Nonlinear Programming in MATLAB

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Feb. 28, 2013

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Optimization Tool Box in MATLAB

- Minimization (solving minimization problems)
  - linprog: linear programming problems
  - quadprog: quadratic programming problems
  - bintprog: binary integer programming problems
  - fminbnd: minimum of single-variable function
  - fseminf: minimum of semi-infinite constrained multivariable function
  - fmincon: minimum of constrained multivariable function
  - ...
- Equation solving
- Curve fitting

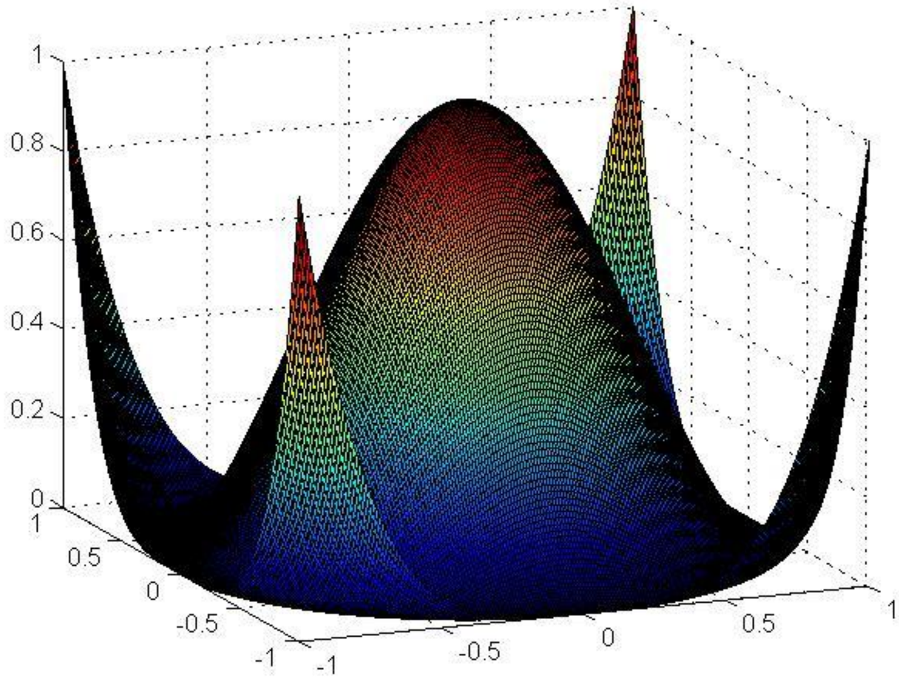
Optimization Tool Box in MATLAB

- GUI for optimization tool box
  - Type command "optimtool" in command window.
- Problem setup
  - Select solver and algorithm
  - Specify objective function
  - Specify constraints
  - Specify options
  - Run solver and check the output

GUI for Optimization Tool Demo

$$\min \quad f(x) = \left(x_1^2 + x_2^2 - 1\right)^2$$

$$\text{s.t.} \quad -1 \le x_1 \le 1, -1 \le x_2 \le 1$$



GUI for Optimization Tool Demo

- Select solver and algorithm
  - "fmincon"
  - "Active set"
- Specify objective function
  - "@(x) (x(1)^2+x(2)^2-1)^2"
  - Starting point "[1; 1]"
- Specify constraints
  - Aeq=[]; beq=[];
  - Lower=[-1; -1]; Upper=[1; 1];
- Run

GUI for Optimization Tool Demo

- Output
  - Objective function value: 1.334051452011463E-9

x(1)	-0.707093867648034
x(2)	-0.707093867648034

  - fmincon stopped because the predicted change in the objective function is less than the default value of the function tolerance and constraints are satisfied to within the default value of the constraint tolerance.

GUI for Optimization Tool Demo

- Change options
  - Set function tolerance to "1e-10"
  - Rerun the problem
  - Objective function value: 1.2266282290633e-14

    x(1) -0.707106742029359

    x(2) -0.707106742029359
  - fmincon stopped because the size of the current search direction is less than twice the default value of the step size tolerance

Input arguments for fmincon

- fmincon solves the problems having the following form

$$
\begin{array}{lll}
\min & f(x) & \\
 & A \cdot x \le b & \text{linear inqualities} \\
 & Aeq \cdot x = beq & \text{linear equalities} \\
\text{s.t.} & lb \le x \le ub & \text{lower and upper bounds} \\
 & c(x) \le 0 & \text{nonlinear inequalities} \\
 & ceq(x) = 0 & \text{nonlinear equalities}
\end{array}
$$

