do, but generally only to serve as a check on the overall analytical method or to "eyeball" the more realistic behavior pattern.

Otherwise, these direct irregular sea tests are somewhat cumbersome. Even to model scale, a very long run is required for each test in order to get reasonable assurance that the statistically maximum wave heights were experienced. Then, it becomes necessary to examine and analyze a long record to establish what values were achieved, with what frequency, and so forth. It can be done, but with much labor and with much room for error.

A further problem is encountered when ship characteristics are sought at forward speeds. Here we ru[n] [i]t of tank length long before we cover the gamut of wave-height variations. Then, finally, add to this the difficulty of utilizing the data from one sea-spectrum test; and, applying it to still another spectrum, we can see that the simpler, quicker, more readily usable data derived from regular-wave tests (later applied to whatever wave spectrum desired) give more satisfying results, particularly so, in view of the fact that repeated tests and checks on the method show it to be accurate.

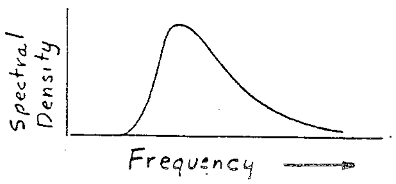

Fig. 7

Quantitative Formulations for Sea Spectrum

In order to apply the foregoing outlined principles of spectral analysis to specific engineering purposes, we need to know the quantitative values of the wave spectrum for different ocean areas and for different wind or seasonal conditions.

We shall not attempt to describe the methods used in deriving wave spectra from actual sea data, a sort of "working backward" technique employing rather arduous mathematics. A fair number of spectrum formulas have been proposed through the years by analysts accomplished in the field, and more formulas are coming. Some will replace older formulations on the basis of new data, some will have more versatile application, some will apply to new ocean areas.

There still exist significant differences among spectrum formulas since the theory is still young, and data and data-taking methods are not the same. Nevertheless, these formulas are the best we have now, and we can accept them as authoritative and applicable to our further purposes.

Unfortunately, there also are significant differences in terminology, notation, and parameters used for the ordinates of the spectral curves. It would be wise to review these differences and help avoid confusion and error when confronted with various forms.

In regard to the value of the ordinate of the curve which does have the generally accepted label "spectral density," Fig. 7, some early investigators used the full *energy* of the component waves. An obvious simplification has been made in all modern presentations; insofar as the energy of the component waves is directly related to the square of their heights, the spectral density can be referred directly to the square of the height or amplitude of the waves. We have several popular choices, to wit:

1 (Amplitude)^2—which has the support of mathematical logic since the statistical laws apply precisely to amplitudes, not to wave heights. This was the most widely used form until recently. For this spectrum the "significant height" has the value: $H_s = 2.83\sqrt{\text{area}}$

2 1/2 (Amplitude)^2—which is now most popular among researchers. The area under the curve is equivalent to the statistical "variance," but its only particular benefit in use is that the Rayleigh constant for significant height is a simpler number; thus, $H_s = 4.0\sqrt{\text{area}}$

3 (Height)^2—which some investigators favor. Since height is what they measure, height is what they use. The area under the curve is equivalent to the mean square under the Rayleigh curve. Then, $H_s = 1.414\sqrt{\text{area}}$

4 2(Height)^2—which is being favored by many practicing naval architects and oceanographers. It has the happy feature that the square root of the area under the curve gives the "significant wave height" directly, with no multiplying factor. Ergo, $H_s = \sqrt{\text{area}}$

(Remember that the ordinate, spectral density, is expressed in terms of square feet-seconds, when using an amplitude or height value plotted against frequency-1/sec. It is the area under the curve that is then expressed only in square feet.)

In regard to the abscissa, there is contention between cyclic frequency, $f = 1/T$, and circular frequency, $\omega = 2\pi/T$, both in units of 1/sec (hardly anybody uses period, T, any longer). Of the two, circular frequency has the edge, primarily since it is the term in the basic sine wave equation $y = H/2 \cos \omega t$, and all ship-motion and wave-force functions are derived on this base, as well. There is nothing difficult or obscure, in any case, in accounting for the factor, 2π, between the two terms.

Now, let us tackle terminology and notation. This would be a formidable task indeed if we attempted to show all the symbolism that has been employed.

In regard to the spectrum terminology, there has been a lack of definitiveness or discrimination between the various forms used. All are still loosely called "energy spectrum" or "wave spectrum," which is all right when indicating the general theory, but we need to be more specific when differentiating between types in use. Corresponding to the order given in the foregoing, the following terminology is suggested.

1 amplitude spectrum
2 amplitude half-spectrum
3 height spectrum
4 height double-spectrum

We may, of course, refer to the symbolic notation used to find the particular parameters used for the spectrum.

