Lecture 3

Topics: Level curves, review of limits and continuity in 1 variable. Limits is 2 variables. Intro to partial derivatives.

- Introduced the concept of a level curve (and level surface). Talked about familiar applications such as the contour lines on topographic maps and isotherms on weather maps and so on.
- Viewed some surface and level curves on maple (code is in the materials sections).
- Emphasized that a curve is the graph of a function of one variable and the level curve of function of two variables. eg y = f(x) = x + 1 or F(x,y) = y -x = 1.
- Reviewed limits in 1 variable. Left and right hand limits. Looked a the example sin(x)/x.
- Reviewed continuity in one variable
- Introduced limits of two variables and noted that there are now infinitely-many paths along which a point can be approached.
- Looked at the examples xy/(x^2 + y^2) and 2x^2y/(x^4 + y^2). In both cases we found the limit at (0,0) does not exist. We then looked at 3D computer plots of the surfaces to understand what the surfaces look like near the point (0,0) which is not in their domains. And also to understand the how limits along different curves give different values. The Maple code and pdf can be found in "materials".
- Showed that limit at (0,0) of x^3/(x^2 + y^2) exists and is equal to zero by using the "Squeeze Theorem".
- Introductory ideas on the "derivative of a function of two variables"
- Extra material on smooth curves (useful for homework 2, #3) - this was not covered during the lecture

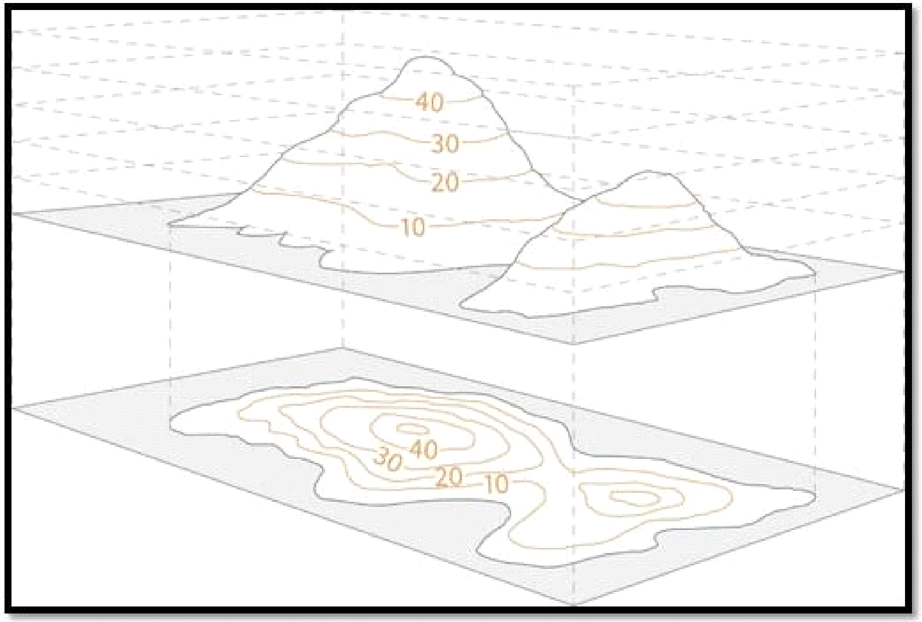
Where to find this material

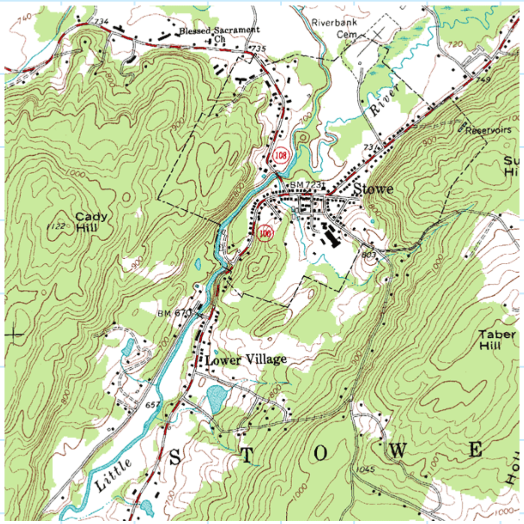
- Adams and Essex
  - 1.2 - 1.5 (we only revied a small part of this but it is background you should know).
  - 12.1 - 12.3
- Corral, 2.1, 2.2
- Guichard, 2.3, 14.1, 14.2, 14.3
- Active Calculus.
  - 9.1, 10.1, 10.2
  - For review of single variable limits see sections 1.2 and 1.7 of this book in the same series https://activecalculus.org/single/]

