Aalto University
School of Science

PHYS-E0420 Many-body Quantum mechanics
Exercise 1: some models

Exercise 1

a)

The time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \tag{1}$$

can be integrated on both sides with respect to time from 0 to t to obtain another form of the equation. The left side becomes

$$i\hbar\int_0^t dt'\frac{\partial}{\partial t'}|\psi(t')\rangle = i\hbar(|\psi(t)\rangle - |\psi(0)\rangle)\ . \tag{2}$$

For the right side, we simply obtain

$$\int_0^t dt'\hat{H}(t')|\psi(t')\rangle \tag{3}$$

Combine the results and reorder and you get

$$|\psi(t)\rangle = |\psi(0)\rangle + \frac{1}{i\hbar}\int_0^t dt'\hat{H}(t')|\psi(t')\rangle\ . \tag{4}$$

b)

We may iterate the alternative form of Schrödinger equation (4) as $|\psi(t)\rangle$ appears both on left side and inside the integral on the right side. After the first step, we obtain

$$|\psi(t)\rangle = |\psi(0)\rangle + \frac{1}{i\hbar}\int_0^t dt'\hat{H}(t')|\psi(0)\rangle + \frac{1}{(i\hbar)^2}\int_0^t dt'\int_0^{t'} dt''\hat{H}(t')\hat{H}(t'')|\psi(t'')\rangle\ . \tag{5}$$

Continuing in this manner, the expressions develops into series

$$|\psi(t)\rangle = \left(\hat{1} + \frac{1}{i\hbar}\int_0^t dt'\hat{H}(t') + \frac{1}{(i\hbar)^2}\int_0^t dt'\int_0^{t'} dt''\hat{H}(t')\hat{H}(t'') + ...\right)|\psi(0)\rangle\ , \tag{6}$$

where the nth term is

$$|\psi^{(n)}(t)\rangle = \frac{1}{(i\hbar)^2}\int_0^t dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_{n-1}} dt_n\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n)|\psi(0)\rangle\ , \tag{7}$$

where by superscript n we denote it being the nth term. This assertion can be proved using induction. We already proved the first step, i.e. the case $n = 1$. The inductive step goes similarly using the iteration of inputting the formula (4) into $|\psi(t_n)\rangle$. The result is the previous last term with $\hat{H}$ and the new term with $|\psi(t_n)\rangle$ If the iteration is still continued, the desired term remains.

c)

To continue, we note that the Hamiltonian operators in (7) are ordered in time such that the rightmost operator comes first in time and the leftmost last (also for any subsequence of the total sequence). Let us introduce the time ordering operator $\hat{T}$ that orders the operators in lowering time order i.e.

$$T(\hat{A}(t)\hat{B}(t')) = \begin{cases} \hat{A}(t)\hat{B}(t') & \text{if } t > t' \\ \hat{B}(t')\hat{A}(t) & \text{if } t < t' \end{cases} \tag{8}$$

where $\hat{A}$, $\hat{B}$ are arbitrary operators, and accordingly for more operators. Furthermore, we have

$$\begin{aligned} &\int_0^t dt' \int_0^t dt'' T(\hat{H}(t')\hat{H}(t''))\,|\psi(0)\rangle \\ =&\int_0^t dt' \int_0^{t'} dt'' \hat{H}(t')\hat{H}(t'')\,|\psi(0)\rangle + \int_0^t dt' \int_{t'}^t dt'' \hat{H}(t'')\hat{H}(t')\,|\psi(0)\rangle \end{aligned} \tag{9}$$

We have to change the order of integration in the second term. This can be done by noting that the region

$$\begin{aligned} 0 \leq t' \leq t \\ t' \leq t'' \leq t \end{aligned} \tag{10}$$

can also be equivalently written as

$$\begin{aligned} 0 \leq t'' \leq t \\ 0 \leq t' \leq t'' \end{aligned} \tag{11}$$

as can be seen that the conditions follow from each other. Therefore, we see by swapping t' and t'' in the second term and using the equivalent description for the region that the terms are equal. Using the result, we get that

$$\int_0^t dt' \int_0^t dt'' T(\hat{H}(t')\hat{H}(t'')) = 2\int_0^t dt' \int_0^{t'} dt'' \hat{H}(t')\hat{H}(t'')\,. \tag{12}$$

