Optics E-5730 Spring 2021

Diffraction I

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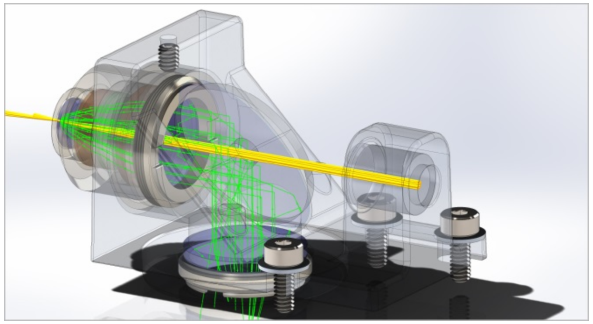
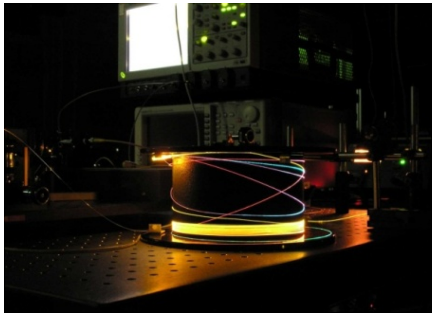

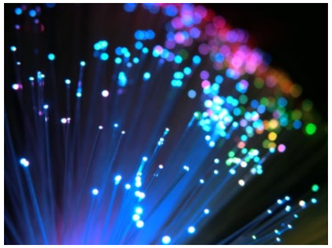
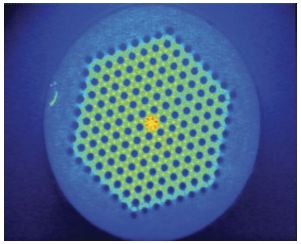
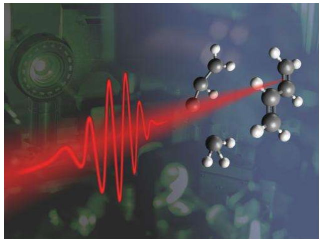

Last Lecture – Fibre Optics & Optical Telecom

- Fibre optics
- Optical telecom

Today's Lecture – Diffraction I

- Diffraction
- Diffraction integral
- Fraunhofer and Fresnel approximations
- Fraunhofer diffraction intensity for one, two and N slits
- Diffraction grating and grating spectrometer

Diffraction Limit

In 1873 Ernst Abbe found out that light with wavelength λ traveling in a medium with refractive index *n* and converging into a spot at an angle θ will form a spot with diameter

$$\emptyset = \frac{\lambda}{n\ sin\theta} = \frac{\lambda}{NA} = \frac{\lambda f}{D}$$

Diffraction Limit

$$\emptyset = \frac{\lambda}{n\ sin\theta} = \frac{\lambda}{NA} = \frac{\lambda f}{D}$$

The denominator nsinθ is the numerical aperture NA and it can reach about 1.4–1.6 using the best modern lenses. Considering green light at 500 nm and a NA of 1 the Abbe limit is roughly equal to λ = 500 nm (0.5 µm) which is small compared to most biological cells (1 µm to 100 µm) but large compared to viruses (100 nm), proteins (10 nm) and less complex molecules (1 nm).

Shorter wavelengths such as UV and X-ray offer better resolution but are expensive, suffer from the lack of contrast in biological samples and are likely to damage the sample.

The Nobel Prize in Chemistry 2014

Photo: Matt Staley/HHMI

Eric Betzig

Prize share: 1/3

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Stefan W. Hell

Prize share: 1/3

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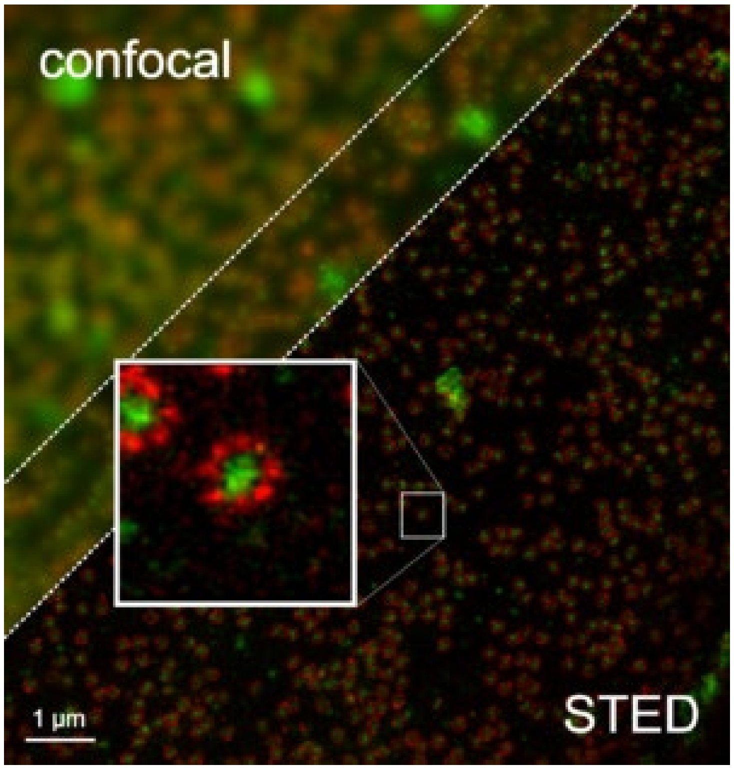
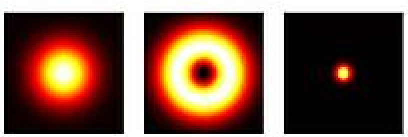
William E. Moerner

Prize share: 1/3

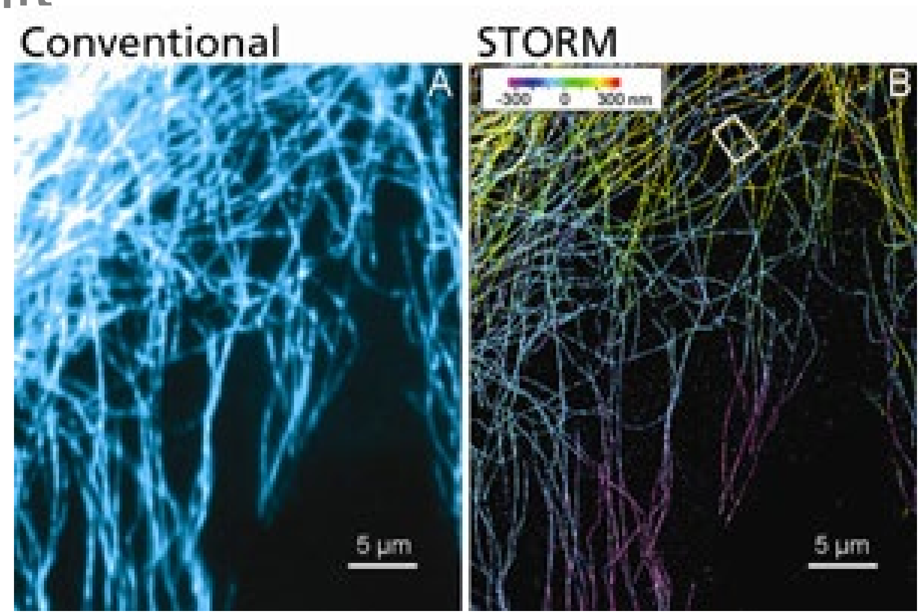
The Nobel Prize in Chemistry 2014 was awarded jointly to Eric Betzig, Stefan W. Hell and William E. Moerner *"for the development of super-resolved fluorescence microscopy".*