But now we are worse off. Not only is the old notation not explicit, but also we keep getting new terms for the same thing. Without belaboring the subject, let us indicate some of the notation that has been used in the more well-known formulations for the "amplitude spectrum" and the "amplitude half-spectrum," so that those being initiated in this field may learn what to expect.

Spectral density ordinate	(amplitude)^2-sec	½(amplitude)^2-sec
Notation for ordinate....	$[r(\omega)]^2$	$S(\omega)$
	$S(\omega)$	$\phi^{+}\xi\xi(\omega)$
	$[A(f)]^2$	$\eta(\omega)$
Notation for area of spectrum...	R	S
	E	$\phi_{\eta\eta}(0)$
	S	

Do not panic! In such situations, where the value assigned to the spectral density ordinate has not been made clear, look for the accompanying Rayleigh distribution factor that is to be used in association with the $\sqrt{}$area that determines the significant wave height (make sure it is *wave height*). Then, you can deduce what the notation represents in way of spectral density values. Also check whether cyclic frequency (f) or circular frequency (ω) is used. Next, immediately place the equation in the form you prefer (by juggling a few constants only) and use the notation you prefer. Thereafter, be as consistent as possible.

One other bit of terminology may be discussed. The term "response amplitude operator" or its abbreviation, RAO, will crop up. This stands for the unit response of the motion of force of a body (per foot of wave height) that we wish to investigate, as described previously. The word "amplitude" applies to the amplitude of the motion or force (i.e., the maximum value measured from zero position). Be sure that you use this unit motion or force per *foot of wave height* when applying the Rayleigh constants that have been derived for *height*, as used herein, and this is true whatever spectral density ordinates are used (since the Rayleigh constants have been so adjusted).

For consistency in the actual values of the spectrum formulas, and because we consider it the best form to use in practical ship applications, we shall give these formulas in terms of the "height double-spectrum" throughout on a base of cyclic frequency, ω, and all boiled down to their simplest form, in English units. We shall use the notations

$2h^2(\omega)$ for the ordinate of spectral density

H_s^2 for the area under the curve

We can immediately place most of the proposed formulas into one of two categories

1 Those that use wind speed in the formulation to derive the spectral density values

2 Those that use significant height and period

The wind-speed formulas represent the classic approach. The contention is that the wind, of a known steady speed, will necessarily build up the sea in a consistent manner, and one may then determine all the sea-wave properties without the a priori need to know what any of the properties are. Unfortunately, most of the effort expended in deriving these equations has been limited in scope, expressly to the North Atlantic Ocean, under fully developed sea conditions (that is, the wind has been blowing steadily for an unlimited time over an unlimited distance).

Nevertheless, they are authoritative and undoubtedly progressively more accurate. We may briefly trace the milestones.

Neumann's spectrum (*the* classic).

$$2h^2(\omega) = 400/\omega^6 e^{-725/V_k^2\omega^2} \qquad (1)$$

and the area

$$H_s^2 = 1.9V_k^5/10^5$$

Neumann's spectrum, modified. As corrected by several researchers who discovered some discrepancies in analysis in the original work and who also noted that the original formula overpredicted the wave heights that should occur. By coincidence only, the equation is practically identical, except for the power of ω

$$2h^2(\omega) = 400/\omega^5 e^{-725/V_k^2\omega^2}$$
$$H_s^2 = 3.8V_k^4/10^4 \qquad (2)$$

Pierson-Moskowitz spectrum. A new formula, based on more comprehensive and methodical analyses of old and new data, is officially accepted by many researchers as *the equation*.

$$2h^2(\omega) = 135/\omega^5 e^{-9.7\times10^4/V_k^4\omega^4}$$
$$H_s^2 = 3.5V_k^4/10^4 \qquad (3)$$

where V_k is the wind speed in knots. For Pierson-Moskowitz, it is stipulated that this be taken at a height of 64 ft above the sea surface. For the other equations, the height is presumably in the order of 20 to 33 ft.