Inspired by this, let us assert that

$$|\psi^{(n)}(t)\rangle = \frac{1}{n!(i\hbar)^n}\int_0^t dt_1 \int_0^t dt_2 \cdots \int_0^t dt_n T(\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n))\,|\psi(0)\rangle \tag{13}$$

We can prove this assertion by induction. The first step is already taken since the case $n = 1$ is trivial. Let us prove that the assertion holds for $n = k + 1$ by assuming the case $n = k$.

$$\begin{aligned} |\psi^{(k+1)}(t)\rangle =& \frac{1}{(i\hbar)^{k+1}}\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_k} dt_{k+1}\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_{k+1})\,|\psi(0)\rangle \\ =& \frac{1}{k!(i\hbar)^{k+1}}\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_1} dt_{k+1} T(\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_{k+1}))\,|\psi(0)\rangle \end{aligned} \tag{14}$$

Where we used the fact $t \geq t_1 \geq t_i$ for any $i > 1$ and so the induction assumption is applicable. Now, as T takes care of symmetrization, only the time ordering of integrals defines the value. If we want to have all the integral boundaries at t we have to relax the assumption that t_1 is last in order. If we do this, then we obtain terms for the t_1 being at any position in the ordering. There are $k + 1$ such terms. All of them are equal because of symmetry: the time ordering takes care of ordering the instances right and each time is similar. Also all the integrals behave symmetrically, and so none of the integrals should behave differently. Therefore, we obtain that we have to divide by $k+1$ to have all t_1 converted to t in the integration boundaries. This proves the claim.

Another way to see this is to note that each possible combination of times that have the time ordering are gone through in the original form indexed $1,2,3$ et cetera. If all the times are permuted and ordered with the time ordering operator, we have that the same integration is done as many time as the t_i can be ordered, i.e. $n!$ times if there are n Hamiltonians. This is the case because the time ordering operator T always orders the operators in the same order, independent of the variables of integration. Thus, in the end we have to divide by $n!$ leading to the result.

Also, similar to the case with $n=2$ there are $n!$ simplexes (simplex=generalize triangle nto arbitrary dimension) in the whole integration as one can choose first one coordinate axis for the first edge, then second, and third et cetera and the order of coordinates chosen defines the simplex. They cover the whole space. Each simplex is however symmetric because of time time-ordering operator. The original integral was over one of such simplex. Thus the overall result, integrating over all t for each integral is $n!$ times the ordered result. Thus, integral over each is the same.

d)

Now, if we take the time ordering operator out of the equation, we obtain for the time evolution operator (the terms that multiply the state at original time to obtain the state at a later time)

$$\begin{aligned} U(t) &= T\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{(i\hbar)^n}\left(\int_0^t dt'\widehat{H}(t')\right)^n \\ &= Te^{-\frac{i}{\hbar}\int_0^t dt'\widehat{H}(t')} \end{aligned} \tag{15}$$

by the definition of exponential function. We have used the fact that in the form with operator T, the integrals do not depend on variables outside so they can be taken to be independent

$$\int_0^t dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_1} dt_{k+1}\,T(\widehat{H}(t_1)\widehat{H}(t_2)\cdots\widehat{H}(t_{k+1})) = \left(\int_0^t dt_1\widehat{H}(t_1)\right)^n. \tag{16}$$

e)

The only case in which the operator T can be omitted from the above equation to obtain

$$U(t) = \exp -\frac{i}{\hbar}\int_0^t d\tau\widehat{H}(\tau) \tag{17}$$

is that the Hamiltonian operators at different times commutate with each other. Then the ordering of any two operators in the expansion can be changed at will.