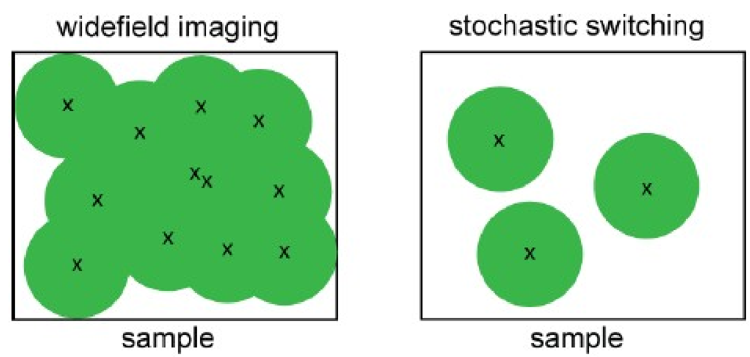
Super-resolution Microscopy – Beyond the Diffraction Limit





Stimulated Emission Depletion - STED





Stochastic Optical Reconstruction Microscopy (STORM)

Diffraction

In the 15th century it was observed that intense light created a shadow that did not have 'sharp edges' as predicted by the geometrical optics ('particle-like' light).

<u>Diffraction</u> – deviation of light from rectlinear propagation (= geometrical optics) due to obstruction

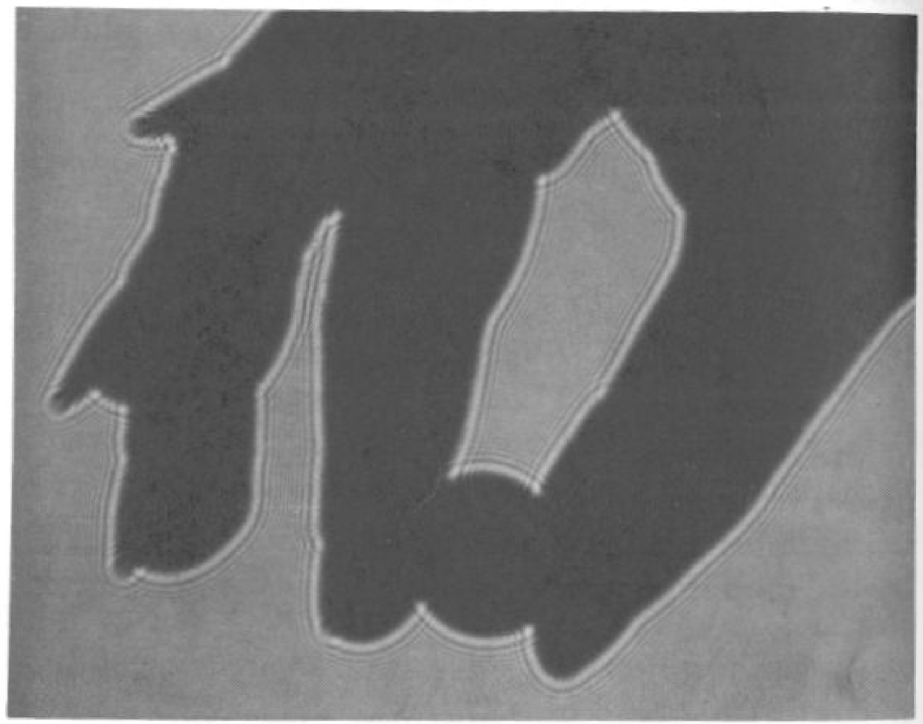
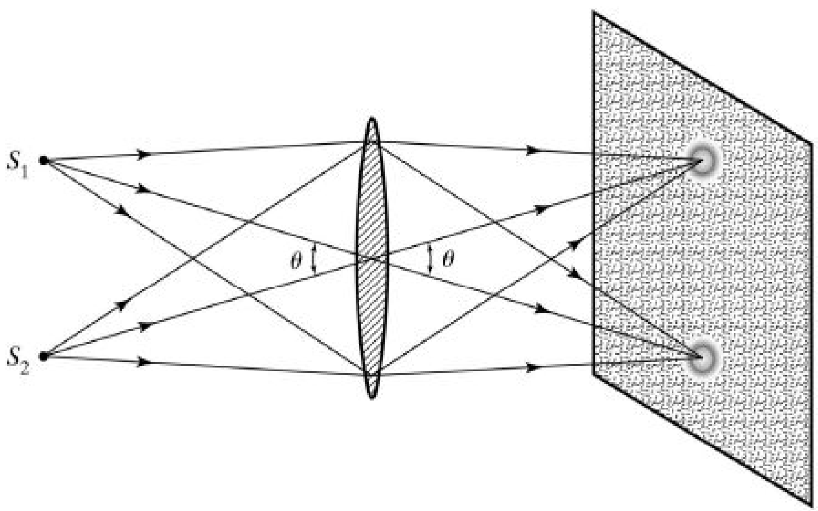
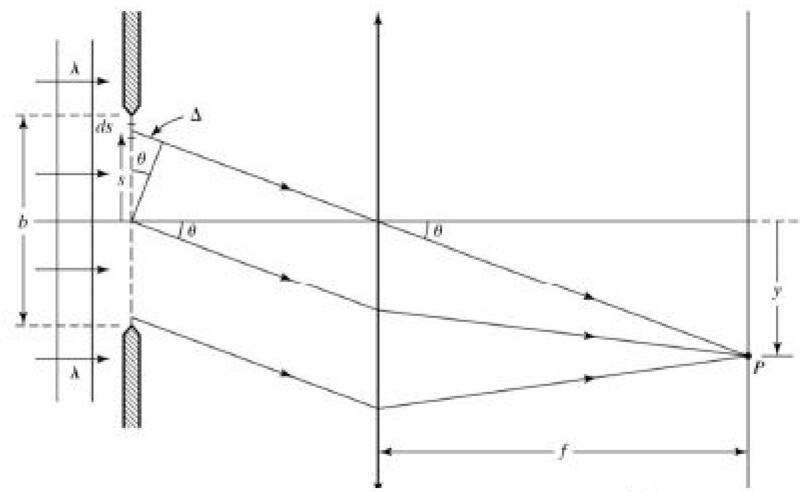


Figure 10.1 The shadow of a hand holding a dime, cast directly on 4 × 5 Polaroid A.S.A. 3000 film using a He–Ne beam and no lenses. (Photo by E.H.)

Interference and Diffraction

- **There is no physical difference between interference and diffraction.**
- However, it has become customary to differentiate the two concepts from each other:
  - **interference** = interplay between a few (discrete) EM waves
  - **diffraction** = interplay between a large number of EM waves (geometrically large, continuous surface of a light source)





Like for interference, wave optics is also a powerful tool to study diffraction.