Then, we come to a never school, those who feel that the spectrum can accurately, and more universally, be defined by the characteristic properties of the sea itself: the significant height and the significant period. Since these properties are more readily determinable for a region under consideration than is the history of the wind's constancy, duration, and distance (both sea and wind statistics are needed to derive the wind-based spectrum), why not dispense with the wind as a parameter? The following have been proposed:

Bretschneider's spectrum. Which is derived on the premise that the wave period follows a Rayleigh distribution, as does the wave height.

$$2h^2(\omega) = 4200H_s^2/T_s^4\omega^5 e^{-1050/T_s^4\omega^4} \qquad (4)$$

ISSC Spectrum. The International Ship Structures Congress' modification of the Bretschneider form:

$$2h^2(\omega) = 2760H_s^2/T_s^4\omega^5 e^{-690/T_s^4\omega^4} \qquad (5)$$

where

H_s = significant wave height (average of the one-third highest waves)

T_s = "significant period," actually the average period of the significant waves

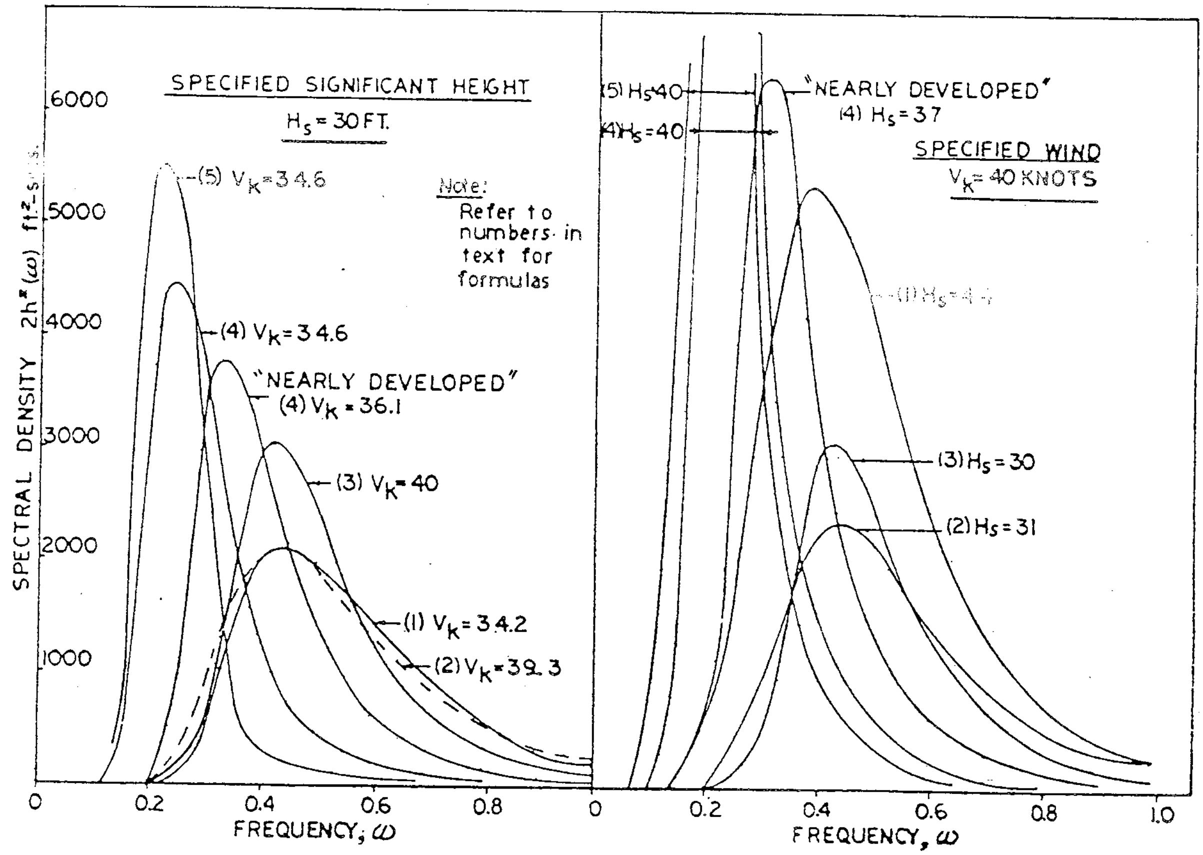

Fig. 8 Comparison of fully developed sea spectrum formulas

It should be noted that Bretschneider's spectrum was derived for a fully developed sea, as were the others. However, from its derivation, it appears to be reasonably acceptable for partially developed seas, as well. We will discuss that point later. In the meantime, let us make a comparison of the various formulas.

We first may note that Bretschneider developed auxiliary relationships between wind speed, wave height and period to apply to his formula (which also is applicable to the ISSC formula). They are given as follows:

For the fully developed sea

$$H_s = 0.025\ V_k^2 \text{ ft}$$
$$T_s = 0.64\ V_k \text{ sec}$$

For the "nearly developed" sea, the earlier relationship of Sverdrup and Munk was considered:

$$H_s = 0.023\ V_k^2 \text{ ft}$$
$$T_s = 0.45\ V_k \text{ sec}$$

which shows a noticeable difference in spectrum shape.

