Lecture 4: The Solow growth model(s)

ECON-C3100 Intermediate Macroeconomics I

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20.1.2020

Introduction

- How can a nation become rich, i.e., initiate a growth process leading to higher GDP per capita Y/L in the long run?
- The basic **Solow model** provides some first answers: It predicts how the evolution and the long-run levels of GDP and consumption per capita depend on structural parameters such as the rate of investment and the growth rate of the labour force
- Key elements of the Solow model
  - In each period, output is determined by capital and labour through the production function
  - Exogenous savings/investment rate, s, and exogenous depreciation rate, δ
  - Explicit description of capital accumulation
  - Accumulation of capital is the main driving force for wealth in the simplest version of the model but later Solow had to add exogenous technolgical growth for the model to give more realistic results

The production side

- is modelled as in Lecture 3
- We will use either the generic form of the macroeconomic production function $Y = F(K, L)$ or the Cobb-Douglas specification $Y = K^{\alpha} L^{1-\alpha}$

Capital accumulation

- Recall the production function in its intensive form: Output per capita/labour $Y/L = y$ only depends on $K/L = k$

$$y = f(k) = k^{\alpha}$$

- This suggests that a good place to start, if we want to explain sustained growth in y, is to understand why and how the capital stock rises over time
- The roots of this rise in the capital stock have to be in investments on new capital and in the savings that are used to finance the new investments

Capital accumulation

- Recall from Lecture 2 the savings-investment balance in the macroeconomic accounts (the key accounting identity)

$$(S - I) + (T - G) = (X - Z)$$

- This can be rearranged to yield

$$I = S + (T - G) + (Z - X)$$

i.e. investments can be financed either by savings by the private sector, net savings of the government (a budget surplus) or the net savings of foreigners

- In the long run (in the steady state), we can assume that the government budget is in balance and that the current account is in balance ($\approx$ net exports $= 0$).
- This means that investments in this economy are ultimately financed by savings by the resident households $I = S$
- First explanation of the growth phenomenon: we save $\Rightarrow$ we invest $\Rightarrow$ we grow

Capital accumulation

- As a first approximation, let s be a constant fraction of GDP which is used to finance investments $I = S = sY$. In intensive form

$$I/L = sY/L = sy = sf(k)$$

- Capital accumulation, or the process by which the capital stock grows as a result of investments in new production equipment, can be described as follows

$$\Delta K = sY - \delta K$$

where δ is the depreciation rate of existing capital. In intensive form (divide through by L)

$$\Delta k = sy - \delta k = sf(k) - \delta k$$

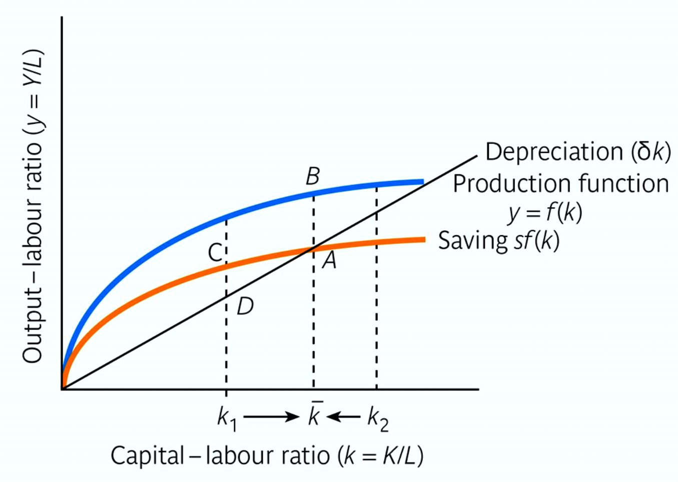
- The net accumulation of capital per unit of labour is positively related to the savings rate s and negatively related to the depreciation rate δ