Diffraction

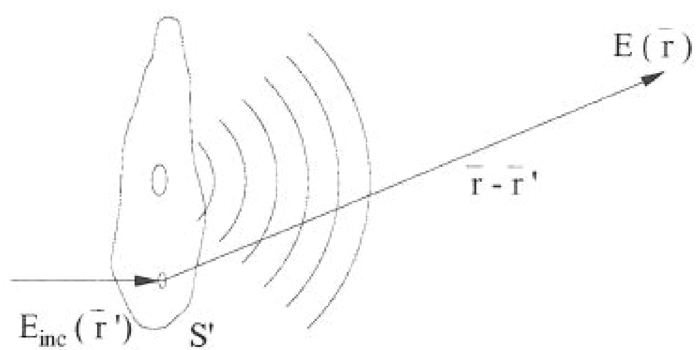
For light diffraction in an aperture S' so called *diffraction integral* can be derived.
Solving the diffraction integral is tedious so approximations have been developed:

In Fraunhofer approximation it is assumed that a plane wave arrives at the aperture S'

- both light source and obervation location are "far away from the aperture"
- $R > a^2/\lambda$, where R is the shortest distance before and after the aperture and a is the largest dimension of the aperture

In Fresnel approximation a spherical wave is assumed on the surface S'

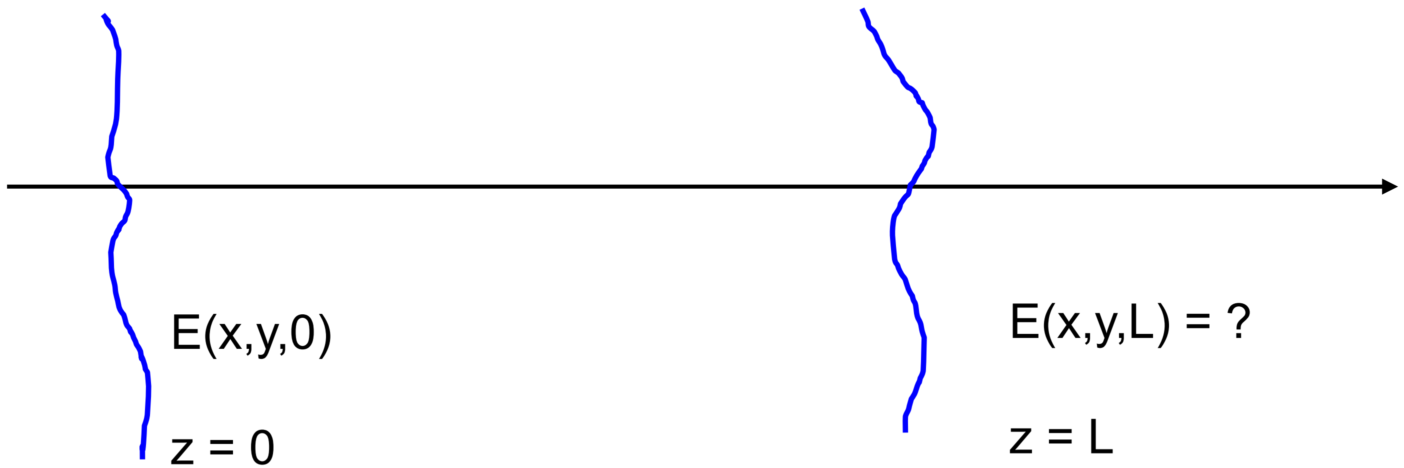
- both light source and obervation location are "close to the aperture"



Other remarks on the solutions of diffraction problems:

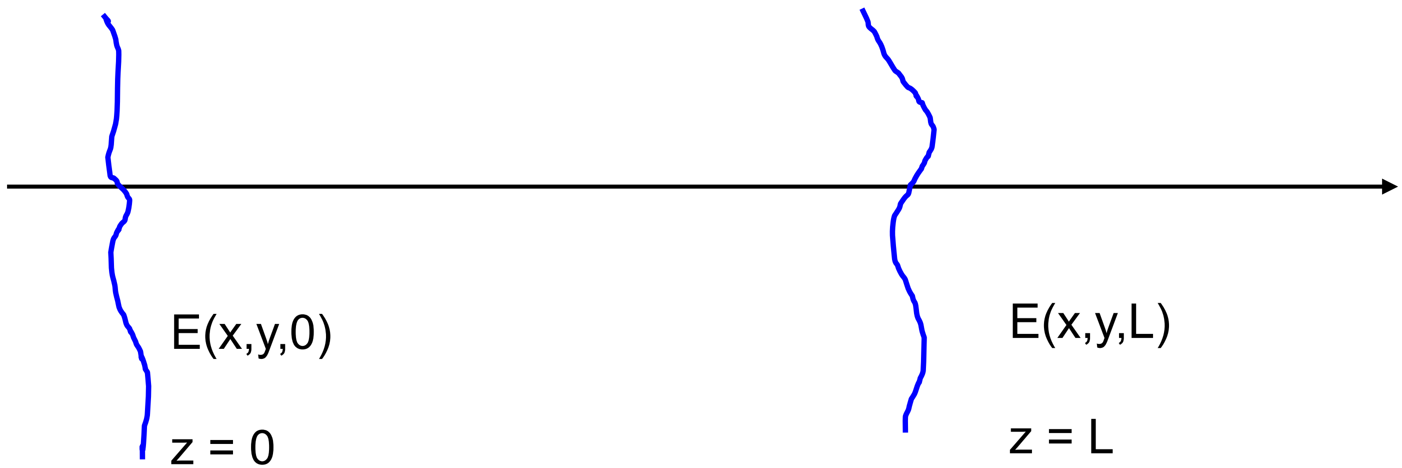
- If S' is large compared with the wavelength of light, the edges of the aperture (electrons) do not significantly contribute to diffraction.
- If S' is small ($\sim\lambda$), the effects due to electrons in the aperture edges must be included.

Diffraction Integral



$$E(x,y,z) = \frac{1}{i\lambda z} e^{-ikz} \iint_{S'} E_{inc}(x',y')\, e^{-ik\frac{\left[(x-x')^2-(y-y')^2\right]}{2z}} dx'dy'$$

