2A Continuous random variables

Class problems

2A1 (A model for wage distribution.) We decide to model the monthly wage (in euros) earned by a randomly chosen employee with a continuous random variable X that has density

$$f(x) \;=\; \begin{cases} \alpha\, c^{\alpha}\, x^{-\alpha-1}, & \text{when } x > c, \\ 0, & \text{otherwise,} \end{cases}$$

with $\alpha = 1.6$ and $c = 1500$.

(a) What is the cumulative distribution function of X? Draw a graph of it.

(b) What are the possible values of X? In particular, how small and how large can wages be?

(c) Calculate the probability that a randomly chosen employee earns more than 15 000 eur.

(d) Find a real number x such that 90% of employees earn at most x eur.

2A2 (Both late.) In this problem we study a *continuous joint distribution* of two variables in a simple case. From multivariate calculus, we only need the following fact: the integral of a *constant* function over a region equals that constant times the area of the region.

Ursula and Vera have agreed to meet for lunch exactly at noon (12:00). However, Ursula arrives U minutes late, and Vera arrives V minutes late. We assume their arrival times are independent, and both are uniformly distributed over the interval $[0, 60]$.

(a) Write down the density functions f_U and f_V, and the joint density function $f_{U,V}$.
Start from the individual densities and then deduce the joint density. Read lecture slides 1B and/or Ross's section 4.3.1. Be careful to express where your functional expression is valid. Where is the joint density zero and where is it nonzero?

(b) Calculate the probability that Ursula arrives before 12:20.
Use the distribution of U only.

(c) Find geometrically the probability that Ursula arrives before 12:15 and Vera arrives between 12:30 and 12:45.
Color the relevant rectangle in the square $[0, 60] \times [0, 60]$. Calculate the area of this rectangle. Recall the joint density and apply the basic fact from multivariate calculus.

(d) Calculate the probability in (c) again, without resorting to geometry, from the probabilities of the individual events $\{U < 15\}$ and $\{30 < V < 45\}$. Exploit the independence of U and V.

(e) Find geometrically the probability that Ursula arrives at least 30 minutes after Vera.
You need a bit of reasoning to find which region contains the points corresponding to the event. Try fixing a value of V, and think what must U then be for the event to occur. After you have found and drawn the region, calculate its area. Recall the area of a triangle.

Home problems

2A3 (Device lifetime) A satellite orbiting the Earth contains a device whose lifetime X (in years) has the *exponential distribution* with parameter $\lambda = 0.5$. Find its density function from lecture slides (or look up Ross's section 5.6).

(a) Show how the cumulative distribution function $F(t)$ is obtained by integrating the density.

(b) Directly from the CDF, calculate the probabilities of the events $A = \{X \leq 1\}$, $B = \{X > 5\}$, and $C = \{5 < X \leq 6\}$. Give the results with at least 6 decimals. Explain in words what this events are.

(c) From the numerical results in (b), and using the definition of conditional probability, calculate $\mathrm{P}(C|B)$. Compare to $\mathrm{P}(A)$ and explain.

(d) Consider a very short interval of $h = 0.01$ years. If the device has lasted to a certain point in time, what is the probability that it breaks during the *next* 0.01 years? Compare your numerical result to the value of λh, and explain why λ is called the *rate parameter*.

2A4 (Fuzzy logic.) In fuzzy logic, a proposition (a claim that something is true) has, instead of a binary truth value (0 or 1), a real-valued truth value in the interval $[0, 1]$. Researchers are modelling a particular proposition's truth value with the random variable X that has density

$$f(x) = \begin{cases} cx(1-x), & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the constant c. The density must integrate to 1.

(b) Determine the cumulative distribution function of X, and draw it.

(c) Calculate the probability that the truth value is at least 0.75.

(d) Find the *mode* of the distribution, that is, the point x where the density $f(x)$ attains its maximum.