The comparisons are shown for the formulas *as given*, in Fig. 8, first on the basis of a given significant height of 30 ft (the corresponding wind-speed differences are noted) and also on the basis of a given wind speed of 40 knots (the resulting significant height differences are noted). The considerable variation evidenced throughout among the various formulas is cause for some concern, particularly in regard to selecting a representative spectrum for application. It is necessary, then, that we evaluate some of these differences:

1 The Pierson-Moskowitz formula (3) is based on wind speeds measured 64 ft above the water, as noted. All others apparently used the "surface" wind, generally in the six to ten-meter range. If the usual correction factors for wind gradient were taken into account, we would find the disparity between this and Bretschneider (4) much less than appears in regard to wind speed and significant height.

2 Once this is considered, we find that the Neumann (1) and the modified Neumann (2) are not in accord with the others—one is too high, the other too low. We seriously may eliminate these from modern consideration in view of the fact that the others are later, more careful, and probably more precise formulations.

3 The big difference left is that of the extreme variation in the ordinate of maximum spectral density between Pierson-Moskowitz (3) and Bretschneider (4) both in their density value and in their frequency. It is difficult to accept Bretschneider's frequency which indicates that the maximum energy of the fully developed spectrum is contained in waves over 3000 ft long, with very little in waves under 1200 ft long (which is about where Pierson-

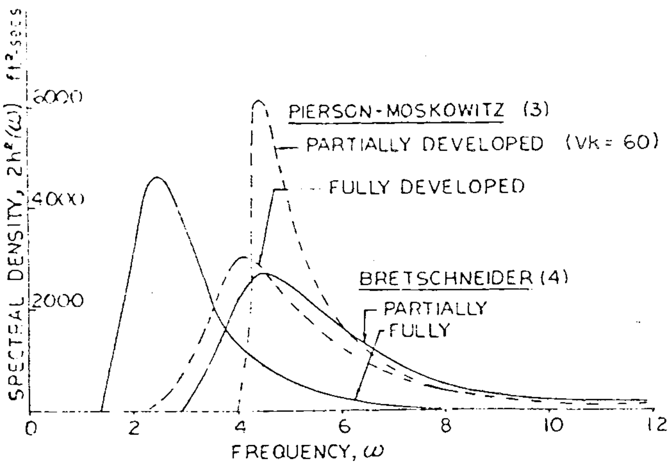

Fig. 9 Thirty-foot significant wave spectrum for partially and fully developed seas

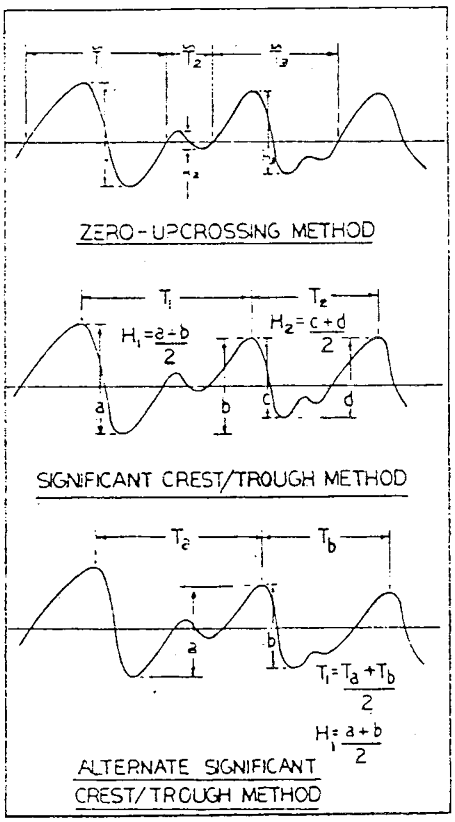

Fig. 10 Method of analyzing data from time record of waves

Moskowitz shows the maximum energy). One must consider that either the waves at sea practically never attain or even approach a fully developed state, or that Bretschneider's data (primarily derived from "model" observations) do not apply to the full-scale sea. In this regard, it is interesting to note that the use of the data of Sverdrup and Munk in Bretschneider's formula results in a spectrum shape approaching that of Pierson-Moskowitz.

In summary, we can see that the biggest problem lies in the selection of the significant period for application to the Breitschneider form of equation. Where "observed" periods are used (and this is the basis on which ISSC promotes its formula), the formula appears fully satisfactory.

Let us also consider the differences among the various spectrum formulas in regard to partially developed seas, that is, where the wind has not blown long enough or over a sufficient distance to develop the sea completely, a condition generally considered to be more prevalent in the oceans than is the fully developed case. This is particularly true in such areas as the North Sea, where fetches are limited, and in the Gulf of Mexico and eastern Pacific Ocean, where hurricanes and typhoons of high wind but short duration and/or fetch are experienced.