Note: The content added during the live lecture is all in orange

Level curves

This is actually a familiar idea. We want to represent a surface in terms of a plot in the xy-plane





A level curve (set) of $f(x,y)$ at level c is the set of points

$$\{ (x,y) \mid f(x,y) = c \}$$

That is, the solutions to $f(x,y) = c$

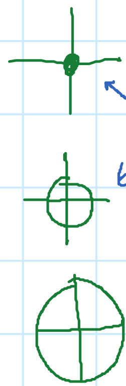
Example Sketch some level curves of $f(x,y) = x^2 + y^2$

$f(x,y) =$ negative value No solutions

$f(x,y) = 0$

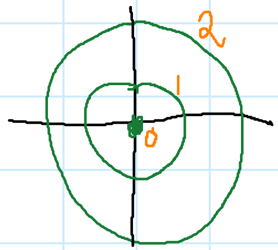
$f(x,y) = 1$

$f(x,y) = 4$



level curves

Combining these



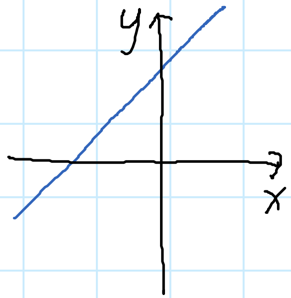
contour plot

Level curves (2)

A conceptual point (useful later in the course)

Let $g(x) = x+1$ which is a function of one variable

The graph of $g(x)$ is

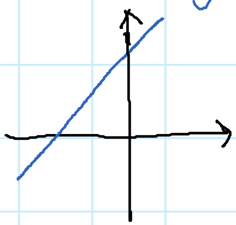


We can rewrite this line as follows

$$y = x+1 \iff \underbrace{y-x}_{f(x,y)} = 1$$

Let $f(x,y) = y-x$ which is a function of two variable

The level curve $f(x,y)=1$ is the same line



Level surfaces

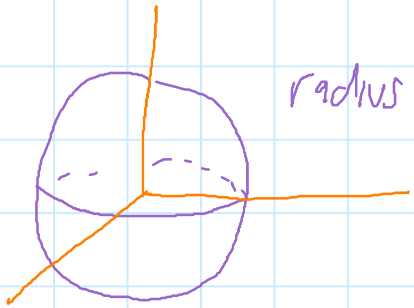
If we have a function of 3 variable, f(x, y, z), then we can not sketch its graph as we would need a 4th dimension to record the values of f.

However, we can sketch the level set (surface) of f(x,y,x)

Example Let $f(x,y,z) = e^{x^2+y^2+z^2}$ sketch the level surface $f=6$

$$e^{x^2+y^2+z^2} = 6$$

$$\Rightarrow \quad x^2+y^2+z^2 = \ln(6) \quad \text{sphere}$$



radius $= \sqrt{\ln(6)}$

Let's now look at some computer plots (code in MyCourses)

Limits (1 variable review)

Intuitive defintion of a limit:

We say $\lim_{x \to a} f(x) = L$ if and only if f(x) can be made as close to L as we wish by choosing x sufficiently close to a.