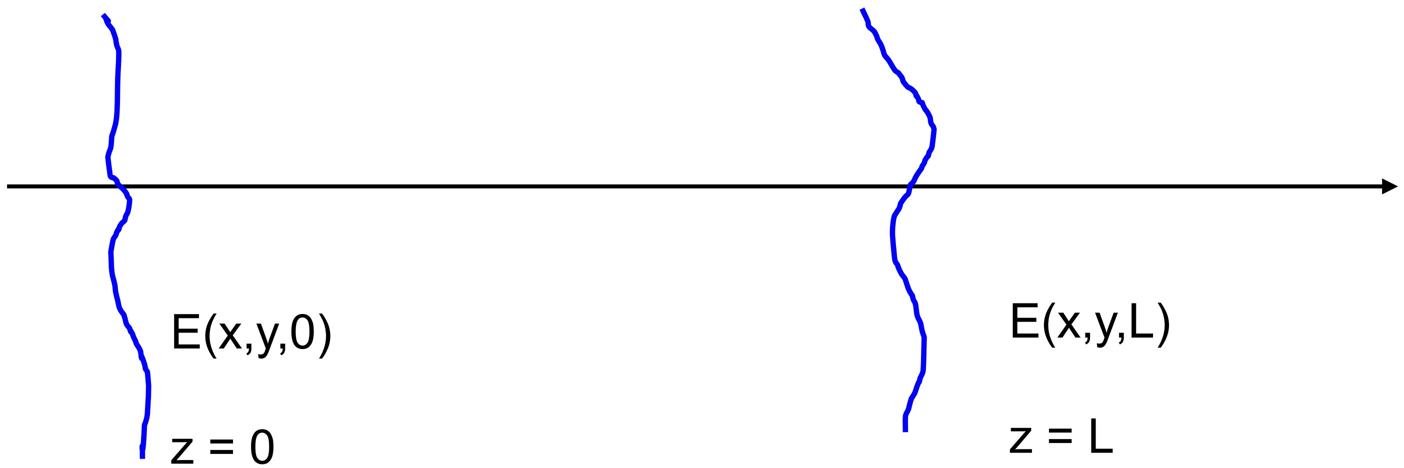
Diffraction Integral



At z = 0 the field amplitude E(x,y,0) can be written as a sum of plane waves (Fourier integral):

$$E(x,y) = \iint A(k_x,k_y)\, e^{-i(k_x x + k_y y)} dk_x dk_y \qquad \text{Eq. (1)}$$

Diffraction Integral



At z = L the field amplitude E(x,y,z) can be expressed as:

$$E(x,y,z) = \iint A\left(k_x, k_y\right) e^{-i\left(k_x x + k_y y\right)} e^{-i k_z z} dk_x dk_y$$

Diffraction Integral

$$E(x,y,z) = \iint A(k_x,k_y)\, e^{-i(k_x x + k_y y)} e^{-ik_z z} dk_x dk_y$$

$$k^2 = \left(\frac{2\pi}{\lambda}\right)^2 = k_x^2 + k_y^2 + k_z^2 \quad \Rightarrow \quad k_z^2 = k^2\left(1 - \frac{k_x^2 + k_y^2}{k^2}\right)$$

for paraxial rays k_x, k_y << k $\Rightarrow$ $k_z \approx k - \dfrac{k_x^2 + k_y^2}{2k}$

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x$$

$$\Rightarrow \quad E(x,y,z) = \iint A(k_x,k_y)\, e^{-i(k_x x + k_y y)} e^{-i\left(k - \frac{k_x^2 + k_y^2}{2k}\right)z} dk_x dk_y$$

From Eq. (1) $\quad A(k_x,k_y) = \dfrac{1}{(2\pi)^2} \displaystyle\iint E(x,y)\, e^{i(k_x x + k_y y)} dx dy$

Diffraction Integral

$$E(x,y,z)=\frac{1}{(2\pi)^2}\iint dx'dy' \iint E(x',y')\, e^{-i\left(k_x(x-x')+k_y(y-y')\right)} e^{-i\left(k-\frac{k_x^2+k_y^2}{2k}\right)z} dk_x dk_y$$

it can be shown using the information $\int_{-\infty}^{\infty} e^{-(\alpha x^2+\beta x)} dx = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{\beta^2}{4\alpha}}$

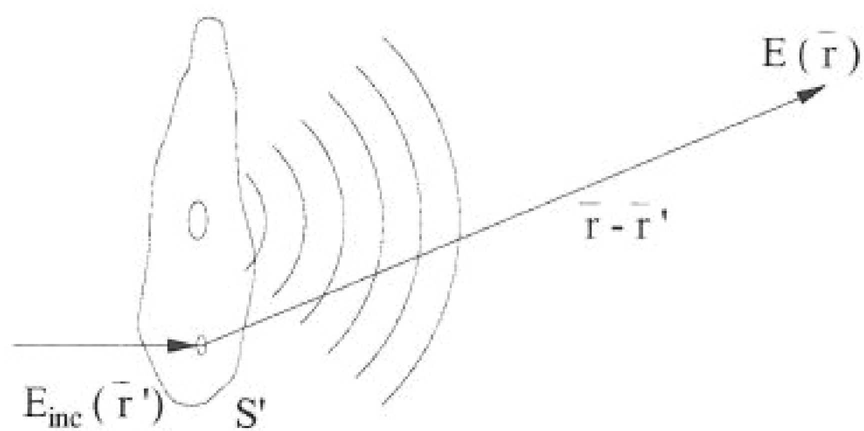
$$\Rightarrow \quad E(x,y,z)=\frac{1}{i\lambda z} e^{-ikz} \iint_{S'} E_{inc}(x',y')\, e^{-ik\frac{\left[(x-x')^2-(y-y')^2\right]}{2z}} dx'dy'$$

Diffraction Integral in Spherical Coordinates

$$E(\vec{r}) = \frac{1}{i\lambda}\int_{S'} E_{inc}(\vec{r}\,')\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} Q dS'$$

$$Q = \frac{1}{2}\left(\cos\theta + \cos\theta'\right)$$

Q is so called inclination factor: for paraxial rays it can be assumed that Q ≈ 1

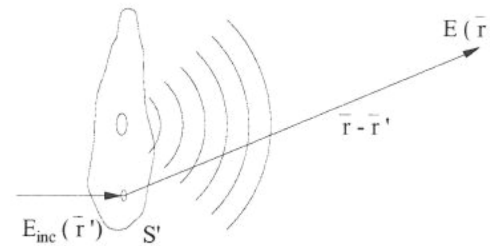


Diffraction

Diffraction integral

$$E(x,y,z) = \frac{1}{i\lambda z} e^{-ikz} \iint_{S'} E_{inc}(x',y')\, e^{-ik\frac{\left[(x-x')^2-(y-y')^2\right]}{2z}} dx'dy'$$

Diffraction integral in spherical coordinates



$$E(\vec{r}) = \frac{1}{i\lambda} \iint_{S'} E_{inc}(\vec{r}') \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} dS'$$

Example: a plane wave focused using a lens

$$E_{inc}(\vec{r}') = \frac{A}{R} e^{-ikR}, \qquad we\ select\ S'\ so\ that\ R = |\vec{r}-\vec{r}'| = constant(distance\ from\ focus)$$

$$E(\vec{r}) = \frac{1}{i\lambda} \iint_{S'} \frac{A}{R} e^{-ikR} \frac{e^{ikR}}{R} dS' = \frac{A}{i\lambda} \iint_{S'} \frac{1}{R^2} dS' = \frac{A}{i\lambda}\Omega$$

the E field amplitude in the focus is not infinite like for an ideal spherical wave E_{inc}

Fraunhofer Approximation

$$E(\vec{r}) = \frac{1}{i\lambda} \iint_{S'} E_{inc}(\vec{r}') \frac{e^{ik|\vec{r}-\vec{r}\prime|}}{|\vec{r}-\vec{r}'|} dS'$$

Assuming that a plane wave arrives at S' so that

- both the light source and obervation location are "far away from the aperture"
- R > a^2/λ, where R is the shortest distance before and after the aperture and a is the largest dimension of the aperture

