Aalto University
School of Electrical
Engineering

Brief Recap of Chapter 2 of Brown et al. (2014)

Simo Särkkä

Contents of Chapter 2 ”Classical Response of a Single Nucleus to a Magnetic Field”

Introduction

- MRI experiment is really a two-step process where:
  1. *The proton 'spin' orientation is manipulated by an assortment of applied magnetic fields*
  2. *Changes in orientation can be measured through the interaction of the proton's magnetic field with a coil detector.*
- Although each proton field is minuscule, a significant signal can be measured resulting from the sum of all fields of all affected protons of the body.

Torque on a Current Loop in a Magnetic Field

- **Lorentz force law:**

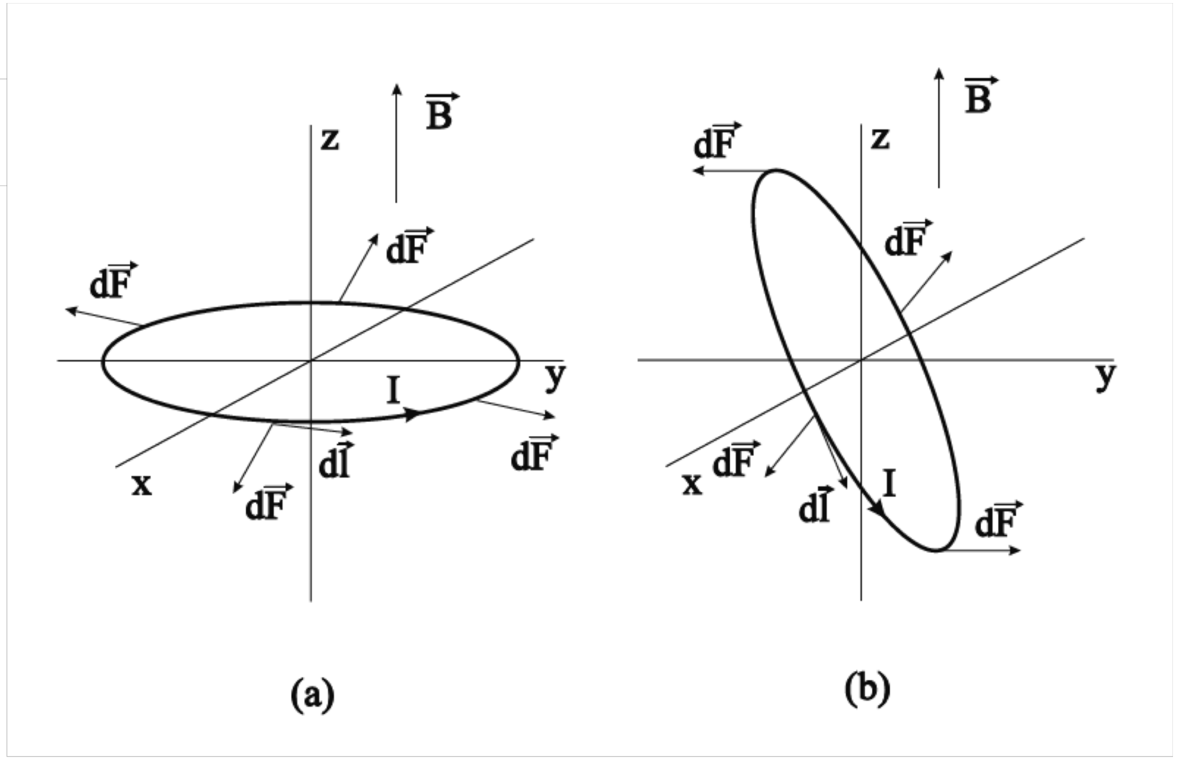
$$d\vec{F} = I d\vec{\ell} \times \vec{B}$$