Depreciation

- A *certain fraction* δ of the previously installed productive equipment/capital (per unit of labour) k depreciates in each period.
- The depreciation rate for the overall economy is fairly stable and will be taken as constant: the more capital is in place, the more depreciation will occur.
- $\Longrightarrow$ The relationship is linear

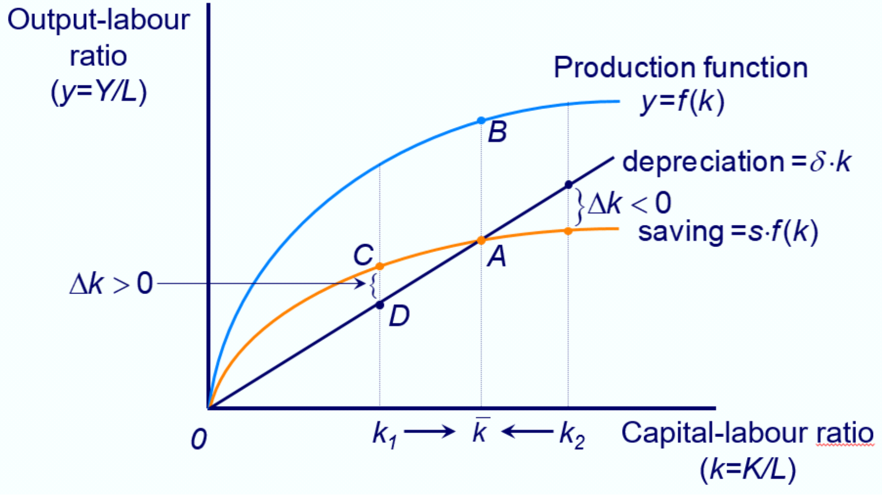
The Steady State of the Solow Model

The Solow Diagram



The Steady State of the Solow Model

The Solow Diagram



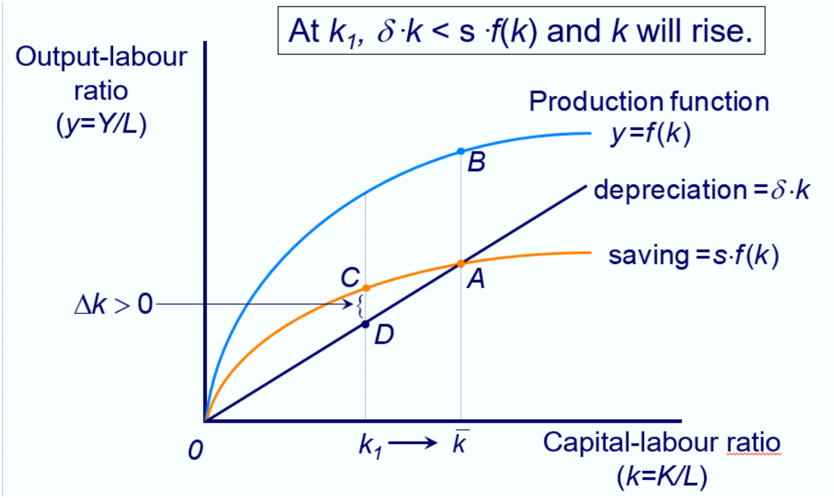
The Steady State of the Solow Model

Interpreting the Solow Diagram

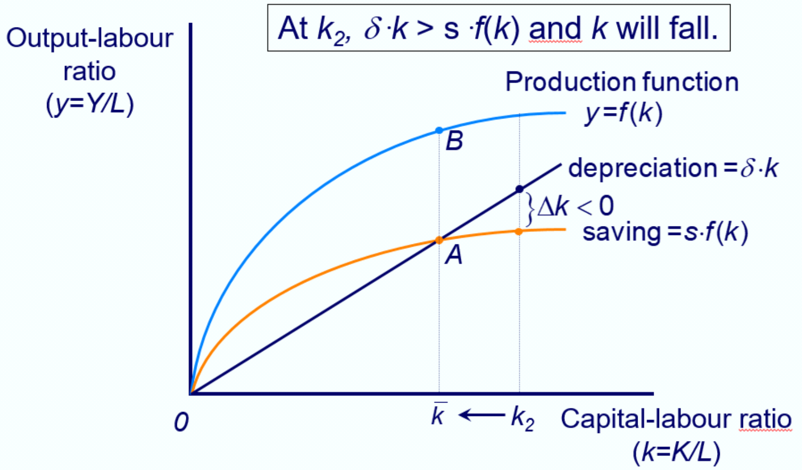
- Δk is the vertical distance between the savings schedule $sf(k)$ and the depreciation line δk
- The sign of Δk tells us in which direction the economy is heading
- At the intersection of the saving schedule and the depreciation line (point A), newly accumulated capital exactly compensates that lost to depreciation, so the capital-labour ratio no longer changes: $\Delta k = 0$
- This is the steady state. Regardless of where it starts, the economy moves towards the steady state and stays there
