2B Expected value

Class problems

2B1 (Expectation of a power) Random variable X has uniform distribution over closed unit interval $[0, 1]$:

$$f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let n be a positive integer.

(a) Determine the cumulative distribution function, and then the density function, of the random variable $Y = X^n$. Hint: What values can X^n take? The CDF $F_Y(t)$ measures the probability of an event. What event? When does that event occur?

(b) Calculate $\mathrm{E}(X^n)$ from the density calculated in (a).

(c) Calculate $\mathrm{E}(X^n)$ using the transformation formula $\mathrm{E}(g(X)) = \int g(x)f(x)dx$ (see e.g. Ross's section 4.5).

(d) Using the formula you now have, calculate $\mathrm{E}(X^n)$ for $n = 1, 2, 3, 4$.

2B2 (Waiting time paradox) Buses arrive to your local bus stop at fixed times, three buses per hour. You arrive to the bus stop at X minutes after 9 o'clock, where X has continuous uniform distribution over the open interval $]0, 60[$ (which does not include the endpoints).

(a) If the buses arrive regularly at 20-minute intervals, what is your expected waiting time for the next bus?

(b) Now from the timetable you learn that the buses arrive at 9:00, 9:10, 9:30, 10:00, ... and so on. Represent your waiting time W for the next bus as a function $W = g(X)$ of your arrival time, and draw a graph of the function. (Hint: define the function by breaking into cases.)

(c) Calculate $\mathrm{E}(W)$. Hint: Transformation formula.

(d) Compare results from (a) and (c), and explain with common sense.

Home problems

2B3 (Repairing the printer) Repairing a jammed printer takes a random time X (in hours) that has density function

$$f(x) = \begin{cases} 1 - x/6, & 2 < x < 4, \\ 0, & \text{otherwise.} \end{cases}$$

(Always between 2 and 4 hours; more probably near the bottom end.) Cost of repairs (in euros) is $g(x) = 100 - 40x + 10x^2$ if repair time is x.

(a) Calculate the expected repair time $\mathrm{E}(X)$.

(b) Calculate the expected repair cost $\mathrm{E}(g(X))$.

(c) Calculate the repair cost *in the case that* the repair time happens to hit its expected value, that is, calculate $g(x)$ if $x = \mathrm{E}(X)$. Is it numerically the same as in (b)? Try to explain with common sense why / why not.

Note. g is not a linear transformation so you cannot use the "shift and scale" formulas. Hint: Lecture 2A; Ross §4.5.

2B4 (Peer grading) 30 students arrive at an exercise session, and each gives their answer sheet to the assistant. The assistant shuffles the sheets thoroughly, and deals them back to the students for grading, one sheet to each student. Determine the expected value of the number of students that receive their own paper.

Hint. Define indicator variables

$$X_i = \begin{cases} 1, & \text{if the } i\text{th student receives his/her own paper,} \\ 0, & \text{otherwise.} \end{cases}$$

Recall expectation of a sum of random variables.