On surface S' (z'=0) the exponent in the nominator:

$$\begin{aligned} |\vec{r}-\vec{r}'| &= \sqrt{z^2 + (x-x')^2 + (y-y')^2} \\ &\approx z\left(1 + \frac{(x-x\prime)^2}{2z^2} + \frac{(y-y\prime)^2}{2z^2}\right) \\ &= z - \frac{xx'}{z} - \frac{yy'}{z} + \frac{x^2+y^2}{2z} + \frac{x'^2+y'^2}{2z} \\ &\approx z - \frac{xx'}{z} - \frac{yy'}{z} + \frac{x^2+y^2}{2z} \end{aligned}$$

approximation (z=L)

$$k(x'^2 + y'^2) \ll 2L$$

denominator: $|\vec{r}-\vec{r}'| \approx z$

Fraunhofer Approximation

$$|\vec{r}-\vec{r}'| \approx z - \frac{xx'}{z} - \frac{yy'}{z} + \frac{x^2+y^2}{2z}$$

$$E(\vec{r}) = \frac{1}{i\lambda} \iint_{S'} E_{inc}(\vec{r}') \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} dS'$$

$$|\vec{r}-\vec{r}'| \approx z$$

$$E(x,y,z) = \frac{e^{ikz+\frac{ik}{2z}(x^2+y^2)}}{i\lambda z} \iint_{S'} E_{inc}(x',y')\, e^{-ik\frac{xx'+yy'}{z}} dx'dy'$$

Using notation $\nu_x = \frac{x}{\lambda z}$ $\nu_y = \frac{y}{\lambda z}$

$$E(x,y,z) = \frac{e^{ikz+\frac{ik}{2z}(x^2+y^2)}}{i\lambda z} \iint_{S'} E_{inc}(x',y')\, e^{-i2\pi(\nu_x x'+\nu_y y')} dx'dy'$$

2D Fourier-transformation

$$\mathcal{F}[E_{inc}(x',y')](\nu_x,\nu_y) = \iint_{S'} E_{inc}(x',y')\, e^{-i2\pi(\nu_x x'+\nu_y y')} dx'dy'$$

$$I(\nu_x,\nu_y) = \frac{1}{\lambda^2 z^2}\left[F[E_{inc}(x',y')](\nu_x,\nu_y)\right]^2$$

Intensity of the diffraction pattern results from 2D-Fourier transformation of the incident field.

Fraunhofer Diffraction of a Single Slit

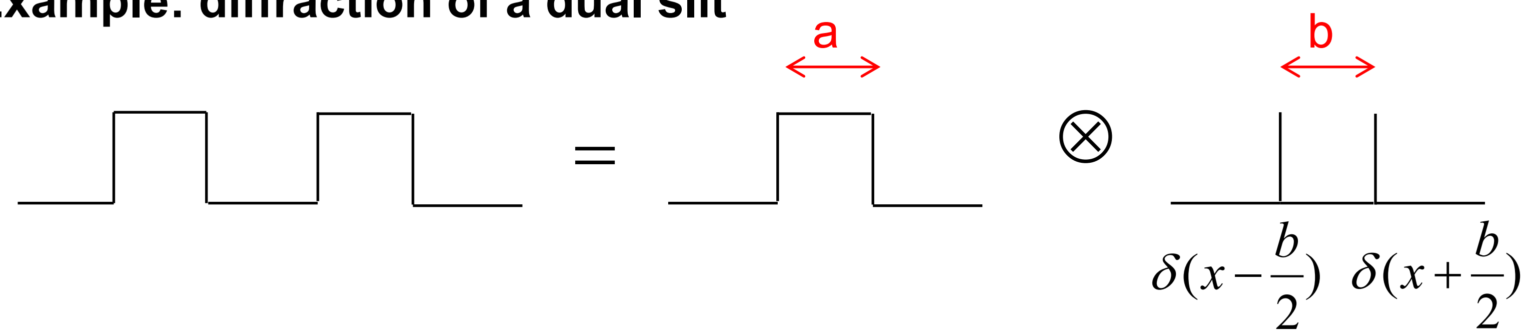
Convolution Theorem

Fourier transformation of convolution of two functions f(x) and g(x) is equal to product of the individual Fourier transformations:

$$\Im\{f(x)\otimes g(x)\} = \Im\{f(x)\}\Im\{g(x)\}$$

where $$f(x)\otimes g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

Example: diffraction of a dual slit



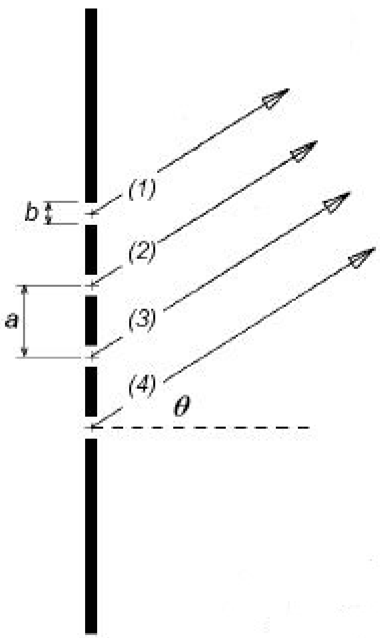
$$\Im\left\{E_{\text{single slit}} \otimes \left[\delta\left(x-\frac{b}{2}\right)+\delta\left(x+\frac{b}{2}\right)\right]\right\} = \Im\{E_{\text{single slit}}\}\Im\left\{\delta\left(x-\frac{b}{2}\right)+\delta\left(x+\frac{b}{2}\right)\right\}$$

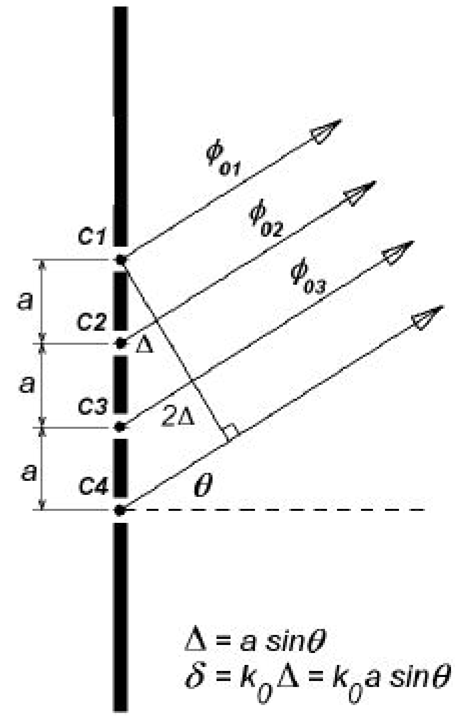
$$\propto \text{sinc}(\pi\nu_x a) \qquad e^{-i\pi\nu_x b}+e^{i\pi\nu_x b} = \cos(\pi\nu_x b)$$

$$I_{double\ slit} \propto \text{sinc}^2(\pi\nu_x a)\cos^2(\pi\nu_x b)$$

Diffracted Intensity of N Slits

slit width ***b***
distance between slits ***a***





P

With respect to an observation point P for each "single slit diffraction field" the relative phase-differences are:

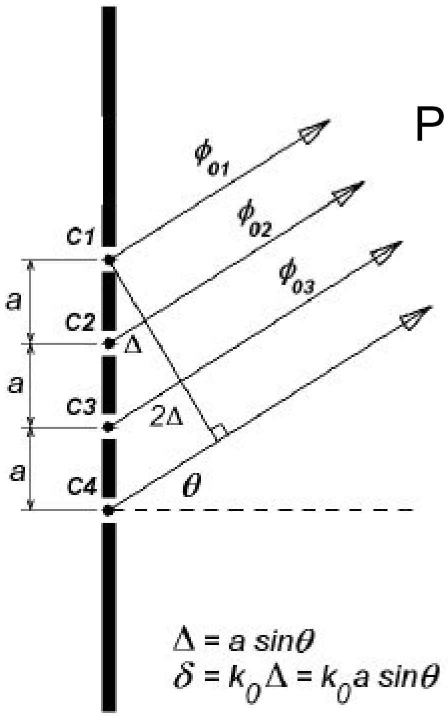
$$E_{tot} = E_0 \mathrm{sinc}(\pi \nu_x b) e^{i\Phi_{01}} + E_0 \mathrm{sinc}(\pi \nu_x b) e^{i\Phi_{02}} + \ldots + E_0 \mathrm{sinc}(\pi \nu_x b) e^{i\Phi_{0n}}$$

We define the δ phase-shift between slits next to each other:

$$\Phi_{02} - \Phi_{01} = k_0 a \sin\theta = \delta, \quad \Phi_{03} - \Phi_{01} = 2k_0 a \sin\theta = 2\delta$$

$$\ldots \Phi_{0N} - \Phi_{01} = N k_0 a \sin\theta = N\delta$$

Diffracted Intensity of N Slits



$$E_{tot} = E_0 e^{i\Phi_{01}} \mathrm{sinc}(\pi \nu_x b)\left[1 + e^{i\delta} + e^{i2\delta} + \ldots + e^{i(N-1)\delta}\right]$$

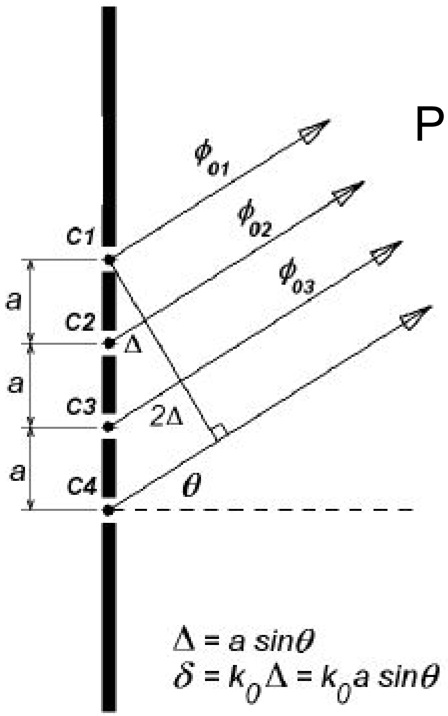
$$\sum_{j=0}^{N-1} x^j = \frac{x^N - 1}{x - 1} \quad \text{where} \quad x = e^{i\delta}$$

$$E_{tot} = E_0 e^{i\Phi_{01}} \mathrm{sinc}(\pi \nu_x b)\left[\frac{e^{iN\delta} - 1}{e^{i\delta} - 1}\right]$$

$$\left[\frac{e^{iN\delta} - 1}{e^{i\delta} - 1}\right] = \frac{e^{iN\delta/2}\left[e^{iN\delta/2} - e^{-iN\delta/2}\right]}{e^{i\delta/2}\left[e^{i\delta/2} - e^{-i\delta/2}\right]} = \frac{e^{iN\delta/2}\, 2\,\mathrm{Im}\left[e^{iN\delta/2}\right]}{e^{i\delta/2}\, 2\,\mathrm{Im}\left[e^{i\delta/2}\right]}$$

$$= e^{i(N-1)\delta/2} \frac{\sin(N\delta/2)}{\sin(\delta/2)}$$

Diffracted Intensity of N Slits



$$E_{tot} = E_0 e^{i\Phi_{off-set}} \mathrm{sinc}(\pi \nu_x a)\left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right]$$

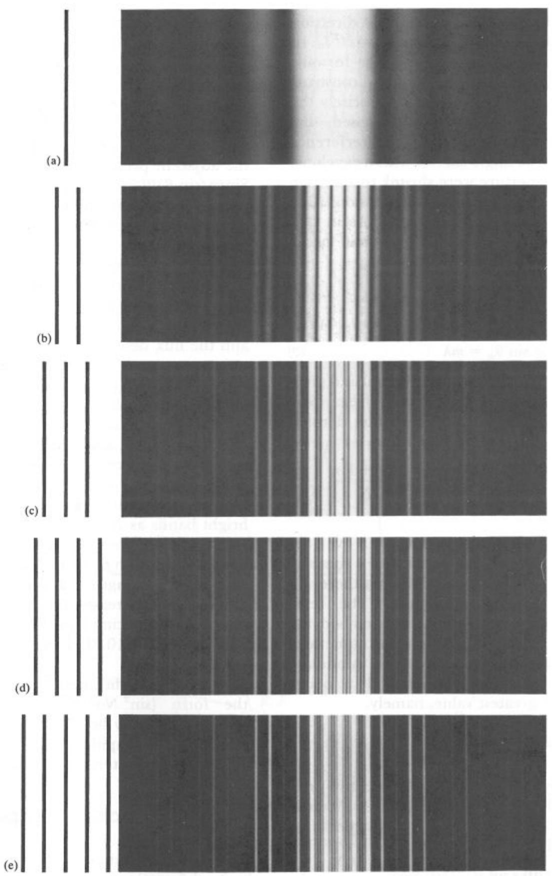
$$I_{\mathrm{N\,slit}} = I_0 \mathrm{sinc}^2(\pi \nu_x b)\left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right]^2$$

$$\delta = k_0 a \sin\theta \approx \frac{2\pi}{\lambda} a\theta \approx \frac{2\pi}{\lambda} a \frac{x}{z} = 2\pi a \nu_x$$

$$I_{\mathrm{N\,slits}} = I_0 \mathrm{sinc}^2(\pi \nu_x b)\left[\frac{\sin(\pi N a \nu_x)}{\sin(\pi a \nu_x)}\right]^2$$