Input arguments for fmincon

- The syntax for fmincon

```
[x,fval,exitflag]=fmincon(objfun,x0,A,b,Aeq,beq,lb,ub,
                                               nonlcon,options);
```

  - x: optimal solution; fval: optimal value; exitflag: exit condition
  - objfun: objective function (usually written in a separate M file)
  - x0: starting point (can be infeasible)
  - A: matrix for linear inequalities; b: RHS vector for linear inequalities
  - Aeq: matrix for linear equalities; beq: RHS vector for linear equalities
  - lb: lower bounds; ub: upper bounds
  - Nonlcon: [c,ceq]=constraintfunction(x)

Construct Nonlinear Objective and Constraint Functions for fmincon

$$\min \quad f(x) = \left(x_1^2 + x_2^2 - 1\right)^2$$

$$-1 \le x_1 \le 1, -1 \le x_2 \le 1,$$

$$\text{s.t.} \quad x_1 + x_2 \ge 1$$

$$x_1 x_2 \ge 1/2, \; x_2 \ge x_1^2, \; x_1 \ge x_2^2$$

$$A = [-1, -1]; \; b = -1;$$

$$lb = [-1; -1]; \; ub = [1; 1];$$

$$c(x) = \begin{bmatrix} \frac{1}{2} - x_1 x_2 \\ x_1^2 - x_2 \\ x_2^2 - x_1 \end{bmatrix}; \; ceq(x) = [];$$

```
% myobj.m
function  f=myobj(x)
f = (x(1)^2+x(2)^2-1)^2;

% mycon.m
function [c, ceq]=mycon(x)
c=[1/2-x(1)*x(2);
   x(1)^2-x(2);
   x(2)^2-x(1)];  % nonlinear inequalities c(x) <= 0;
ceq=[ ];   % nonlinear equalities ceq(x) = 0;

% main file for fmincon
[x,fval] = fmincon(@myobj,xo,A,b,[],[],lb,ub,
                        @mycon,options);
```

Construct objective functions with parameters

$$\min \quad f(x) = \left(x_1^2 + x_2^2 - a\right)^2$$

$$-1 \le x_1 \le 1, -1 \le x_2 \le 1,$$

$$\text{s.t.} \quad x_1 + x_2 \ge 1$$

$$x_1 x_2 \ge 1/2, \; x_2 \ge x_1^2, \; x_1 \ge x_2^2$$

```
% myobj.m
function  f=myobj(x, a)
f = (x(1)^2+x(2)^2-a)^2;
% main file for fmincon
a = 1;
[x,fval] = fmincon(@(x) myobj(x,a),xo,A,b,[],[],lb,ub,
                                                @mycon,options);
```

Provide gradient information

- Provide gradient information could accelerate the solver and improve the accuracy.

$$f(x)=\left(x_1^2+x_2^2-a\right)^2 \Rightarrow \nabla f(x)=\left[4x_1\left(x_1^2+x_2^2-a\right) \quad 4x_2\left(x_1^2+x_2^2-a\right)\right]$$

$$c(x)=\begin{bmatrix} c_1(x) \\ c_2(x) \\ c_3(x) \end{bmatrix}=\begin{bmatrix} \frac{1}{2}-x_1x_2 \\ x_1^2-x_2 \\ x_2^2-x_1 \end{bmatrix} \Rightarrow \nabla c(x)=\begin{bmatrix} \frac{\partial c_1(x)}{\partial x_1} & \frac{\partial c_2(x)}{\partial x_1} & \frac{\partial c_3(x)}{\partial x_1} \\ \frac{\partial c_1(x)}{\partial x_2} & \frac{\partial c_2(x)}{\partial x_2} & \frac{\partial c_3(x)}{\partial x_2} \end{bmatrix}=\begin{bmatrix} -x_2 & 2x_1 & -1 \\ -x_1 & -1 & 2x_2 \end{bmatrix}$$