- **Newton's law:**

$$\vec{F} = \frac{d\vec{p}}{dt}$$

- **Differential torque:**

$$d\vec{N} = \vec{r} \times d\vec{F}$$



Torque on a Current Loop in a Magnetic Field

- **Torque in terms of magnetic dipole moment $\vec{\mu}$:**

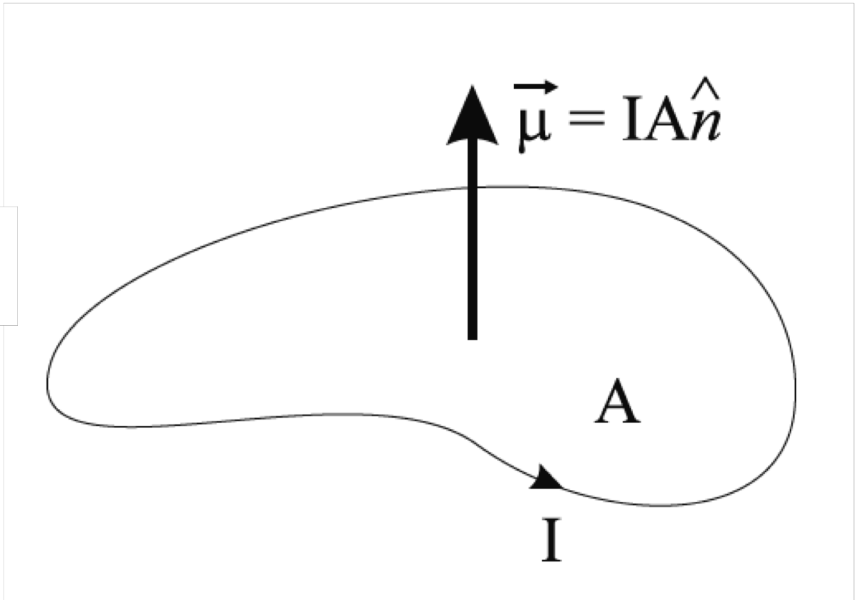
$$\vec{N} = \vec{\mu} \times \vec{B}$$

- **Which is**

$$d\vec{N} = \vec{r} \times (Id\vec{\ell} \times \vec{B}) = Id\vec{\ell}\,(\vec{B} \cdot \vec{r}) - I\vec{B}\,(d\vec{\ell} \cdot \vec{r})$$

- **We can derive from this:**

$$\vec{\mu} = I\pi R^2 \hat{z}$$



Magnetic Moment with Spin: Equation of Motion

- **Differential equation for total angular momentum:**

$$\frac{d\vec{J}}{dt} = \vec{N}$$

- **(Spin) Angular momentum of proton:**

$$\vec{\mu} = \gamma \vec{J}$$

- **The gyromagnetic ratio for proton:**

$$\gamma = 2.675 \times 10^8 \text{ rad/s/T}$$

$$\bar{\gamma} \equiv \frac{\gamma}{2\pi} = 42.58 \text{ MHz/T}$$

- **Cannot be derived from classical physics, quantum needed**

Problem 2.1

a) Show that the angular momentum $\vec{r} \times \vec{p}$ of the circulating particle with respect to the center is $mrv\hat{n}$ where $\hat{n}$ is a unit vector perpendicular to the plane of the circle. Here, $\hat{n}$ points in a direction given by the right-hand rule applied to the particle's motion.

b) Show that the magnetic moment associated with the motion of the point charge is $qvr/2$ and thus that the gyromagnetic ratio is given by (2.19).

c) Evaluate numerically the gyromagnetic ratio γ (2.19), choosing the same mass (1.67×10^{-27} kg) and charge (1.60×10^{-19} C) as for a proton. The difference between your answer and (2.17) is due to the more complicated motion of the proton constituents, the 'quarks.' For related reasons, a neutron has a nonvanishing magnetic moment despite its zero overall charge.