Fraunhofer Diffraction of 1 - N Slits

$$E(x, y, z) = \Im\{E_{inc}(x', y')\}$$



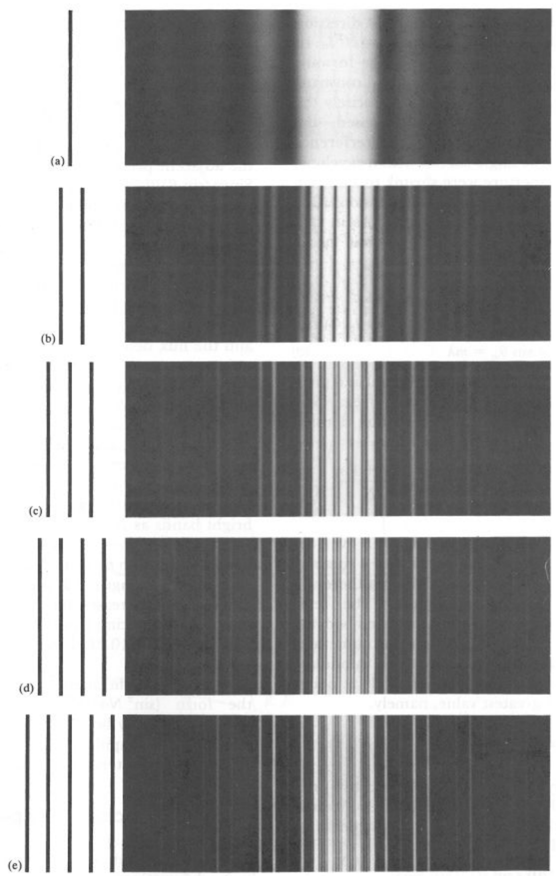
$$I_{\text{single}} \propto \text{sinc}^2(\alpha)$$

$$I_{double} \propto \text{sinc}^2(\alpha)\cos^2(\beta)$$

$$I_N \propto \frac{\sin^2(Na)}{\sin^2(a)}\text{sinc}^2(\alpha)$$

Fraunhofer Diffraction of Slits in 1D and 2D

one, two, …, N slits



1D

2D

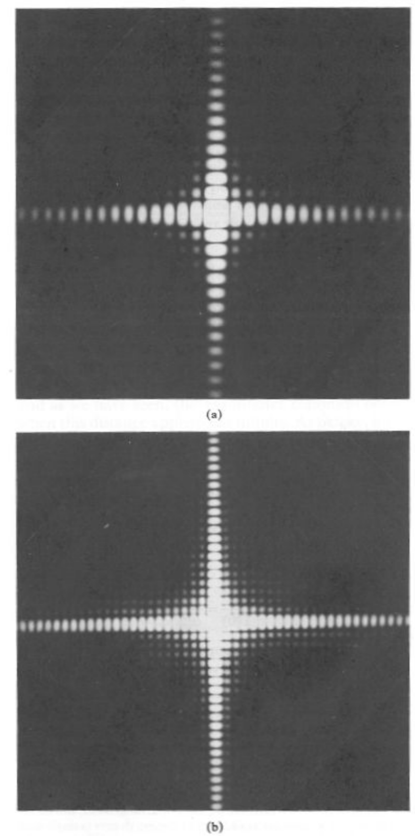
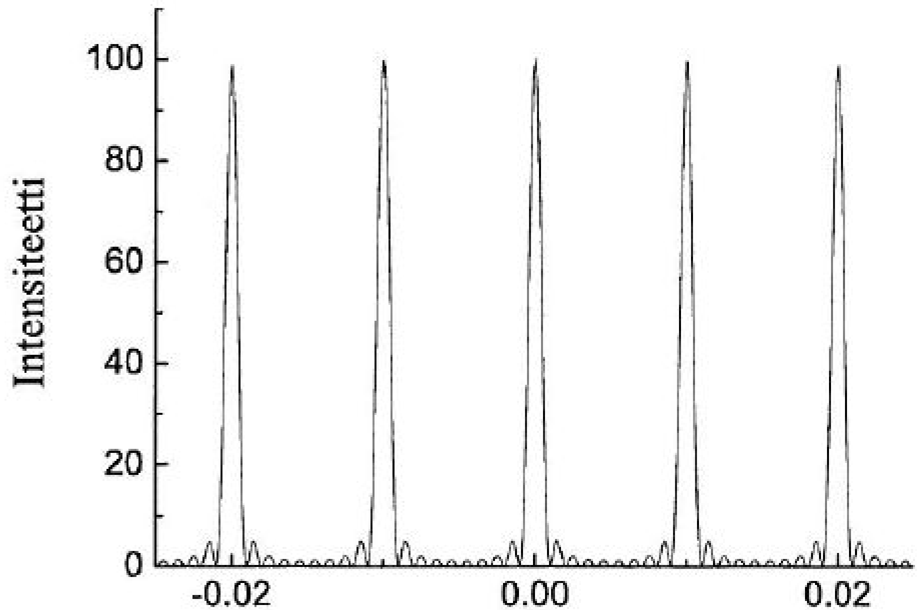


Figure 10.24 (a) Fraunhofer pattern of a square aperture. (b) The same pattern further exposed to bring out some of the faint terms. (Photos by E. H.)

Diffraction Grating

$$I(x/z) = \frac{\sin^2(\pi N \frac{d}{\lambda} \frac{x}{z})}{\sin^2(\pi \frac{d}{\lambda} \frac{x}{z})} \operatorname{sinc}^2\left(2\pi \frac{x_0}{\lambda} \frac{x}{z}\right)$$



Intensity gets maximum value when:

$$d\nu_x = \frac{dx}{\lambda L} = m$$

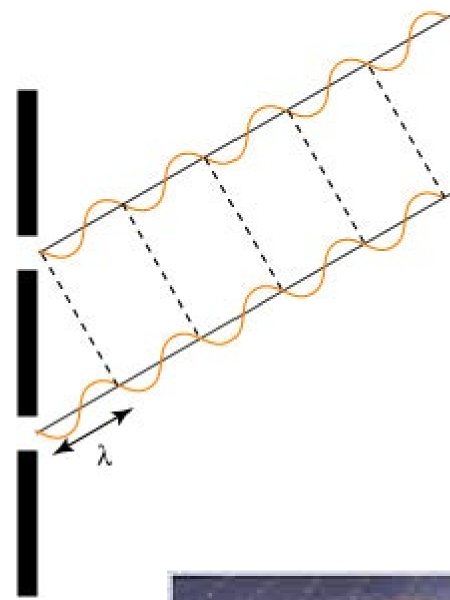
where m is an integer and d the distance between slits