Provide gradient information

```
% objective function with gradient information
function [f,G]=myobj(x,a)
f = (x(1)^2+x(2)^2-a)^2;
% Gradient of objective function
if nargout > 1
    G = [4*x(1)*(x(1)^2+x(2)^2-a),
          4*x(2)*(x(1)^2+x(2)^2-a)];
end
```

```
% constraint function with gradient information
function [c, ceq, DC, DCeq]=mycon(x)
c=[1/2-x(1)*x(2);
   x(1)^2-x(2);
   x(2)^2-x(1)];  % nonlinear inequalities c(x)
ceq=[];   % nonlinear equalities ceq(x) = 0;
% gradient of contraint function
if nargout > 2
    DC=[-x(2),  -x(1);
        2*x(1), -1;
        -1,     2*x(2)]';
    DCeq=[];
end
```

Problems in fmincon

- Results could be wrong
  - Sometimes, fmincon find a local maximum instead of local minimum!
- Different algorithms or starting points could return different results.
- It's unstable for non-differentiable objective or constraint functions.
- For NLP, fmincon does not guarantee to return the global minimum.

Comments

- Better formulation for your problem
  - Continuous and differentiable
  - Convex
- Different starting points
- Different solvers and algorithms
  - Select the solver appropriate for your problems
  - Provide gradient or Hessian information if possible

CVX – Convex Optimization Package for MATLAB

- Convex Optimization

$$\begin{aligned} &\min \ f(x) \\ &\text{s.t. } \ g_i(x) \leq 0 \ \text{ for } i = 1, \dots, m. \end{aligned}$$

where $f(x)$ and $g_i(x)$ are convex functions.

- Global optimal solution is guaranteed by theory
- Stable algorithm for well-posed problems

CVX – Modeling Systems

- Less strict modeling syntax
  - What you saw is what you get
- Transform the problem into standard form (LP, SDP or SOCP) automatically
- Return the solver's status (optimal, infeasible etc.)
- Transforms the solution back to original form

CVX – A simple example

- Constrained least square problem

$$\begin{aligned} \min \quad & \| Ax - b \| \\ \text{s.t.} \quad & x^T x \le 1 \end{aligned}$$

- where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

CVX – A simple example

```
% CVX least square ||Ax-b|| demo
A = [1 2 3; 2 4 6; 7 8 9]; % matrix A;
b = [1; 1; 1];  % right-hand side vector b;

cvx_begin  % start of CVX
    variable x(3);  % declare variables
    minimize( norm(A*x-b) );   % declare
objective function; note to use parentheses
    subject to         % can be omitted;
starting of constraints
    x'*x <= 1; % or norm(x)^2 <= 1
cvx_end % end of CVX
```

CVX – Correlation Matrix Verification

Given three random variables $A,\ B$ and C with the correlation coefficients ρ_{AB}, ρ_{AC} and ρ_{BC}, respectively. Suppose we know from some prior knowledge (e.g., empirical results of experiments) that $-0.2 \leq \rho_{AB} \leq -0.1$ and $0.4 \leq \rho_{BC} \leq 0.5$. What are the smallest and largest values that ρ_{AC} can take?

Hint

The correlation coefficients are valid if and only if

$$\begin{bmatrix} 1 & \rho_{AB} & \rho_{AC} \\ \rho_{AB} & 1 & \rho_{BC} \\ \rho_{AC} & \rho_{BC} & 1 \end{bmatrix} \succeq 0$$

CVX – Correlation Matrix Verification

SDP formulation

The above problem can be formulated as following problem:

$$
\begin{aligned}
Min/Max \quad & \rho_{AC} \\
s.t. \quad & -0.2 \leq \rho_{AB} \leq -0.1 \\
& 0.4 \leq \rho_{BC} \leq 0.5 \\
& \rho_{AA} = \rho_{BB} = \rho_{CC} = 1 \\
& \begin{bmatrix} \rho_{AA} & \rho_{AB} & \rho_{AC} \\ \rho_{AB} & \rho_{BB} & \rho_{BC} \\ \rho_{AC} & \rho_{BC} & \rho_{CC} \end{bmatrix} \in \mathcal{S}_+^3
\end{aligned}
$$

CVX – Correlation Matrix Verification

```
% CVX correlation matrix verification
cvx_begin
cvx_precision best;  % set precision to be BEST
cvx_solver sedumi;  % select solver as SeDuMi instead of SDPT3
    variable rho(3,3) symmetric;   % declare variable matrix rho
    minimize rho(3,1)   % specifying objective function
    subject to  % start of constraints
        rho(1,2) <= -0.1;
        rho(1,2) >= -0.2;
        rho(2,3) <= 0.5;
        rho(2,3) >= 0.4;
        rho(1,1) == 1;   % note equal "==" not "="
        rho(2,2) == 1;
        rho(3,3) == 1;
        rho == semidefinite(3); % matrix rho is positive
semidefinite
cvx_end
```

CVX – Torricelli Point Problem

The problem was proposed by Pierre de Fermat in 17th century. Given three points $a,\ b$ and c on the $\mathbb{R}^2$ plane, find the point in the plane that minimizes the total distance to the three given points. The solution method was found by Torricelli, hence know as Torricelli point.