$$\gamma = 2.675 \times 10^8 \text{ rad/s/T} \tag{2.17}$$

$$\gamma \text{ (point charge in circular motion)} = \frac{q}{2m} \tag{2.19}$$

Magnetic Moment with Spin: Equation of Motion

- **Different gyromagnetic ratios, e.g. for electron:**

$$\frac{|\gamma_e|}{\gamma_p} = 658$$

- **Fundamental differential equation for magnetic moment:**

$$\frac{d\vec{\mu}}{dt} = \gamma \vec{\mu} \times \vec{B}$$

- **Recall:**

$$\omega \equiv \left| \frac{d\phi}{dt} \right| = \gamma B$$

A solution:

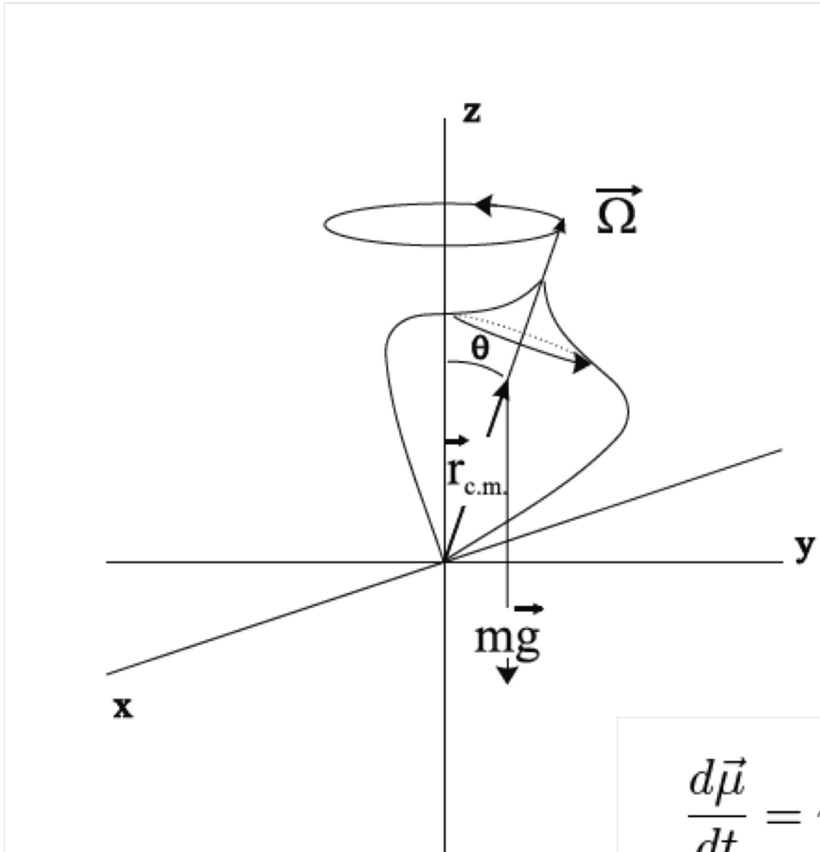
$$\vec{\mu}(t) = \mu_x(t)\hat{x} + \mu_y(t)\hat{y} + \mu_z(t)\hat{z}$$

