Aalto University
School of Science

PHYS-E0420 Many-body Quantum mechanics
Exercise 1
by 22nd of January 2021 12:15

You get a bonus for your midterm score based on how many problems you have solved. As an extra encouragement, some problems give double points in this scoring, see for instance problem 3 in this set. Grading by self evaluation on an integer scale 0 ... 2. Occasionally we will also collect the papers to check that your self-evaluation scales make sense.

1. a) Show that the time-dependent Schrödinger equation may be written as

$$|\psi(t)\rangle = |\psi(0)\rangle + \frac{1}{i\hbar}\int_0^t d\,t'\,H(t')\,|\psi(t')\rangle$$

b) Show that the preceding equation leads to the following formal series (It is the Neumann-Liouville expansion.)

$$|\psi(t)\rangle = \left(1 + \frac{1}{i\hbar}\int_0^t d\,t'\,H(t') + \frac{1}{(i\hbar)^2}\int_0^t d\,t'\int_0^{t'} d\,t''\,H(t')H(t'') + ...\right)|\psi(0)\rangle$$

c) Show that the nth order term in this series may be written as

$$\left|\psi^{(n)}(t)\right\rangle = \frac{1}{n!(i\hbar)^n}\int_0^t \cdots \int_0^t d\,t_1 \cdots d\,t_n\,T(H(t_1)\cdots H(t_n))\,|\psi(0)\rangle\,,$$

where T is the time ordering operator defined so that for the operators A and B

$$\begin{aligned} T(A(t)B(t')) &= A(t)B(t') \quad \text{if } t > t' \\ T(A(t)B(t')) &= B(t')A(t) \quad \text{if } t' > t \end{aligned}$$

d) Show that this leads to the form $U(t) = T\exp(-\frac{i}{\hbar}\int_0^t d\,\tau\,H(\tau))$ given in equation (2.7) in the lecture notes.

e) In some cases, a simpler formula $U(t) = \exp(-\frac{i}{\hbar}\int_0^t d\,\tau\,H(\tau))$ without T is valid. When? (Hint: When is time ordering irrelevant?)

2. Derive the Heisenberg equation of motion *i.e.* equation (2.19) in the lecture notes. (Hint: To obtain $\frac{d}{dt}U^{-1}$ take the time derivative of $UU^{-1} = I$ in which I is the identity operator.)

3. (*Double points.*) Consider a perturbed (1D) harmonic oscillator $H = H_0 + V(t)$, where $H_0 = \hbar\omega_0\left(\hat{n} + \frac{1}{2}\right)$ and $V(t) = V_0(\hat{a} + \hat{a}^\dagger)f(t)$. Let us assume that the initial state is the nth eigenstate of H_0, $|\psi(0)\rangle = |n\rangle$.

a) Calculate the state $|\psi_I(t)\rangle$ using first order time dependent perturbation theory.

b) Show that $|\langle m|\psi(t)\rangle|^2 = |\langle m|\psi_I(t)\rangle|^2$ *i.e.* the transition probability is the same in Schrödinger and interaction pictures. (You don't have to use perturbation theory here. $|m\rangle$ is a basis state of H_0)

(If you do not remember the basics of the 1D harmonic oscillator take a look at the previous courses or course books.)

4. a) Show that

$$\frac{d}{dt} T \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} d\tau\, H(\tau)\right) = -\frac{i}{\hbar} H(t) T \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} d\tau\, H(\tau)\right).$$

b) Write down Eq. (2.26) in the lecture notes starting from $U_I(t) = T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} d\tau\, V_I(\tau)\right)$ in which $V_I = \lambda H'$.

5. a) Plot the function $\sin(A(\omega)t)/A(\omega)$ and search from the internet/literature (or figure out otherwise) what are the limits

$$\lim_{t\to\infty} \frac{\sin A(\omega)t}{A(\omega)},$$

when $A(\omega) \neq 0$ and when $A(\omega) = 0$. How does this relate to energy (non-) conservation in Eq. (3.11) in the lecture notes. (Note: in Eq. (3.11) $A(\omega) = (E_k - E_l - \hbar\omega)/2\hbar$.)

b) Briefly read about the connection between uncertainty relation and Fourier transforms. How does the uncertainty relation appear from quantum mechanical wavefunctions and explain qualitatively how uncertainty relation between energy and time would appear from such considerations from Eq. (3.11). (Note: connection between time-frequency uncertainty exists also for "classical" signals in optics for example.)

Self-evalution: Mark how well you think you did. Use an integer scale 0 ... 2 for each problem

Your name	Std. number	Prob. 1	Prob. 2	Prob. 3	Prob. 4	Prob. 5

Questions? Contact the assistant jani-petri.martikainen@aalto.fi.