1. a) Schrödinger eq. $i\hbar \frac{d\psi}{dt} = H(t)\psi(t)$

$$\Rightarrow \int_0^t dt'\, i\hbar \frac{d}{dt'}\psi(t') = \int_0^t dt' H(t')\psi(t')$$

$$\Rightarrow i\hbar(\psi(t) - \psi(0)) = \int_0^t dt' H(t')\psi(t')$$

$$\Rightarrow \psi(t) = \psi_0 + \frac{1}{i\hbar}\int_0^t dt' H(t')\psi(t') \qquad \square$$

b) $\psi(t) = \psi(0) + \frac{1}{i\hbar}\int_0^t dt' H(t')\underbrace{\psi(t')}_{\psi(0) + \frac{1}{i\hbar}\int_0^{t'} dt'' H(t'')\psi(t'')}$ etc.

$$= \left(\sum_{n=0}^{\infty}\left(\frac{1}{i\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, H(t_1)H(t_2)\cdots H(t_n)\right)\psi(0)$$

c) n:th order term $\left(\frac{1}{i\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, H(t_1)\cdots H(t_n)\psi(0)$

Integration limits such that $t_1 \geq t_2 \geq \ldots \geq t_n$

$$= \left(\frac{1}{i\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, T\left(H(t_1)H(t_2)\ldots H(t_n)\right)\psi(0)$$

Same thing $n!$ times. π = permutation

$$\Rightarrow \left(\frac{1}{i\hbar}\right)^n \frac{1}{n!}\sum_{\pi\in S_n} \underbrace{\int_0^t dt_{\pi(1)} \int_0^{t_{\pi(1)}} dt_{\pi(2)} \cdots \int_0^{t_{\pi(n-1)}} dt_{\pi(n)}}_{\text{every point in } [0,t]^n \text{ here once ... never mind the boundaries.}} \underbrace{T\left(H(t_{\pi(1)})H(t_{\pi(2)})\cdots\right)}_{T(H(t_1)H(t_2)\ldots)} \times\psi(0)$$

$$= \left(\frac{1}{i\hbar}\right)^n \frac{1}{n!}\int_0^t dt_1 \int_0^t dt_2 \cdots \int_0^t dt_n\, T\left(H(t_1)H(t_2)\cdots H(t_n)\right)\psi(0) \qquad \square$$

① d) $|\psi(t)\rangle = \sum_n \frac{1}{n!}\left(\frac{1}{i\hbar}\right)^n \int_0^t \cdots \int_0^t dt_1 \ldots dt_n \, T(H(t_1)\cdots H(t_n))|\psi(0)\rangle$

$= T \sum_n \frac{1}{n!}\left(\frac{1}{i\hbar}\int_0^t dt' H(t')\right)^n |\psi(0)\rangle$

$= \underbrace{T \exp\left(\frac{1}{i\hbar}\int_0^t dt' H(t')\right)}_{U(t)} |\psi(0)\rangle$

e) if $[H(t), H(t')] = 0 \quad \forall t, t'$

$\Rightarrow T(H(t_1)\cdots H(t_n)) = H(t_1)\ldots H(t_n)$

② $i\hbar \frac{d}{dt} A_H(t) = [A_H, H_H]$??

Schrödinger: $i\hbar \frac{d}{dt}|\psi\rangle_S = H_S|\psi\rangle_S$

$i\hbar \frac{d}{dt} U(t)|\psi_0\rangle = H_S U(t)|\psi_0\rangle \quad (\forall |\psi_0\rangle)$

$i\hbar \frac{d}{dt} U(t) = H_S U(t) = \underbrace{U(t) H_H U^{-1}(t)}_{H_S} U(t) = U(t) H_H$

$i\hbar \frac{d}{dt} \mathbb{1} = 0 = i\hbar \frac{d}{dt}(U(t)U^{-1}(t)) = \left(i\hbar \frac{d}{dt}U(t)\right)U^{-1}(t) + U(t)\underbrace{i\hbar\frac{d}{dt}U^{-1}(t)}$

$\Rightarrow i\hbar \frac{d}{dt} U^{-1}(t) = -U^{-1}(t)\underbrace{\left(i\hbar\frac{d}{dt}U(t)\right)}_{U(t)H_H} U^{-1}(t) = -H_H U^{-1}(t)$?? $\Rightarrow$ solve

then finally: $i\hbar \frac{d}{dt} A_H(t) = i\hbar \frac{d}{dt}(U^{-1}(t) A_S U(t))$

$= \left(i\hbar \frac{d}{dt} U^{-1}(t)\right) A_S U(t) + U^{-1} A_S \, i\hbar \frac{d}{dt} U$

$= -H_H \underbrace{U^{-1} A_S U}_{A_H} + U^{-1} A_S U H_H = -H_H A_H + A_H H_H$

$= [A_H, H_H]$ □

(3.) $H = H_0 + \hat{V}(t)$, $\hat{V}(t) = V_0(\hat{a} + \hat{a}^\dagger) f(t)$, $H_0 = \hbar\omega_0(\hat{a}^\dagger\hat{a} + 1/2)$

$|\Psi(0)\rangle = |n\rangle$

a) $|\Psi_I(t)\rangle = ?$ in 1st order time dependent perturbation theory.