(note: this says nothing about that falue $f(a)$, in fact it does not even need to be defined)

Formal defintion (not required for this course)

$\forall \varepsilon > 0, \exists \delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Fact:

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = L$

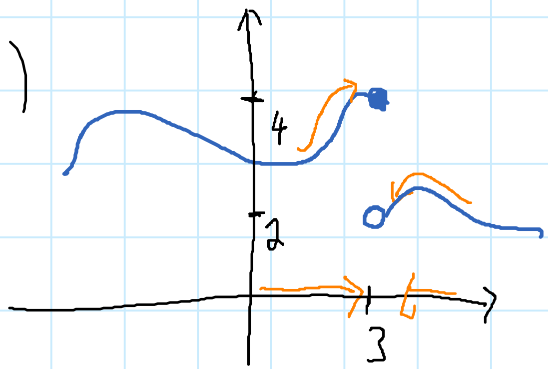
from the right   from the left

Continuity:

$f(x)$ is continuous at $x = a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$

Examples

(1)



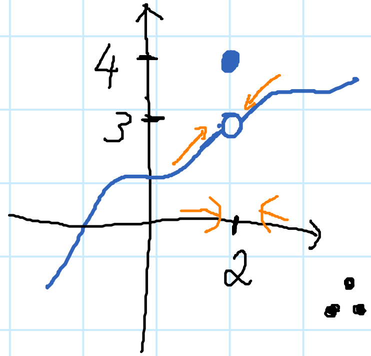
$\lim_{x \to 3^-} f(x) = 4$

$\lim_{x \to 3^+} f(x) = 2$

$\therefore \lim_{x \to 3} f(x)$ does not exist

and $f(x)$ is not continuous at $x=3$

(2)



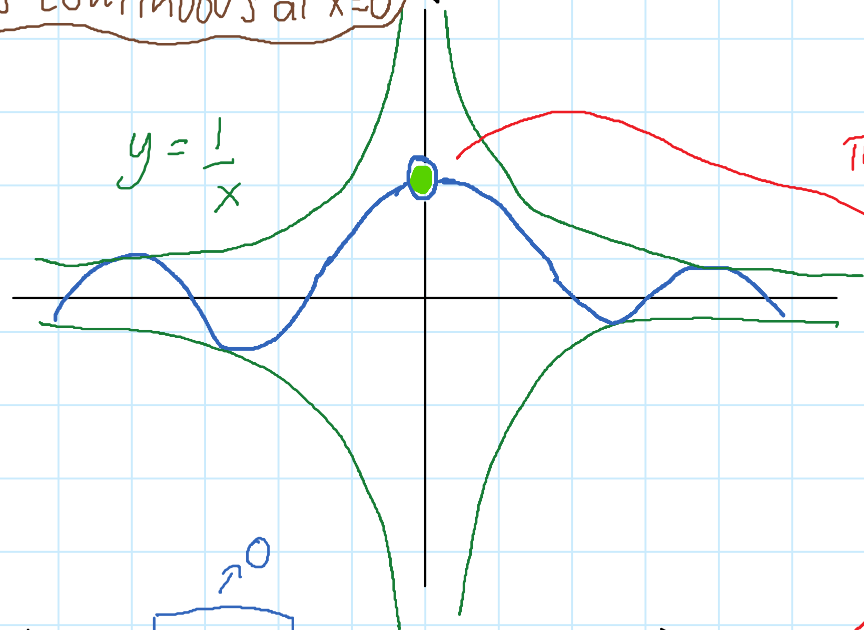
$\lim_{x \to 2} f(x) = 3$

$f(2) = 4$

$\therefore$ f is not continuous at $x=2$

(3) Let $f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Show that f(x) is continuous at x=0