- The steady state condition is

$$\begin{aligned} \Delta k &= sy - \delta k = sf(k) - \delta k = 0 \\ \iff & \quad sf(k) = \delta k \end{aligned}$$

The Steady State: k below the steady state value

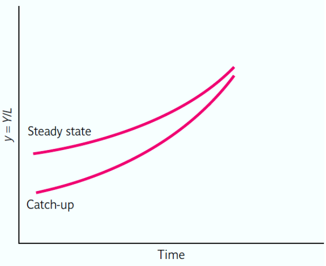


The Steady State: k above the steady state value



Convergence

- As the economy gets closer to point A, net investment becomes smaller and smaller, and equals zero when $k = \overline{k}$
- At the same time, additional output from increasing k gets smaller (diminishing returns!)
- Catching up occurs when a country starts below its steady state GDP (lower curve) and embarks on a faster growth path, approaching the steady state (the upper curve) from below



The Role of Savings gor Growth

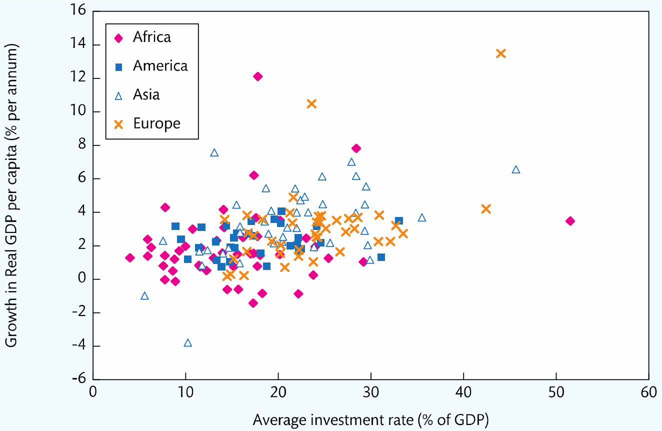
- First explanation of the growth phenomenon: we save $\Rightarrow$ we invest $\Rightarrow$ we grow
- Would suggest that countries that save and invest a lot have high per capita incomes
- Is this proposition supported by the data?

Investment and GDP per capita

Investment rate and real GDP per capita (level)