As previously indicated, the spectrum builds from the high-frequency end, so that for a partially developed sea, few of the low-frequency components have arisen. This is treated in the wind-type spectral approach by terminating the curve more or less abruptly, the area of the remaining intact curve representing H_s^2 of the partially developed sea. The height/period spectrum, on the other hand, retains its full formulation but with height and period values altered to suit the situation.

It is evident that two parameters are needed to define either type of spectrum. For the wind-type spectrum, both the significant wave height and the wind are required; for the height/period spectrum, it is the significant height and period.

In the latter case, however, Bretschneider has derived a relationship between height and period that holds approximately for the partially developed sea, at least in the range of hurricane activity. This is

$$H_s = 0.222 T_s^2$$

and the Bretschneider spectrum may thus be defined, once the significant height is specified.

A comparison of the two types of spectrum for a partially developed sea, generating a 30-ft-high significant wave, is made in Fig. 9, using the Pierson-Moskowitz formula (with a wind speed of 60 knots) and the Bretschneider formula. Note the similarity between the partial spectrum of Bretschneider and the full spectrum of Pierson-Moskowitz. Perhaps we do not attain "fully developed" seas, after all, or perhaps it is coincidence. As yet, we are not sure.

Part of the problem in correlating the various spectrum formulas may well have to do with the way in which various investigators measure their sea data. In general, the wind-speed formulas reflect the use of the "zero-upcrossing" method in determining heights and periods. As illustrated, Fig. 10, the time between two successive rises of the water surface above the mean level is taken as the wave period, and the distance fr crest to trough in that interval is taken as the wave height.

The height/period formulas, however, have utilized only the "significant" crests and troughs for determining the wave periods, averaging out two successive crest/-trough heights to give the height of the included wave.

The differences in evaluating the periods from the wave data are reflected in the shape of the resulting spectrum. The "zero-upcrossing" method contains more lower period (higher frequency) components, and the spectra derived on this basis are somewhat broader (with maximum spectral density also at a higher frequency) than those spectra derived from "significant" measurements.

The differences in evaluating wave heights from the wave data may not be serious, however. The "zero-upcrossing" method gives a wider range of heights, but it is likely that the overall average is closely the same for either method.

The controversy still rages over which gives the most practical information for application to ship-behavior problems.

Which Spectrum Formula to Use?

It is difficult to make a full evaluation of the various proposed spectra and determine which is right (if any can be determined to be right), for all occasions. Even among the analysts themselves all the differences in shape and area cannot be resolved. Further, unless the significant parameters of wave height, period, and wind are all defined, there are differences within the individual spectrum forms themselves. If we were to show additional spectra that also are currently used, we still would find further differences.

Perhaps the best philosophy to adopt at this time is the one expressed by the International Ship Structures Committee, which presented its formula (5) in association with an assembly of data on wave heights and periods representative of ocean areas all over the world. In effect, they stated that perhaps their wave data were not precise nor their formula exact; nevertheless, with the data and the data-use techniques developed for this spectral analysis theory, we are far closer to the accurate determination of the forces and motions of a body in the sea than we ever were using the old regular-wave methods.

So, in counseling on the relative merits of the various spectra, it can only be said "You pays your money, and you takes your choice."

In our practice we have favored the Bretschneider form, finding it somewhat more amenable to practical usage than the other types. Most specifications or bills of requirements and most feedback information from ships and from oceanographic reports are in terms of wave height, and it is most satisfying to be able to apply this directly. Further, we can use the formula directly without considering whether the sea is fully developed or not, particularly where significant periods are also given for the ocean area under consideration, or where we manipulate the period to study the maximum effect on the behavior of a vessel. As yet, however, we still will run a check using the wind-based spectrum, such as Pierson-Moskowitz (3), just to make sure that all bets are covered, but for the bulk of any such investigation we find the height/period form a better tool.

Some Final Points on Application

It should be noted that there is a practical limit to the number of waves one need consider in determining the maximum heights from the Rayleigh distribution. Otherwise, if we took the ultimate, we would end up with a fantastic height even though its chance of occurrence were very slim. One should remember that the Rayleigh distribution is only a convenient mathematic fit to the histogram of actual wave measurements, which do not show extreme wave heights (probably since breakers result from any tendency toward extreme height). Notwithstanding the freak catastrophic build-up of height that is reported on rare occasions, sound practice indicates a realistic limit should be applied. It is generally accepted that 1000 waves are sufficiently representative of the entire spectrum, and the "most probable" value of the 1/1000 highest wave represents the maximum. Some investigators use only 600 as an upper limit; some take the total number of waves that may pass a point in one hour (dividing one hour by the significant wave period). Using 1000 waves should cover all requirements.