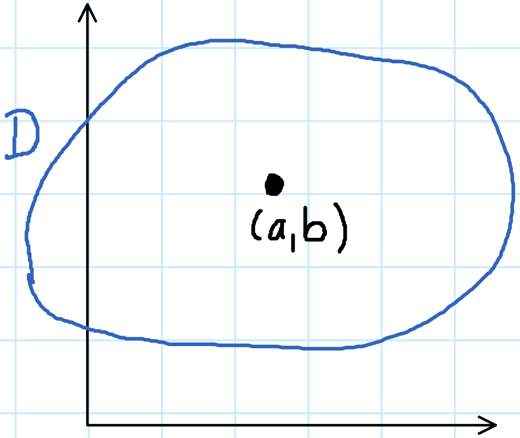
Therefore

$$\lim_{x \to 0} \frac{\boxed{\sin(x)}^{\nearrow 0}}{\boxed{x}_{\searrow 0}} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$$

by L'hopital's rule

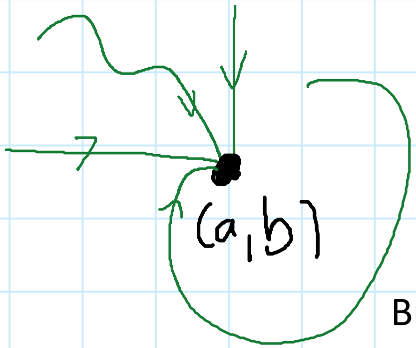
Limits in 2 variables

Let $f(x,y)$ be a function with domain $D \subseteq \mathbb{R}^2$
Let (a,b) be a point in D.



We say
$$\lim_{(x,y)\to(a,b)} f(x,y) = L \text{ if}$$
and only if f(x,y) can be made as close to L as we wish by choosing (x,y) sufficiently close to (a,b).

Important difference to the 1 variable case!!
There are infinitely-many ways to appraoch (a,b)



So we can NOT show that a **limit exists** by testing paths

But, we can show a **limit does not exist** by finding path with different limits (analagous to the left/right limits for 1-variable)

Examples

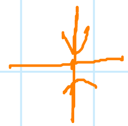
① $$\lim_{(x,y)\to(0,1)} \frac{\underbrace{x}_{\to 0} + \underbrace{x^2y}_{\to 0} + \underbrace{3}_{\to 3}}{\underbrace{\sin(\underbrace{xy}_{\to 0})}_{\to 0} + \underbrace{4y}_{\to 4}} = \frac{3}{4}$$

② $$\lim_{(x,y)\to(0,0)} \frac{\underbrace{xy}_{\to 0}}{\underbrace{x^2+y^2}_{\to 0}}$$

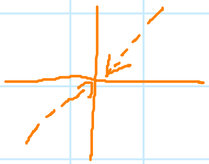
Test paths

- Along $y=0$ (x-axis): $\displaystyle\lim_{\substack{x\to 0\\ y=0}} \frac{\underbrace{xy}_{=0}}{x^2+y^2} = 0$

- Along $x=0$ (y-axis) $\displaystyle\lim_{\substack{y\to 0\\ x=0}} \frac{\underbrace{xy}_{\to 0}}{x^2+y^2} = 0$



- Along the line $y = x$ $\displaystyle\lim_{x\to 0} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0}\frac{x^2}{2x^2} = \frac{1}{2}$

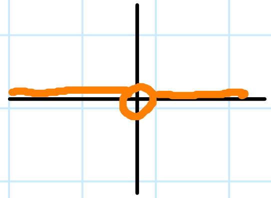


We found different limits along different paths, and therefore **the limit does not exist**.

Limits in 2 variables (2)

③ $$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$$

TEST PATHS :

graph 

- Along the x-axis ($y=0$): $\lim_{x\to 0} \frac{0}{x^4} = 0$

- Along the y-axis: $\lim_{y\to 0} \frac{0}{y^2} = 0$

- Along the line $y=x$: $\lim_{x\to 0} \frac{2x^3}{x^4+x^2}$

$$= \lim_{x\to 0} \frac{2x}{x^2+1}$$ Factor out x^2

$$= 0$$

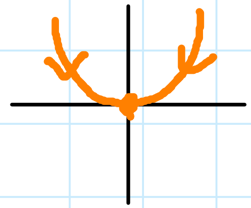
ok.. this is getting boring.. let's be smart and test all lines at once (except $x=0$) by setting $y = kx$

- Along the line $y = kx$:

$$\lim_{x\to 0} \frac{2x^2(kx)}{x^4+k^2x^2} = \lim_{x\to 0} \frac{2kx}{x^2+k^2}$$

$$= 0 \text{ for all values of } k$$

So we see that the limit is 0 along all lines. But it would be a mistake to conclude that the limit is in fact 0. How can this be!!!!!