$$\theta \approx \frac{x}{L} \approx \sin\theta$$

$$\Rightarrow$$

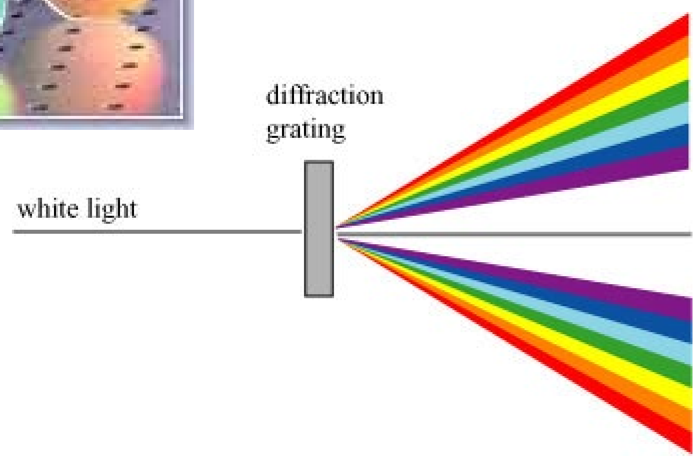
$$d \sin\theta_m = m\lambda$$

Diffraction Grating – Physical Approach



transmission grating

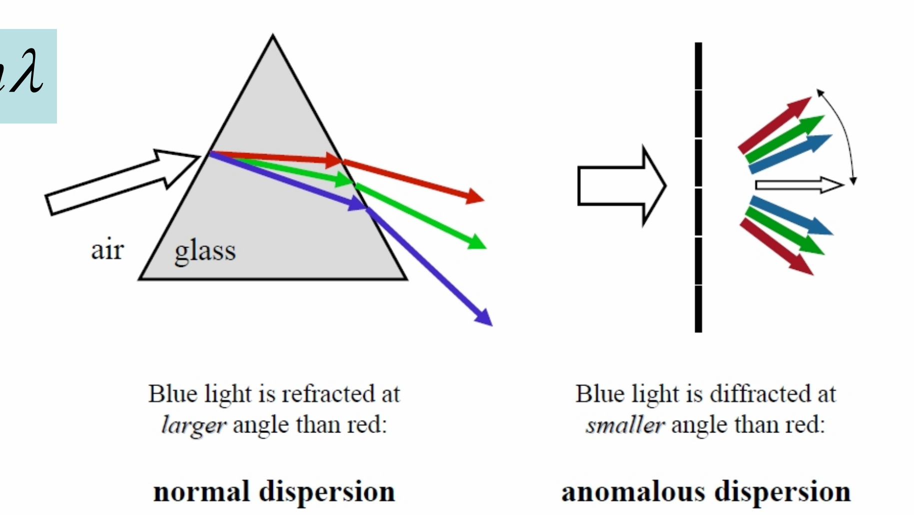
reflection grating



Properties of Diffraction Grating

grating equation: $a(\sin\theta_m - \sin\theta_i) = m\lambda$

angular dispersion: $\frac{d\theta_m}{d\lambda} = \frac{m\cos\theta_m}{a}$



wavelength resolution or smallest measurable change in wavelength:

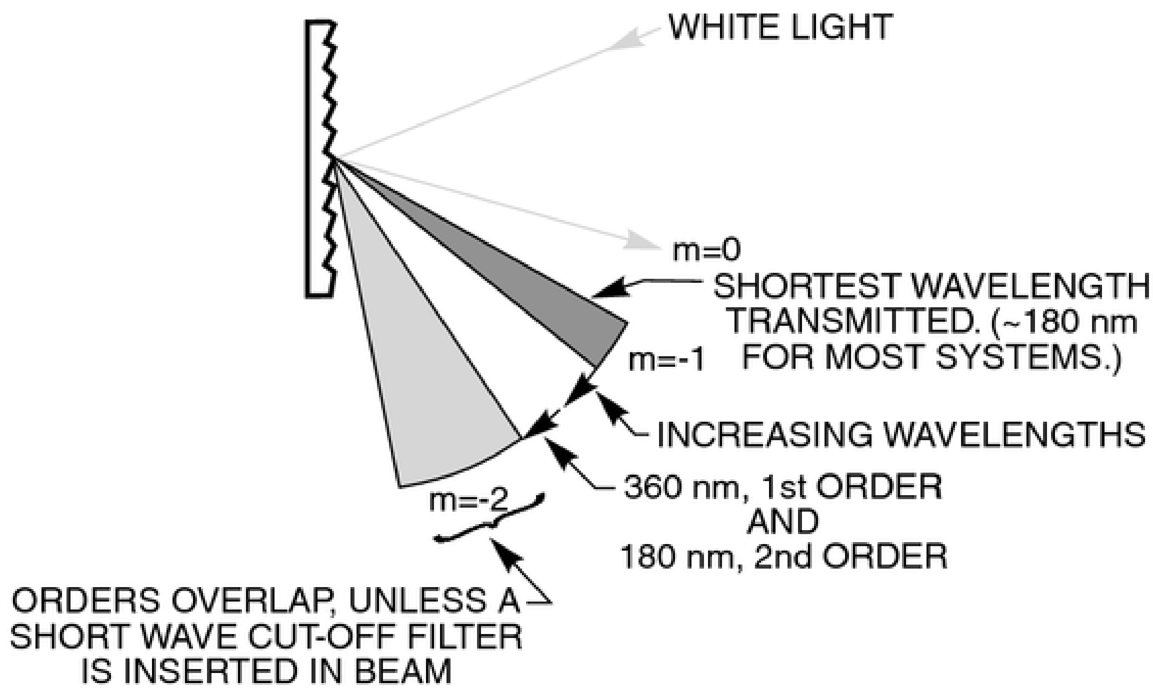
$$\frac{\Delta\lambda_{MIN}}{\lambda_0} = \frac{\lambda}{Na(\sin\theta_m - \sin\theta_i)}$$

where N is the number of slits illuminated

Example: 150 mm wide grating which has 600 grooves/mm is completely illuminated. N = 90 000 so for the 2nd order diffraction (m=2) $\Delta\lambda/\lambda_0$=180 000. If the centre wavelength of incident light is 540 nm then the wavelength resolution of this measurement is 0.003 nm or 3 picometers (cf. FTIR resolution 1/Δx cm^{-1}).

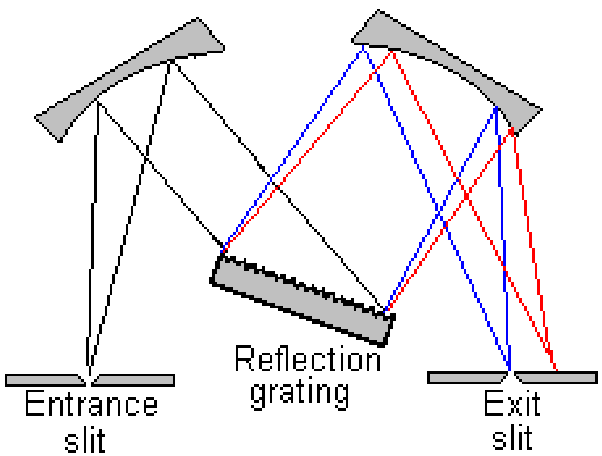
Properties of Diffraction Grating

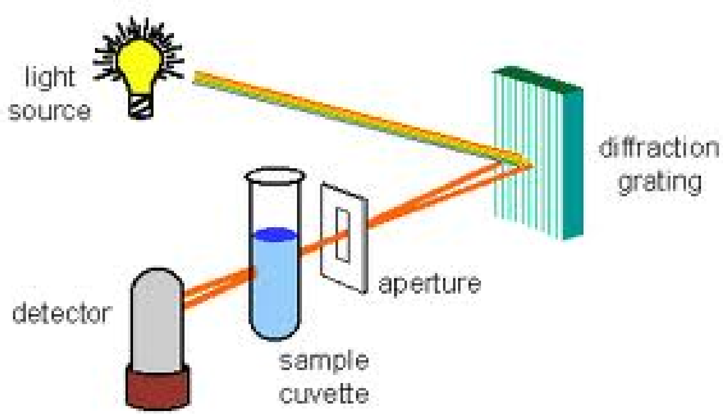
Limitation of diffraction grating: for spectrally broadband light the diffraction orders overlap if the lights' spectral bandwidth is larger than one octave, e.g, 400-800 nm covers one octave:



Grating Monochromator and Grating Spectrometer

monochromator





spectrometer