Further, we need stress the point of examining the wave spectrum and its effect on motions and forces to see what may result from a shift in significant wave period. The wave heights are fairly consistently measured and accountable; the determination of the significant periods is, as we have seen, rather nebulous and questionable. It behooves any conscientious investigator to search all likely spectrum shapes for the maximum effects on the vessel with which he is concerned.

Summary

The foregoing is an attempt to present the basic fundamentals of the sea-spectrum theory and the principles of its application to ship behavior in a manner that would make it easy to grasp, at the same time providing a sound background for advancement into the further ramifications of the subject for those who so desire.

For many of today's practical problems, the scope of the aforementioned treatment is sufficient and far more precise than the methods previously used to analyze ship motions and forces. Yet, in the continued drive toward greater accuracy and more factual representation of prevalent conditions, we need to go down the road toward greater precision and accountability of additional factors.

For instance, some of the things we have not dwelt on, and which would make for a more complete treatment of the subject, are

1 The effect of ship speed and direction relative to the sea.

2 The confused sea (several commingling seas).

3 The long-term spectra (and predictions thereof).
4 Shallow water effects.
5 Nonlinear ship responses.
6 Corrections to the wave-height distribution for spectral shape.

[illegible]ion, considering their growing complexity and the continual research that makes consolidation of knowledge a difficult, [illegible] possible, task.

This presentation may be likened, then, to the situation of mastering a new dance. You have to go easy at first, step-by-step, until the rudiments are under control. This in itself might be sufficient for those of us who only take one or two turns around the floor, but it is also invaluable for those who go on to perform the greater intricacies.

Bibliography

SNAME Publications

M. St. Denis and W. J. Pierson, "On the Motions of Ships in Confused Seas," *Trans. SNAME*, vol. 61, 1953, pp. 280–357.

The original work bringing to our attention the wave spectrum theory and its application to ship behavior. Its fundamentals are still sound, including extensions beyond this present treatise into directional seas and effects of forward ship speed.

N. H. Jasper, "Statistical Distribution Patterns of Ocean Waves, and of Wave-Induced Ship Stresses and Motions, with Engineering Applications," *Trans. SNAME*, vol. 64, 1956, pp. 375–432.

This is a study of actual ship responses over a long period of time, indicating that the Rayleigh distribution holds for a discreet interval but that the intensities in time vary as the log normal distribution.

B. V. Korvin-Kroukovsky, *Theory of Seakeeping*, The Society of Naval Architects and Marine Engineers, 1961, 352 pages.

This traces the development of the theory of sea waves and the origins of sea spectral theory. It also gives basic data on ship responses and forces applicable to the theory. An excellent reference work.

Wilbur Marks, "The Application of Spectral Analysis and Statistics to Seakeeping," Technical and Research Bulletin No. 1-24, 1963.

The fundamentals of the spectrum theory and its application to ships are discussed. It moves off into the methods by which wave records are analyzed to determine spectral forms. It offers a relatively easy introduction into statistical language as referred to the sea spectrum.

M. K. Ochi, "Extreme Behavior of a Ship in Rough Seas—Slamming and Shipping of Green Water," *Trans. SNAME*, vol. 72, 1964, pp. 143–202.

The sea spectral techniques are applied to extreme motions and slamming of a ship, with excellent correlation shown. A very clear and satisfying presentation.

N. Hamlin and R. Compton, "Assessment of Seakeepability," *Marine Technology*, vol. 3, No. 4, 1966, pp. 454–468.

Comparative motions of ships having different dimensions are studied, by sea spectrum analyses.

M. A. Abkowitz, L. A. Vassilopoulos, and F. H. Sellars, "Recent Developments in Seakeeping Research and its Application to Design," *Trans. SNAME*, vol. 74, 1966, pp. 194–259.

Several of the latest spectrum formulas are presented, and the trends of present research methods in applying spectral techniques to seakeeping problems are elaborated.

Other Selected Publications

W. J. Pierson, G. Neumann, and R. W. James, "Observing and Forecasting Ocean Waves," Hydrographic Office Publication No. 603, 1960.

An excellent work giving an explanation of the wave spectrum in simple and practical terms. It does not go deeply into consequent ship behavior, but concentrates on the method of predicting sea states utilizing Neumann's original spectrum formula. Highly recommended for clarity and basics.

C. L. Bretschneider, "Wave Variability and Wave Spectra for Wind-Generated Gravity Waves," Beach Erosion Board, T. M. No. 118, 1959.

A comprehensive work covering the development of spectral relationships and the Bretschneider form. With much supporting data and comparisons of various spectrum formulas, it has valuable content. Even though filled with equations and mathematical analysis it is well-ordered and easy to follow.