- Along the path $y = x^2$ 

$$\lim_{\substack{x\to 0\\ y=}} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^4}{x^4+x^4}$$

$$= \lim_{x\to 0} 1$$

$$= 1$$

We conclude that The limit does not exist ......

Limits in 2 variables (3)

(4) $\lim_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2}$

We will use the squeeze theorem to show that this limit exists.

If $a(x) \le b(x) \le c(x)$ and $\lim_{x\to x_0} a(x) = \lim_{x\to x_0} c(x) = L$ then $\lim_{x\to x_0} b(x) = L$

Similarly in 2-variables

Think!

- $y^2 \geq 0$, so $x^2+y^2 \geq x^2$
  $\Rightarrow \dfrac{1}{x^2+y^2} \le \dfrac{1}{x^2}$
- $0 \le \left|\dfrac{x^3}{x^2+y^2}\right| \le \left|\dfrac{x^3}{x^2}\right| = |x|$

  $0 = 0$, $\quad |x| \to 0$ as $x \to 0$

So by the squeeze theorem

$$\frac{x^3}{x^2+y^2} \to 0 \quad \text{as } (x,y)\to(0,0)$$

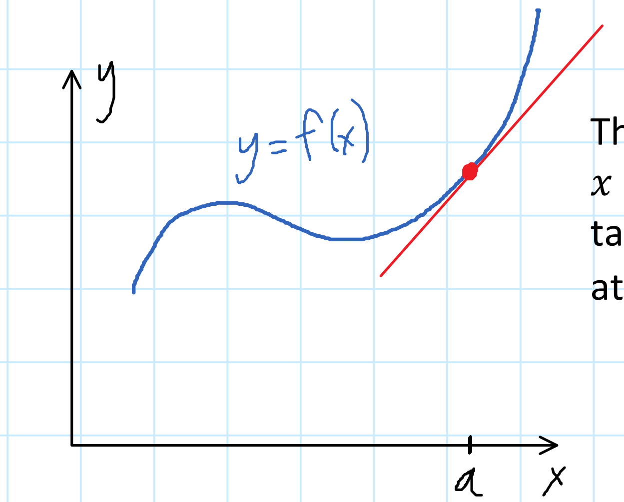
For the examples we just did let's now:

1. Look at plots of the surfaces to help understand what is going on
2. Also look at the level curves to get a different way of understanding. (This is related to Assigments 2, question #6)

The Maple code and output plots are available on MyCourse

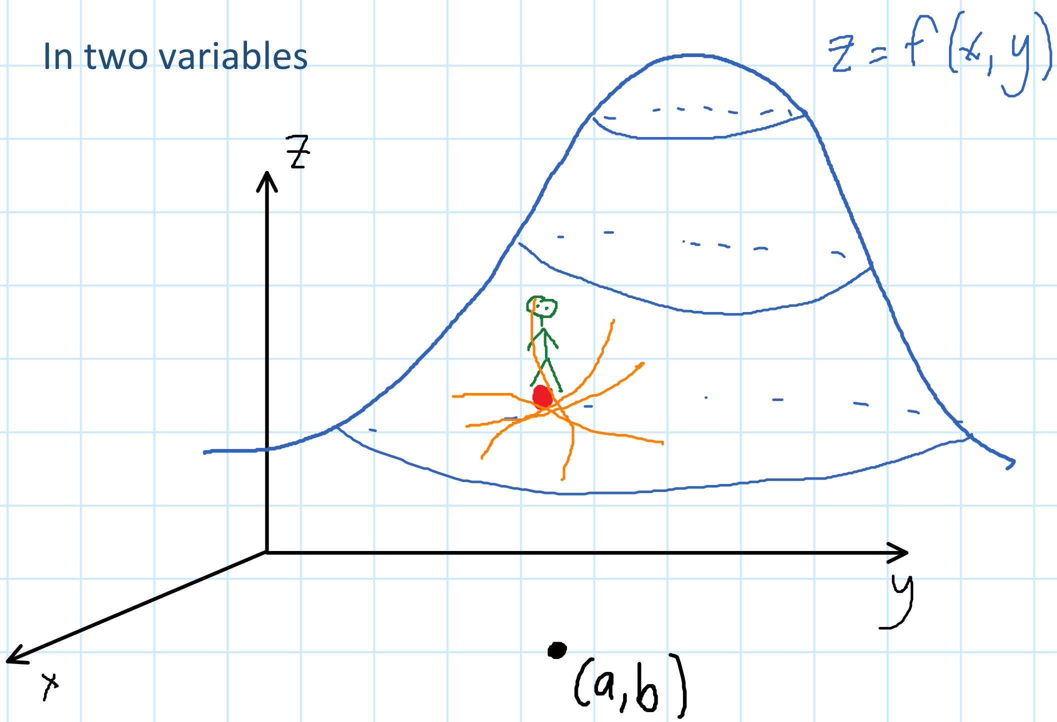
Intro to derivatives of a function of 2 variables