Investment and Growth in GDP per capita

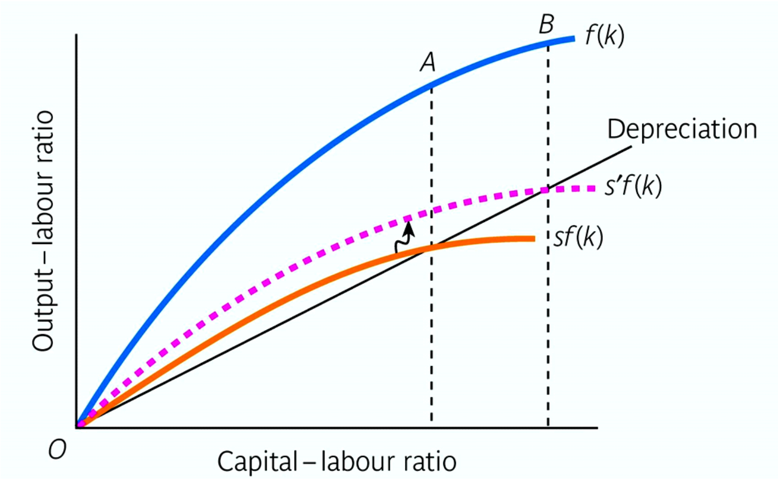
Investment rate and real growth in GDP per capita (% per annum)



Savings and Growth

- Link between savings and growth not very strong empirically
- Seems that savings and investment are more closely linked to steady state *levels* of GDP rather than to the *growth* of GDP
- Countries which save more should have higher economic well-being but they do not necessarily grow faster, indefinitely
- This is exactly what the Solow growth model suggests

Effect of an increase in the savings rate



Effect of an increase in the savings rate

- The savings investment schedule shifts upwards
- The new capital-labour ratio and output per labour ratio are both higher in the new steady state (point B), but adjustment takes time
- During adjustment growth is faster because the economy starts from below the new (higher) steady state -> convergence
- However, increased savings do not affect long-run growth

The 'government sector' in the Solow model

- The constant saving rate s of the Solow model can be interpreted to be the sum of private and public consumption
- The government can determine the saving rate s by taxation and public saving
- Is there a welfare-maximizing level of the saving rate?
- To what level should the government adjust the saving rate if it has the political freedom to do so?
- We call this case **the Golden Rule**

The Golden Rule

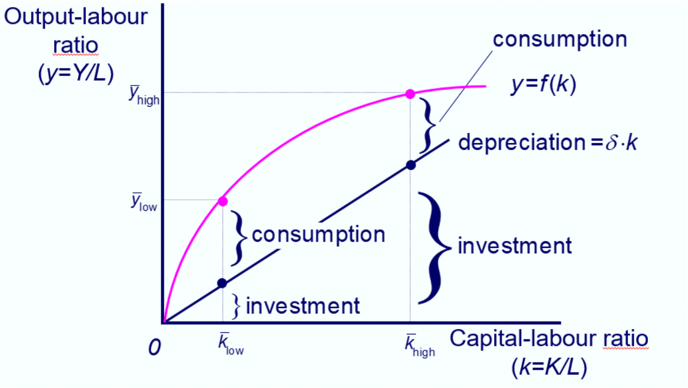
- In the model, consumption per worker, $c = 1 - s$, can be used as a measure of welfare
- In the steady state, per worker consumption is equal to $y - sy$, i.e. per worker income y minus per worker savings

$$\overline{c} = \overline{y} - s\overline{y} = f\left(\overline{k}\right) - \delta\overline{k}$$

- The Golden Rule, i.e. the government's optimal public policy could be found by maximizing consumption wrt the capital-output ratio

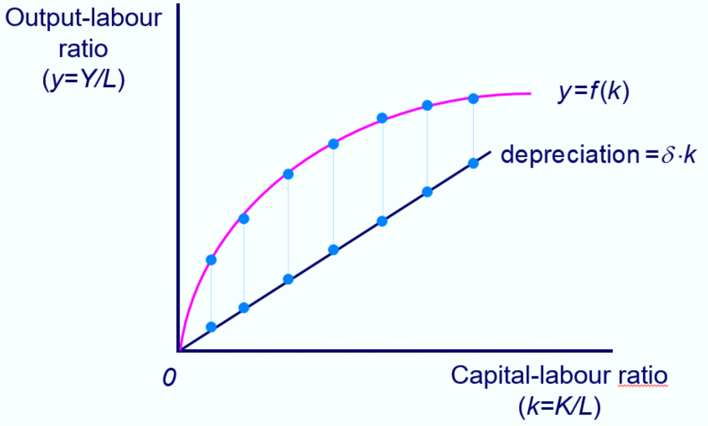
The Golden Rule

Steady state consumption is the vertical distance between the production function and the depreciation line