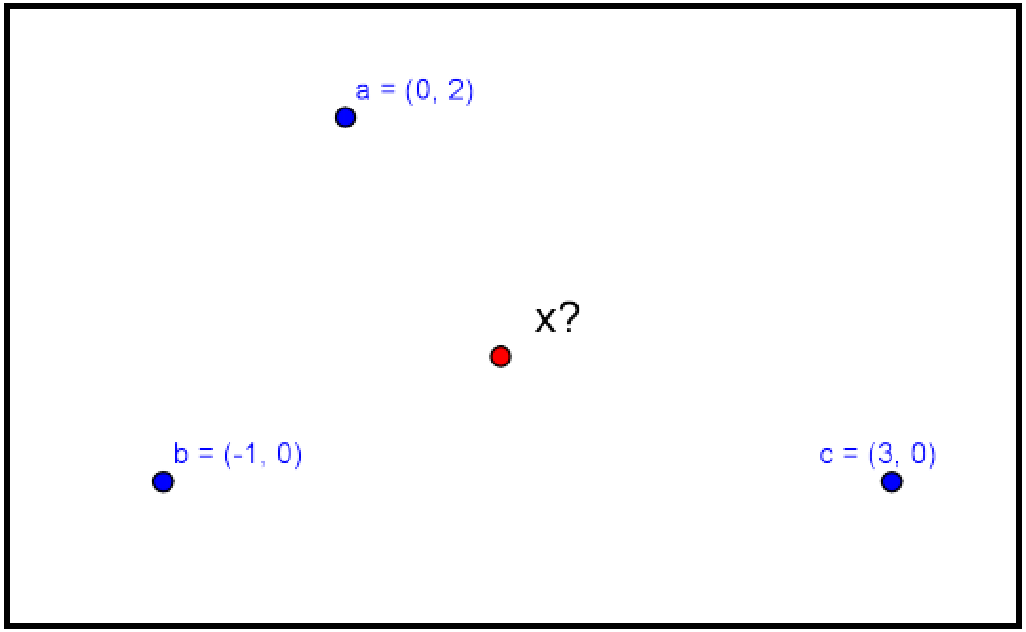


Figure: Torricelli Point Problem

CVX – Torricelli Point Problem

Hint

$$t_1 \geq \|x-a\|_2 \Leftrightarrow \begin{bmatrix} x-a \\ t_1 \end{bmatrix} \in \mathcal{L}^3,$$

$$t_2 \geq \|x-b\|_2 \Leftrightarrow \begin{bmatrix} x-b \\ t_2 \end{bmatrix} \in \mathcal{L}^3,$$

$$t_3 \geq \|x-c\|_2 \Leftrightarrow \begin{bmatrix} x-c \\ t_3 \end{bmatrix} \in \mathcal{L}^3.$$

SOCP Formulation

$$\begin{array}{ll} Min & t_1 + t_2 + t_3 \\ s.t. & \begin{bmatrix} x-a \\ t_1 \end{bmatrix} \in \mathcal{L}^3, \begin{bmatrix} x-b \\ t_2 \end{bmatrix} \in \mathcal{L}^3, \begin{bmatrix} x-c \\ t_3 \end{bmatrix} \in \mathcal{L}^3 \end{array}$$

CVX – Torricelli Point Problem

```
% CVX Torricelli Point Problem
a=[0;2]; b=[-1;0]; c=[3;0];  % location of three points

cvx_begin
cvx_precision best;
cvx_solver sedumi;
    variables t(3) x(2);  % declare multiple variables
    minimize ( sum(t) );
    subject to
        {x-a, t(1)} <In> lorentz(2); % SOC constraint
        {x-b, t(2)} <In> lorentz(2); % note the dimension
        {x-c, t(3)} <In> lorentz(2);
cvx_end
%% One more straight forward formulation
cvx_begin
cvx_precision best;
cvx_solver sdpt3;
    variable x(2)
    minimize ( sum(norms( x*ones(1,3) - [a,b,c] )) );
cvx_end
```

END

THANK YOU