In one variable



The derivative of $f(x)$ at $x = a$ is the slope of the tangent line to the curve at the point $(a, f(a))$.

In two variables



Question:

What is the derivative of $f(x, y)$ at the point $(x, y) = (a, b)$? (go to zulip poll)

Does not make sense !

What does make sense:

What is the slope in a given direction

- "DIRECTIONAL DERIVATIVE"

Special cases - x-direction

y-direction

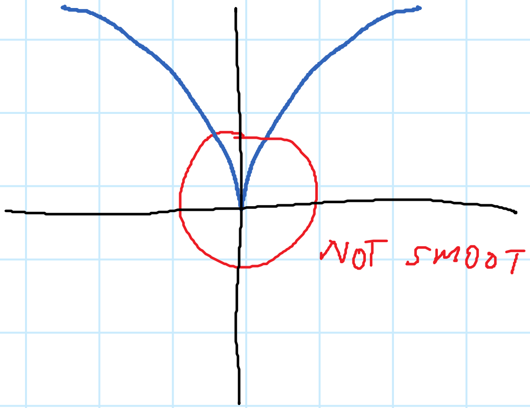
partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

Note about smooth curves

(Not covered during lecture - might help with exercises sheet 2, #3)

Example



$$y = x^{2/3} \left[= (x^2)^{1/3} \right]$$

$$y'(x) = \frac{2}{x^{1/3}}$$

$$\to \infty$$

as $x \to 0$

So $y'(x)$ is not defined at $x = 0$

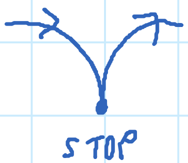
Parametrize $\quad x = t^3, \quad y = t^2$

these are smooth functions of t.

But $\quad \vec{r}\,'(t) = \langle 3t^2, 2t \rangle$

$$\vec{r}\,'(0) = \langle 0, 0 \rangle$$

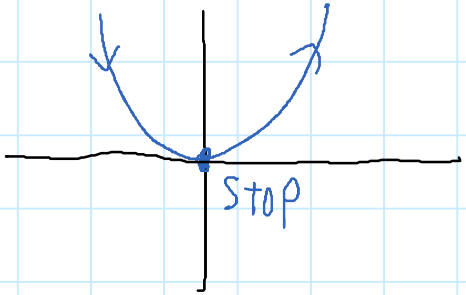
Motion comes to a stop at the non-smooth point



Example $\quad x(t) = t^2, \; y(t) = t^4$

$$\vec{r}\,'(0) = \langle 0, 0 \rangle$$

but this curve is smooth (parabola)



Fact:
At a point where the curve is not smooth, $\vec{r}\,'(t) = \langle 0,0 \rangle$ or $\vec{r}\,'(t)$ is not defined (does not exist)

But the converse is not true, as illustrated by the above example.