In interaction picture $\hat{V}_I(t) = e^{iH_0 t/\hbar}\hat{V}(t)e^{-iH_0 t/\hbar}$

(remember $\begin{cases} H_0|n\rangle = \hbar\omega_0(n+1/2)|n\rangle = E_n|n\rangle \\ a|n\rangle = \sqrt{n}\,|n-1\rangle,\ a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \end{cases}$)

Eq 3.2 $\Rightarrow$ $|\Psi_I(t)\rangle = |\Psi_I(0)\rangle + \frac{1}{i\hbar}\int_0^t \hat{V}_I(t')|\Psi_I(0)\rangle dt' + \dots$ higher order

$$= |n\rangle + \frac{V_0}{i\hbar}\int_0^t dt' f(t')\, e^{iH_0 t'/\hbar}(\hat{a} + \hat{a}^\dagger)\underbrace{e^{-iH_0 t'/\hbar}|n\rangle}_{e^{-iE_n t'/\hbar}|n\rangle}$$

$$= |n\rangle + \frac{V_0}{i\hbar}\int_0^t dt' f(t') e^{-iE_n t'/\hbar} e^{iH_0 t'/\hbar}(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle)$$

$$= |n\rangle + \frac{V_0}{i\hbar}\int_0^t dt' f(t')\, e^{i(E_{n-1}-E_n)t'/\hbar}\sqrt{n}\,|n-1\rangle$$
$$+ \frac{V_0}{i\hbar}\int_0^t dt' f(t')\, e^{i(E_{n+1}-E_n)t'/\hbar}\sqrt{n+1}\,|n+1\rangle, \quad E_{n-1}-E_n = -\hbar\omega_0$$

$$= |n\rangle + \frac{V_0}{i\hbar}\sqrt{n}\int_0^t dt' f(t')\, e^{-i\hbar\omega_0 t'/\hbar}|n-1\rangle$$
$$+ \frac{V_0}{i\hbar}\sqrt{n+1}\int_0^t dt' f(t')\, e^{i\hbar\omega_0 t'/\hbar}|n+1\rangle \quad \square$$

b) $|\Psi_I(t)\rangle = e^{iH_0 t/\hbar}|\Psi(t)\rangle \Rightarrow \langle m|\Psi_I(t)\rangle = \langle m|e^{iH_0 t/\hbar}|\Psi(t)\rangle$

$= e^{iE_m t/\hbar}\langle m|\Psi(t)\rangle$

$\Rightarrow$ prob. $|\langle m|\Psi_I(t)\rangle|^2 = |e^{iE_m t/\hbar} \cdot e^{-iE_m t/\hbar}|\,|\langle m|\Psi(t)\rangle|^2$

$= |\langle m|\Psi(t)\rangle|^2 \Rightarrow$ the sceme $\square$

Exercise 4

In this exercise we calculate the time derivative of time evolution operator and consider some following properties.

a)

Let us use the series definition of exponential

$$\begin{aligned}\frac{d}{dt}T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}d\tau H(\tau)\right) &= \frac{d}{dt}T\sum_{n=0}^{\infty}\frac{1}{n!(i\hbar)^n}\left(\int_{t_0}^{t}d\tau H(\tau)\right)^n \\ &= T\sum_{n=1}^{\infty}\frac{1}{(n-1)!(i\hbar)^{n-1}}\frac{1}{i\hbar}\frac{d}{dt}\int_{t_0}^{t}d\tau H(\tau)\left(\int_{t_0}^{t}d\tau H(\tau)\right)^{n-1} ,\end{aligned} \tag{18}$$

where we have used the property that a convergent series can be differentiated term by term. The derivative of the integral is simply $H(t)$. By making a change in summation variable $n-1 \to n$, we obtain

$$\frac{d}{dt}T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}d\tau H(\tau)\right) = \frac{1}{i\hbar}T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}d\tau H(\tau)\right)H(t) = -\frac{i}{\hbar}T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}d\tau H(\tau)\right) \tag{19}$$

as was to be shown.