"Environmental Conditions," Committee No. 1, *Proceedings of the Second International Ship Structures Congress*, Delft, Netherlands, 1964.

The problems of practical ship designers trying to find an acceptable approach *now* is clearly stated. Most valuable are the data given for all ocean areas throughout the world, suitable for application.

Appendix 1

Rayleigh Distribution Factors

Multiply the square root of the area under the spectrum by the factors given in Table 1.

Spectra of Other Densities

The factors given in Table 1 must be multiplied by the following additional factor:

Amplitude spectrum	2.83
Amplitude half-spectrum	4.0
Height spectrum	1.41

Appendix 2

Design Example

A floating drill rig of multicaisson configuration, Fig. 11, is at 65 ft draft, firmly anchored in 600 ft of water. The maximum anticipated sea conditions are given

Table 1

This applies to all spectra - wave height, ship motions, ship forces, etc., - that are derived on the basis of the "height double spectrum"; that is, whose wave spectrum area is equal to H_s^2.

TEN PERCENT RANGES

(10 percent of the height, or motion, or forces, etc., will be between the following values)

10% between	0.0 and 0.23
" "	0.23 and 0.33
" "	0.33 and 0.42
" "	0.42 and 0.50
" "	0.50 and 0.59
" "	0.59 and 0.68
" "	0.68 and 0.78
" "	0.78 and 0.90
" "	0.90 and 1.08
10 percent over	1.08

AVERAGE VALUES

(For Height, Forces, Motions, Etc.)

The most frequent value	0.50
The average value	0.625
The significant value, (Average of 1/3 highest)	1.00
Average of 1/10 highest values	1.27

PROBABLE MAXIMUM VALUE

N (Number of waves)	Factor
20	1.22
50	1.40
100	1.52
200	1.63
500	1.76
1000	1.86

Note: Value = $\sqrt{\frac{1}{2} \log_e N}$

$H_s = 30$ ft, $T_s = 11.5$ sec. Assuming the rig movements are negligibly small due to the firm anchoring system, find:

(*a*) The maximum force on one caisson/hull section (one side).

(*b*) The maximum force on one anchor chain (combined force of both sides).

First, consider that the water is sufficiently deep for the deep-water wave relationships to be applicable. Second, consider the inertial forces only (inertia coefficient = 2.0), neglecting the relatively much smaller drag forces.

The horizontal acceleration of a level, z, below the water surface is, for a regular wave of frequency, ω, and height, h

$$\alpha = \frac{h}{2} \omega^2 e^{-\omega^2 z/g} \cos(\omega t - \omega^2 x/g)$$

The maximum inertial force (in long tons) for one caisson/hull section is then:

1 For the hull at $z = 50$ ft

$$F_h = \frac{(80 \times 30^2\pi/4)2.0}{35g} \frac{h\omega^2}{2} e^{-1.55\omega^2}$$

$$= 50\, h\omega^2 e^{-1.55\omega^2} \text{ (tons)}$$

2 For the caisson between $z = 0$ to 50 ft

$$F_c = \frac{(30^2\pi/4)2.0}{35g} \frac{h\omega^2}{2} \int_0^{50} e^{-\omega^2 z/g}\, dz$$

$$= 20h(1 - e^{-1.55\omega^2}) \text{ (tons)}$$

(*a*) The force on one caisson/hull section, taking $x = 0$ for convenience, is

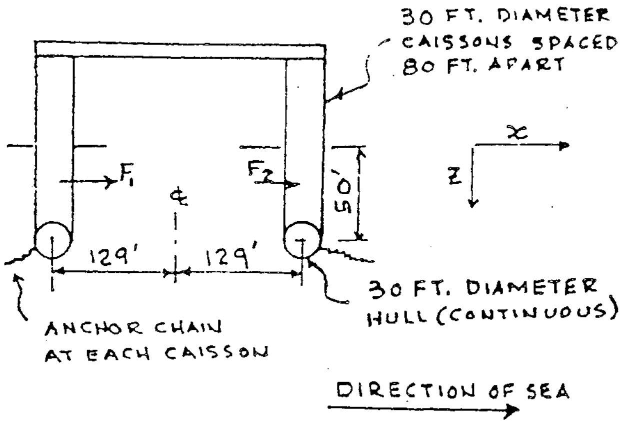

Fig. 11

$$F_1 = (F_h + F_c) \cos \omega t$$

and the response amplitude operator is then

$$(F_{1/h}) = 50\omega^2 e^{-1.55\omega^2} + 20(1 - e^{-1.55\omega^2})$$

(*b*) The force on the anchor chain (with one side slack), taking $x = \pm 129$ ft, as measured from CL