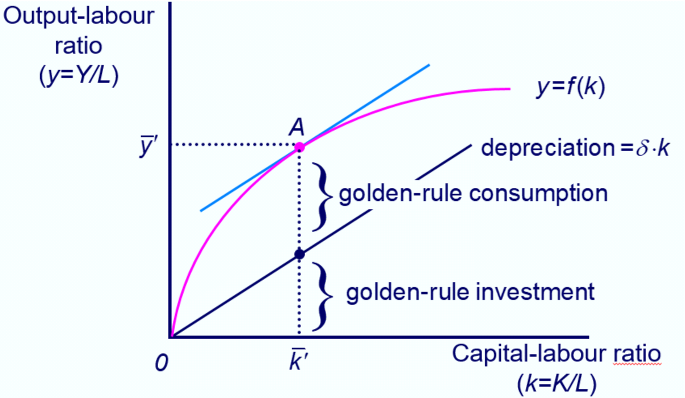
The Golden Rule

Steady state consumption is at maximum when the distance between the production function and the depreciation line is largest



The Golden Rule

The maximum can be found in point A where the slope of the production funtion, the marginal productivity of capital, is equal to δ, the slope of the depreciation line



The Golden Rule

The capital-labour ratio $\overline{k}$ that will maximize per worker consumption $f\left(\overline{k}\right) - \delta\overline{k}$ is determined by the first-order condition

$$\partial\overline{c}/\partial\overline{k} = f'(\overline{k}) - \delta = 0$$

This is equivalent to

$$f'(\overline{k}) = \delta$$

or

$$MPK = \delta$$

Steady state consumption is at maximum when the slope of the production function, the marginal productivity of capital, is equal to the depreciation rate (the slope of the depreciation line)

The Golden Rule

In the steady state it also holds true that (the steady state condition)

$$\bar{s}f\left(\bar{k}\right) = \delta\bar{k}$$

Combining these conditions yields

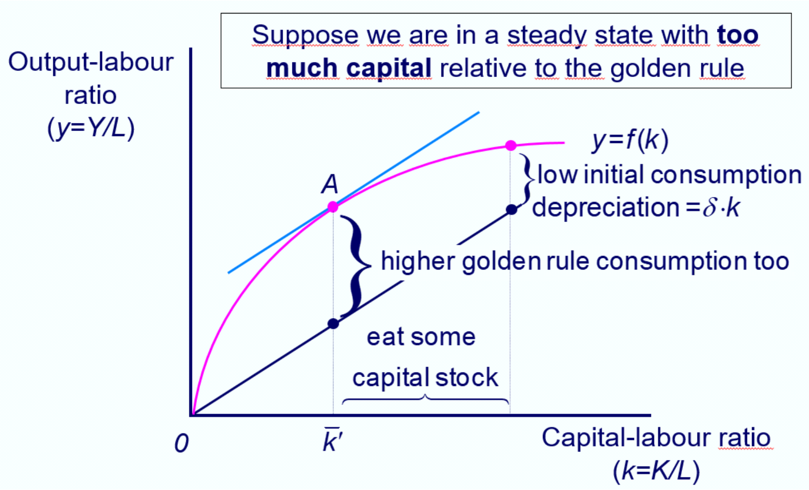
$$\bar{s}f\left(\bar{k}\right) = f'(\bar{k})\bar{k}$$

Solving for $\bar{s}$ gives the optimal saving by the government

$$\bar{s} = \frac{f'(\bar{k})\bar{k}}{f\left(\bar{k}\right)}$$

With a specific functional form, such as the Cobb-Douglas form, this can be computed explicitly