b)

Let us write $U_I(t)$ in another form, starting from

$$U_I(t) = T\exp\left(-i\hbar\int_{t_0}^{t}d\tau V_I(\tau)\right) . \tag{20}$$

This may be written as

$$\begin{aligned}U_I(t) &= T\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\int_{t_0}^{t}d\tau V_I(\tau)\right)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{(i\hbar)^n}\int_{t_0}^{t}d\tau_n\int_{t_0}^{\tau_n}d\tau_{n-1}\cdots\int_{t_0}^{\tau_2}d\tau_1 V_I(\tau_n)V_I(\tau_{n-1})\cdots V_I(\tau_1) \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n}{(i\hbar)^n}\int_{t_0}^{t}d\tau_n\int_{t_0}^{\tau_n}d\tau_{n-1}\cdots\int_{t_0}^{\tau_2}d\tau_1 \\ &\quad U_0(0,\tau_n)H_S'(\tau_n)U_0^{-1}(0,\tau_n)U_0(0,\tau_{n-1})H_S'(\tau_{n-1})U_0^{-1}(0,\tau_{n-1})\cdots U_0(0,\tau_1)H_S'(\tau_1)U_0^{-1}(0,\tau_1) \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n}{(i\hbar)^n}\int_{t_0}^{t}d\tau_n\int_{t_0}^{\tau_n}d\tau_{n-1}\cdots\int_{t_0}^{\tau_2}d\tau_1 \\ &\quad U_0(0,\tau_n)H_S'(\tau_n)U_0(\tau_n,\tau_{n-1})H_S'(\tau_{n-1})\cdots U_0(\tau_2,\tau_1)H_S'(\tau_1)U_0(\tau_1,0)\end{aligned} \tag{21}$$

where the connection between interaction and Schrödinger pictures and properties of U_0 were used. Also, the third equality follows from the earlier considerations.

4.

a) $\frac{d}{dt} T\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} d\tau\, H(\tau)\right) = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t}\cdots\int_{t_0}^{t} T(H(\tau_1)\cdots H(\tau_n)) \times d\tau_1\cdots d\tau_n$

$$= \frac{d}{dt}\sum\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{0}^{\tau_{n-1}} d\tau_n\, H(\tau_1)\cdots H(\tau_n) \quad \text{(prob. 1)}$$

$$= \sum\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} d\tau_2 \int\cdots\int_{t_0}^{\tau_{n-1}} d\tau_n\, H(t)\cdots H(\tau_n),$$

where $\frac{d}{dt}\int_{t_0}^{t} d\tau\, f(\tau) = f(t)$ was used.

No Integration over t -> $H(t)$ can be moved outside

$$\Rightarrow -\frac{i}{\hbar} H(t)\sum\left(-\frac{i}{\hbar}\right)^{n-1}\int_{t_0}^{t} d\tau_1 \int_{0}^{\tau_1} d\tau_2\cdots\int_{t_0}^{\tau_{n-2}} d\tau_{n-1}\, H(\tau_1)\cdots H(\tau_{n-1})$$

where we relabelled $\tau_2 \to \tau_1$, $\tau_3 \to \tau_2$ etc.

$$\Rightarrow -\frac{i}{\hbar} H(t)\, T\exp\left(-\frac{i}{\hbar}\int_{t_0} d\tau\, H(\tau)\right)$$

b) note:

$$\begin{cases} |\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle \quad \forall t, t' \\ \Rightarrow |\psi(t)\rangle = U(t,t')|\psi(t')\rangle = U(t,t')U(t',t_0)|\psi(t_0)\rangle \\ \Rightarrow U(t,t_0) = U(t,t')U(t',t_0) \quad \forall t \\ \mathbb{1} = U(t,t) = U(t,t')U(t',t) \Rightarrow U(t,t') = U(t',t)^{-1} \end{cases}$$

$$U_I(t,t_0) = \sum\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} d\tau_1 \cdots \int_{t_0}^{\tau_{n-1}} d\tau_n\, \cancel{H(\tau_1)H(\tau_2)\cdots H(\tau_n)}\; V_I(\tau_1)V_I(\tau_2)\cdots V_I(\tau_n)$$