$$F_1 + F_2 = (F_h + F_c)[\cos(\omega t - 129\omega^2/g) + \cos(\omega t + 129\omega^2/g)]$$

$$= [(F_h + F_c)2 \cos 4\omega^2] \cos \omega t$$

and the RAO is

(1) Frequency, ω	.2	.3	.4	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4
(2) F_1/h	3.08	6.52	10.64	14.92	18.86	22.29	24.43	25.84	26.36	26.19	25.40	24.84	23.95
(3) $(F_1+F_2)/h$	6.06	12.20	17.04	16.12	5.00	18.30	41.00	51.20	33.80	6.40	44.00	44.40	0
(4) $2h^2(\omega)$	0	54	2025	2650	1740	1000	571	335	204	129	84	57	39
(5) $[(2)^2 \times (4)]$	0	2300	230,000	590,000	619,000	497,000	342,000	224,000	142,000	88,400	54,100	35,200	22,400
(6) $[(3)^2 \times (4)]$	0	8000	588,000	688,000	43,600	334,400	960,000	878,000	213,200	5,300	162,400	112,400	0

"Significant" Force F_1: Area (5) = 284,800 Tons^2 $F_{1(S)}$ = 534 T

"Significant" Force (F_1+F_2): Area(6) = 385,500 Ton^2 $(F_1+F_2)_S$ = 622 T

Note: For the "Regular" wave of H = 30 ft., T = 11.5 secs. we would get F_1 = 504 tons, (F_1+F_2) = 358 tons

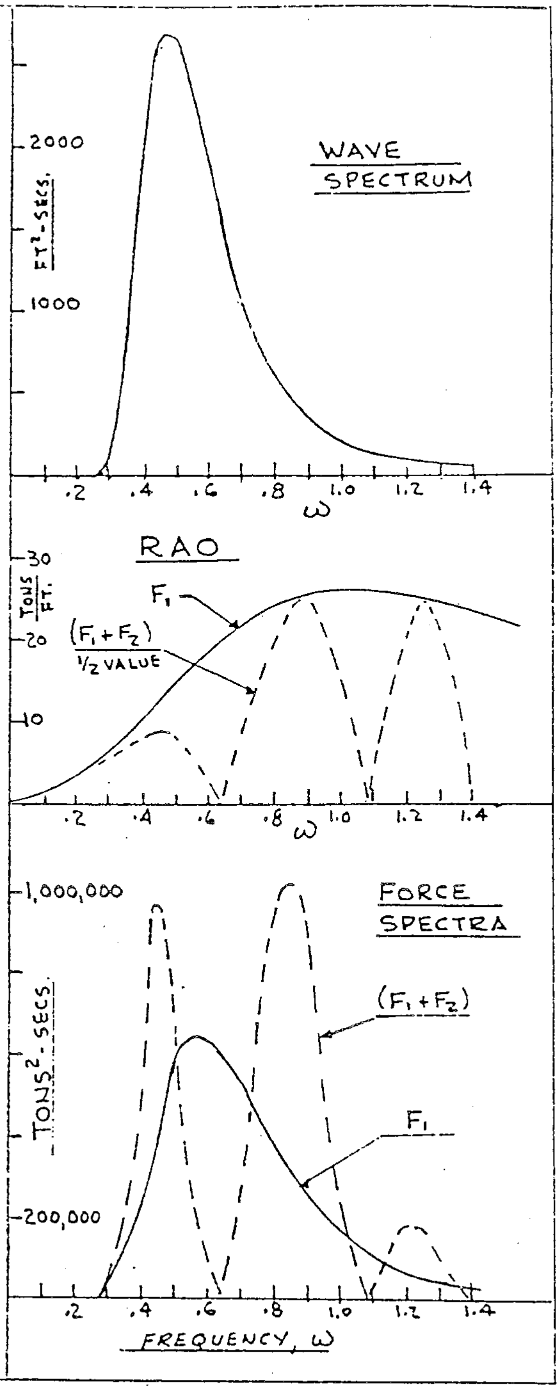

Fig. 12

$$\left(\frac{F_1 + F_2}{h}\right) = 2[50\omega^2 e^{-1.55\omega^2} + 20(1 - e^{-1.55\omega^2})] \cos 4\omega^2$$

Wave Spectrum: Use Bretschneider (4)

$$2h^2(\omega) = \frac{4200 \times 30^2}{11.5^4}, \frac{1}{\omega^5} e^{-1050/11.5^4\omega^4}$$

$$= \frac{216}{\omega^5} e^{-0.06/\omega^4}$$

The RAO values, wave spectrum, and resulting force spectra are calculated through a range of frequency, with results as tabulated in Fig. 12.

Discussers

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