Dynamically inefficient saving rate



The Solow model and population growth

- Until now, we have implicitly assumed that L is constant. Now suppose

$$L_{t+1} = (1+n)\,L_t, \quad n > -1$$

- The capital accumulation equation with time subscripts

$$K_{t+1} = K_t + sY_t - \delta K_t$$

- Rewrite this capital accumulation equation wrt. the capital-labour ratio $k_t = \frac{K_t}{L_t}$ (divide through by $L_{t+1} = (1+n)L_t$)

$$\begin{aligned} k_{t+1} &= \frac{K_{t+1}}{L_{t+1}} = \frac{K_t + sY_t - \delta K_t}{(1+n)L_t} = \frac{k_t + sf(k_t) - \delta k_t}{1+n} \\ &= \frac{1}{1+n}\left(sf(k_t) + (1-\delta)k_t\right). \end{aligned}$$

The Solow model and population growth

- This is the basic law of motion, or the **transition equation,** for the capital-intensity k_t following from the basic Solow model
- The transition equation determines the full dynamic sequence of capital intensities
- Now rewrite the economy's law of motion as the **change** in the capital-labour ratio (subtract k_t from both sides of the transition equation)

$$\begin{aligned} k_{t+1} - k_t &= \frac{1}{1+n}\left(sf(k_t) + (1-\delta)k_t - (1+n)k_t\right) \\ &= \frac{1}{1+n}\left(sf(k_t) - (n+\delta)k_t\right) \end{aligned}$$

The Solow model and population growth

- This is the **Solow equation** which says that the capital-labour ratio increases with higher savings per capita and decreases with higher depreciation of capital or higher growth of the labour force
- The steady state is reached when $k_{t+1} = k_t$. This leads to the **steady state condition**

$$\frac{1}{1+n}\left(sf\left(k_t\right) - \left(n+\delta\right)k_t\right) = 0$$

$$\Longleftrightarrow sf\left(k_t\right) = \left(n+\delta\right)k_t$$

Savings have to equal replacement investment. In addition to depreciation, this has to cover now also the so-called capital-widening

The Steady State with Population Growth

- Using the same notation as previously (without time subscripts), we can note that the change in the capital-labour ratio is approximately equal to the change in capital minus the change in labour

$$\frac{\Delta k}{k} = \frac{\Delta K}{K} - \frac{\Delta L}{L}$$

- After substituting $\Delta K = I - \delta K$ and $\frac{\Delta L}{L} = n$ and setting $I = sY$, the equation simplifies to

$$\frac{\Delta k}{k} = \frac{sY - \delta K}{K} - n = s\frac{Y}{kL} - \delta - n = s\frac{y}{k} - \delta - n$$

$$\iff \Delta k = sy - \delta k - nk = sf\left(k\right) - \left(n + \delta\right)k$$

