Basic problems:

1. [Dasgupta et al., Ex. 7.10] For the following network, with edge capacities as shown, find the maximum flow from $S$ to $T$, along with a matching cut:

2. [Dasgupta et al., Ex. 5.1/5.2] Consider the following graph.

   (a) Run Kruskal’s algorithm on this graph. In what order are the edges added to the MST? For each edge in this sequence, give a cut that justifies its addition.

   (b) Run Prim’s algorithm on the same graph. Whenever there is a choice of vertices, always use alphabetic ordering (e.g. start from vertex $A$). Draw a table showing the intermediate values of the cost array.

   (c) How many minimum spanning trees does the graph have altogether?

3. [Dasgupta et al., Ex. 5.3] Consider the following task.

   Input: A connected, undirected graph $G$.

   Question: Is there an edge you can remove from $G$ while still leaving $G$ connected?

   Can you decide on the existence of such an edge in time $O(|V|)$? How about finding one?

4. [Dasgupta et al., Ex. 5.20] Give a linear-time algorithm that takes as input a tree and determines whether it has a perfect matching: a set of edges that touches each vertex exactly once.
Advanced problems:

5. [Dasgupta et al., Ex. 5.21] A feedback edge set of an undirected graph $G = (V, E)$ is a subset of edges $E' \subseteq E$ that intersects every cycle of the graph. Thus, removing the edges $E'$ will render the graph acyclic.

Give an efficient algorithm for the following problem:

*Input:* Undirected graph $G = (V, E)$ with positive edge weights $w_e$.

*Output:* A feedback edge set $E' \subseteq E$ of minimum total weight $\sum_{e \in E'} w_e$.

6. [Dasgupta et al., Ex. 7.24] Direct bipartite matching. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph (so that each edge has one endpoint in $V_1$ and one in $V_2$), and let $M \subseteq E$ be a matching in the graph (that is, a set of edges that don’t touch). A vertex is said to be covered by $M$ if it is the endpoint of one of the edges in $M$. An alternating path is a path of odd length that starts and ends with a non-covered vertex, and whose edges alternate between $M$ and $E \setminus M$.

(a) In the bipartite graph below, a matching $M$ is shown in bold. Find an alternating path.

![Bipartite Graph](image)

(b) Prove that a matching $M$ is maximum if and only if there does not exist an alternating path with respect to it.

(c) Design an algorithm that finds an alternating path in $O(|V| + |E|)$ time using a variant of breadth-first search.

(d) Give a direct $O(|V| \cdot |E|)$ algorithm for finding a maximum matching in a bipartite graph. *(Hint: Note that $\min\{a, b\} \cdot (a + b) \leq 2ab$.)*