$$= \sum\left(-\frac{i\lambda}{\hbar}\right)^n \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{n-1}} d\tau_n\, \underbrace{H'(\tau_1)\cdots H'(\tau_n)}_{(*)}$$

$$(*) = \underbrace{U_0^{-1}(\tau_1,t_0)}_{U_0(t_0,\tau_1)} H'_S(\tau_1) U_0(\tau_1,t_0)\, \underbrace{U_0^{-1}(\tau_2,t_0)}_{U_0(t_0,\tau_2)} H'_S(\tau_2) U_0(\tau_2,t_0)\cdots$$

4. ...

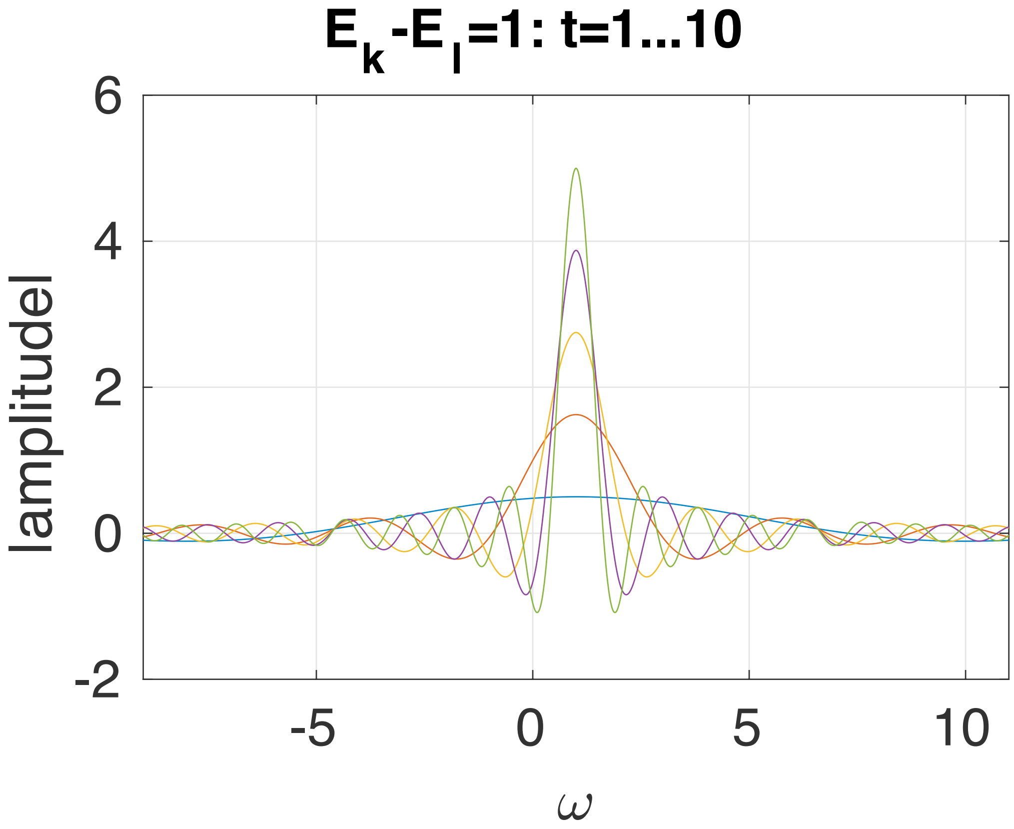
$$(*) = U_0(t_0,\tau_1)\, H_S'(\tau_1) \underbrace{U_0(\tau_1,t_0)\, U_0(t_0,\tau_2)}_{U_0(\tau_1,\tau_2)} H_S(\tau_2)\, U_0(\tau_2,t_0)\cdots$$

$$\Rightarrow \sum\left(-\frac{i\lambda}{\hbar}\right)^n \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{n-1}} d\tau_n\; U_0(t_0,\tau_1) H_S(\tau_1) U_0(\tau_1,\tau_2) H_S(\tau_2)$$
$$\cdot\, U_0(\tau_2,\tau_3) H_S(\tau_3) \cdots\; U_0(\tau_n,t_0)$$

choose $t_0 = 0$

relabel : $\tau_1 \to \tau_n$
$\tau_2 \to \tau_{n-1}$ $\Rightarrow$ (Eq. 2.26)
etc.