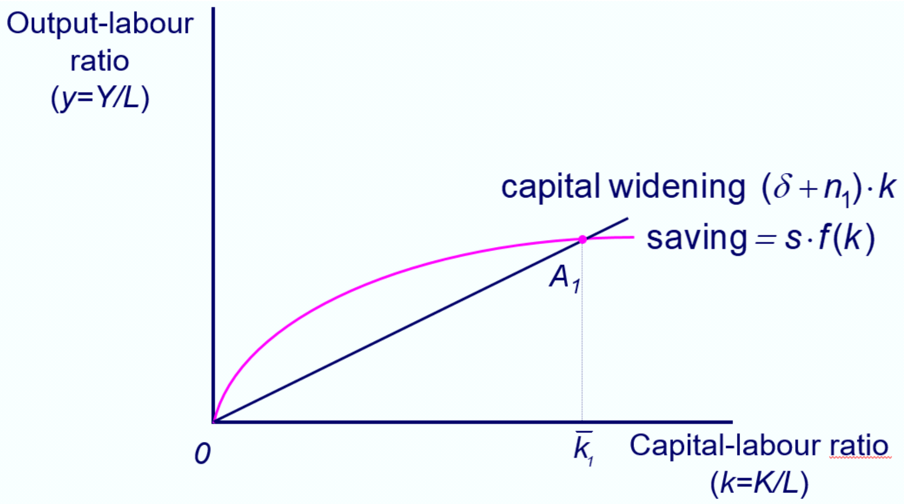
Steady State with Population Growth

- The steady state is reached when $\Delta k = 0$. The steady state condition is

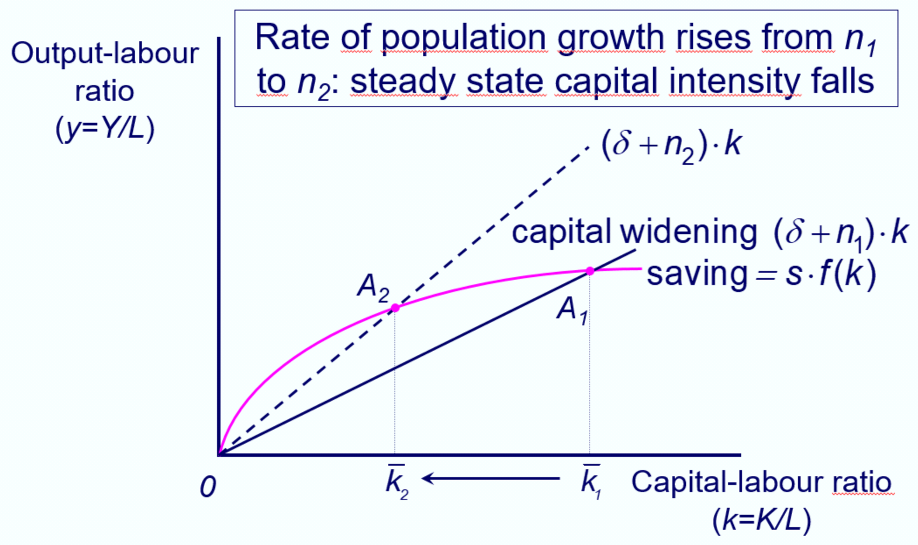
$$sf(k) = (n + \delta) k$$

- For the capital-labour ratio to increase, investment must not only compensate for depreciation, it must also provide new workers with the same equipment as those already employed

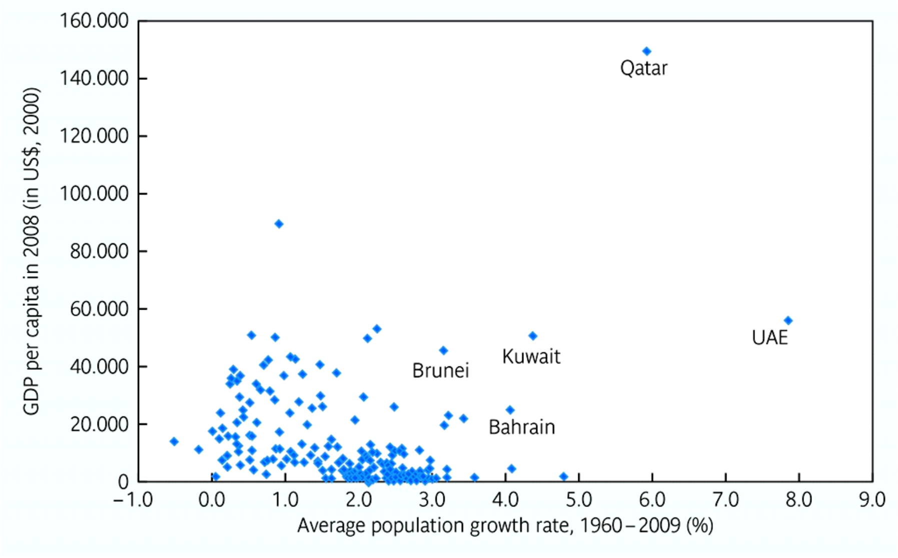
Steady State with Population Growth



Increase in the rate of population growth



Population Growth and GDP per Capita (190 Countries)



