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2 Random variables and measurable functions

In this exercise you continue to practice analyzing probability measures on general spaces, and you become familiar with the connections between measurable sets, measurable functions, and random variables.

2.1 *Truncation and conditioning.*

(a) Let ν be a measure on a measurable space $(S, \mathscr{S})$ and let $B \in \mathscr{S}$. Show that $A \mapsto \nu[A \cap B]$ defines a measure on $(S, \mathscr{S})$.

(b) Let P be a probability measure on $(\Omega, \mathscr{F})$, and let B be an event such that $\mathsf{P}[B] > 0$. Show that the conditional probability $A \mapsto \mathsf{P}\big[A \,\big|\, B\big] := \frac{\mathsf{P}[A \cap B]}{\mathsf{P}[B]}$ is a probability measure.

2.2 *Indicators of sets.* The *indicator function* 1_A of a set $A \subset S$ is defined by

$$1_A(x) \;=\; \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Let $\mathscr{S}$ be a sigma-algebra on S. Show that 1_A a $\mathscr{S}$-measurable function if and only if A is a $\mathscr{S}$-measurable set.

(b) Show that $1_{A \cap B} = 1_A \, 1_B$.

(c) When is it true that $1_{A \cup B} = 1_A + 1_B$?

2.3 *Sigma-algebra generated by a random variable.* Let $\sigma(Y)$ be the sigma-algebra generated by a real-valued random variable Y defined on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$.

(a) Show that the $\sigma(Y)$ coincides with the σ-algebra $\sigma\big(Y^{-1}(\mathscr{B})\big)$ generated by the collection of events $Y^{-1}(\mathscr{B}) = \{Y^{-1}(B) \,\big|\, B \in \mathscr{B}\}$.

(b) Show that we in fact have the equality $\sigma(Y) = Y^{-1}(\mathscr{B})$.

(c) Show that for any Borel function $h\colon \mathbb{R} \to \mathbb{R}$, the random variable $h(Y)$ is $\sigma(Y)$-measurable.

2.4 *Sums and mixtures.* The sum of two measures μ and ν on $(S, \mathcal{S})$ is a set function $\mu + \nu$ defined by $(\mu + \nu)(A) = \mu(A) + \nu(A)$ for $A \in \mathcal{S}$. For $a \geq 0$ we define the scalar product $a\mu$ as the set function $A \mapsto a\mu(A)$.

(a) Prove that $\mu + \nu$ is a measure on $(S, \mathcal{S})$.

(b) Prove that $a\mu$ is a measure on $(S, \mathcal{S})$.

(c) If $P_1, \dots, P_n$ are probability measures on $(S, \mathcal{S})$ and $a_1, \dots, a_n \geq 0$ satisfy $a_1 + \cdots + a_n = 1$, prove that $a_1 P_1 + \cdots + a_n P_n$ is a probability measure on $(S, \mathcal{S})$.

(d) Define $P = \frac{1}{3}P_1 + \frac{2}{3}P_2$ where $P_1 = \delta_0$ is the Dirac measure at 0 and $P_2(A) = \frac{1}{10}\Lambda(A \cap [0, 10])$ is the uniform distribution on the continuous interval $[0, 10]$ (here Λ is the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Compute the probabilities $P((0, 7))$, $P((0, 7])$, $P([0, 7))$, and $P([0, 7])$.

2.5 *Constructing a random variable with a given cumulative distribution function.* Suppose that a function $F \colon \mathbb{R} \to [0, 1]$ satisfies the following properties:

(i) F is increasing: if $x \leq y$ then $F(x) \leq F(y)$.

(ii) F is right-continuous: if $x_n \downarrow x$ then $F(x_n) \downarrow F(x)$.

(iii) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$.

Let $U \colon \Omega \to \mathbb{R}$ be a random variable on some probability space $(\Omega, \mathscr{F}, \mathsf{P})$ following the uniform distribution on the unit interval, so that $\mathsf{P}\big[\{\omega \in \Omega \mid a < U(\omega) < b\}\big] = b - a$ whenever $0 \leq a \leq b \leq 1$. Define $X \colon \Omega \to \mathbb{R}$ by

$$X(\omega) = \inf \{z \in \mathbb{R} \mid F(z) \geq U(\omega)\}.$$

(a) Show that for any $c \in \mathbb{R}$ we have $X(\omega) \leq c$ if and only if $U(\omega) \leq F(c)$.

(b) Using (a), show that for any $c \in \mathbb{R}$ we have $X^{-1}\big((-\infty, c]\big) \in \mathscr{F}$.

(c) Conclude that X is a random variable on probability space $(\Omega, \mathscr{F}, \mathsf{P})$.

(d) Show that the cumulative distribution function of X equals F, that is, $\mathsf{P}[X \leq c] = F(c)$ for all $c \in \mathbb{R}$.