Exercise 1.5



Ex 1.5

a) see figure. As ω approaches $(E_k - E_l)/\hbar$ function becomes strongly peaked.

There is a representation of Dirac-delta

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{\sin x/\varepsilon}{\pi x}$$ which is relevant here.

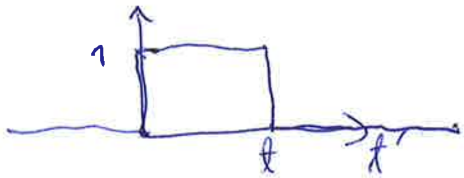
$$t = 1/\varepsilon \to \delta(x) = \lim_{t \to \infty} \frac{\sin x t}{\pi x}.$$

b) 3.10 term responsible for absorption

$$\langle k | \Psi(t) \rangle_I \approx \frac{\lambda \langle k | H' | l \rangle}{i\hbar} \underbrace{\int_0^t dt' \, e^{i(E_k - E_l)t'/\hbar} e^{-i\omega t'}}_{(*)}$$

$$(*) = \int_{-\infty}^{\infty} dt' \, S(t', t) \, e^{i(E_k - E_l)t'/\hbar} e^{-i\omega t'}$$

where $S(t', t)$ is a step function



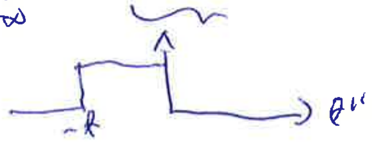
Fourier transform from time to frequency

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t') e^{i\omega t'} dt'$$

(Sign in the exponent is sort of irrelevant as long as you know your convention.)

$(*) \cdot \frac{1}{\sqrt{2\pi}}$ & $t'' = -t'$, $dt' = -dt''$, $t'' \in [\infty, -\infty]$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt'' \, \underbrace{S(t'', t)} \, e^{-i(E_k - E_l)t''/\hbar} e^{i\omega t''}$$



Ex 1.5

b) to cast this into a bit more symmetric form. let us still move the step around origin so $t''' = t'' + t/2$ so the integral becomes

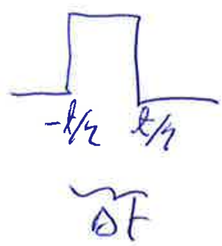
$$\frac{1}{\sqrt{2\pi}} \int_{-t/2}^{t/2} dt''' \, e^{-i(E_k-E_\ell)t'''} \, e^{+i(E_k-E_\ell)/\hbar\,(t/2)} \, e^{i\omega t'''} \, e^{-i\omega t/2}$$

$$= \frac{1}{\sqrt{2\pi}} \underbrace{e^{i(E_k-E_\ell-\hbar\omega)t/2\hbar}}_{\text{irrelevant phase: doesn't affect probabilities}} \int_{-t/2}^{t/2} dt''' \, e^{-i(E_k-E_\ell-\hbar\omega)t'''/\hbar}$$

$$= \frac{1}{\sqrt{2\pi}} e^{i(E_k-E_\ell-\hbar\omega)t/2\hbar}\left[\frac{\hbar i}{(E_k-E_\ell-\omega\hbar)}\left(e^{-i(E_k-E_\ell-\hbar\omega)t/2\hbar} - e^{+i(E_k-E_\ell-\hbar\omega)t/2\hbar}\right)\right]$$

$$= \frac{+2}{\sqrt{2\pi}} e^{i(E_k-E_\ell-\hbar\omega)t/2\hbar} \frac{\hbar}{(E_k-E_\ell-\hbar\omega)} \sin\left[\left(\frac{E_k-E_\ell-\hbar\omega}{2\hbar}\right)t\right]$$

This is essentially a F-transform of



If $\Delta t = t$ is small then the Fourier transform is broad. If you were to measure (E_k-E_ℓ) by looking at the population of level k, then resolution would be poor for small t (compared to $\sim \hbar/(E_k-E_\ell)$). If t becomes large, F-transform is narrow and energy resolution is high. $\Delta t \cdot \Delta E \gtrsim \hbar$. (However, we don't have Hermitian operator for t!)