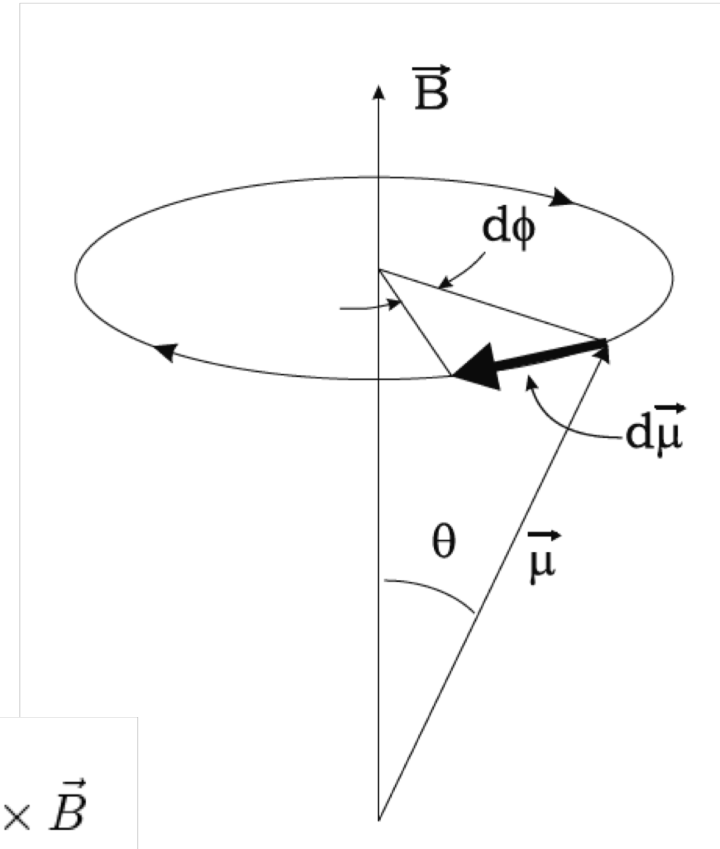
$$\begin{aligned} \mu_x(t) &= \mu_x(0)\cos\omega_0 t + \mu_y(0)\sin\omega_0 t \\ \mu_y(t) &= \mu_y(0)\cos\omega_0 t - \mu_x(0)\sin\omega_0 t \\ \mu_z(t) &= \mu_z(0) \end{aligned}$$

Gyromagnetic ratios

Nucleus	Spin	Magnetic moment	γ	Abundance in human body
hydrogen ^{1}H	1/2	2.7928	42.58	88 M
sodium ^{23}Na	3/2	2.2175	11.27	80 mM
phosphorus ^{31}P	1/2	1.1316	17.25	75 mM
oxygen ^{17}O	5/2	−1.8938	−5.77	17 mM
fluorine ^{19}F	1/2	2.6289	40.08	4 μM

Precession





$$\frac{d\vec{\mu}}{dt} = \gamma\vec{\mu} \times \vec{B}$$

Magnetic Moment with Spin: Equation of Motion

- **Matrix representation:**

$$\vec{\mu}(t) = R_z(\omega_0 t)\vec{\mu}(0)$$

$$\vec{\mu}(t) = \begin{pmatrix} \mu_x(t) \\ \mu_y(t) \\ \mu_z(t) \end{pmatrix}$$

- **Complex representation:**

$$\mu_+(t) = \mu_x(t) + i\mu_y(t)$$

$$\frac{d\mu_+}{dt} = -i\omega_0\mu_+$$

$$\mu_+(t) = \mu_+(0)e^{-i\omega_0 t}$$

Problem 2.2

It will be useful in later discussions to have the answer (2.33) rederived as a solution to the differential equation (2.24).

a) For $\vec{B} = B_0\hat{z}$, show that the vector differential equation (2.24) decomposes into the three Cartesian equations

$$\begin{aligned}\frac{d\mu_x}{dt} &= \gamma\mu_y B_0 = \omega_0\mu_y \\ \frac{d\mu_y}{dt} &= -\gamma\mu_x B_0 = -\omega_0\mu_x \\ \frac{d\mu_z}{dt} &= 0.\end{aligned} \tag{2.34}$$

b) By taking additional derivatives, show that the first two equations in (2.34) can be decoupled to give

$$\begin{aligned}\frac{d^2\mu_x}{dt^2} &= -\omega_0^2\mu_x \\ \frac{d^2\mu_y}{dt^2} &= -\omega_0^2\mu_y\end{aligned} \tag{2.35}$$

These decoupled second-order differential equations have familiar solutions of the general form $C_1\cos\omega_0 t + C_2\sin\omega_0 t$.

$$\frac{d\vec{\mu}}{dt} = \gamma\vec{\mu}\times\vec{B} \tag{2.24}$$

$$\begin{aligned}\mu_x(t) &= \mu_x(0)\cos\omega_0 t + \mu_y(0)\sin\omega_0 t \\ \mu_y(t) &= \mu_y(0)\cos\omega_0 t - \mu_x(0)\sin\omega_0 t \\ \mu_z(t) &= \mu_z(0)\end{aligned} \tag{2.33}$$