Growth in the basic Solow model

- The long run prediction of the Solow model is its steady state. What is the growth rate of GDP per capita in steady state?
- One way to find this:
- Take the steady state condition $s\bar{y} = (\delta + n)\bar{k}$ and insert a Cobb-Douglas production function (in intensive form $\bar{y} = \bar{k}^{\alpha}$)

$$s\bar{k}^{\alpha} = (\delta + n)\bar{k}$$

Growth in the basic Solow model

- Solve for the steady state capital/labour ratio

$$\bar{k} = \left(\frac{s}{\delta + n} \right)^{\frac{1}{1-\alpha}}$$

and, correspondingly, the steady state income per worker is

$$\bar{y} = \left(\frac{s}{\delta + n} \right)^{\frac{\alpha}{1-\alpha}}$$

- The steady state income per capita depends on constant parameters only...

Growth in the basic Solow model

- ...implying that steady state growth in income per capita y according to the basic Solow model is **Zero**! Not in accordance with stylized facts
- What is then the growth rate of Y in the steady state? Since GDP per capita $y = \frac{Y}{L}$ is constant, Y and L must grow at the same rate.
- GDP grows, but only at the same rate n as the labour force. Why is that?

Growth in the basic Solow model

- Assume that the economy is initially below the steady state, implying that $k < \bar{k}$. Then savings are larger than replacement investment $sf(k) > (n + \delta)k$, implying that capital per worker increases. But, because of **diminishing returns,** each additional unit of capital per worker will generate ever smaller increases in income and gross savings per worker.
- At the same time, the additional savings needed to compensate for population growth and depreciation increase, since k is increasing
- The growth in k and y will ultimately cease
- However, there is **transitory growth**
- In the Solow model, the transition towards steady state is at least as important as the steady state itself. During this transition there is growth in k and y. Hence, the basic Solow model can be regarded as a growth model

Summarizing the outcomes of the basic Solow model

- The basic Solow model implies that capital per worker and output per worker converge to particular *constant* values in the long run
- The convergence point defines the economy's steady state
- The expressions for the steady state values of the key variables predict how income per worker depends on underlying parameters in the long run
- Gives some answers to the question: What kind of policies can make a nation richer in the long run?
  - Increase the savings rate
  - Reduce the growth rate of the labour force
  - Reduce the rate of depreciation, i.e., invest better

Summarizing the outcomes of the basic Solow model

- According to the basic Solow model, to create a high level of consumption per person, the savings and investment rate should not exceed capital's share of income in production. = Golden Rule
- In the model's steady state there is no positive growth in GDP per worker, consumption per worker or the real wage. This is at odds with the stylized facts of growth
- Remedying this shortcoming is one of the main purposes of later growth models
- These models will contain some form of technological progress, such that total factor productivity (TFP) increases over time
- In the basic Solow model, transitory growth is important

The missing growth factor

- Strictly speaking, the basic Solow model is no growth model at all since the standard of living does not increase at all in the long run
- The simplest way of correcting this defect is to assume *technological change*
- Over time, increased knowledge and more sophisticated production equipment make workers and the equipment they work with more productive
- With a minor change to the aggregate production function, the Solow model can explain ever-rising living standards
- Think of technology as an additional factor of production

$$Y = F(\underset{+}{A}, \underset{+}{K}, \underset{+}{L})$$

- Technical progress is usually introduced in the following way

$$Y = F\left(K, AL\right)$$

Modifications to the basic Solow model

- The productivity (or efficiency) of labour is introduced as a multiplier A of *physical labour* L in the production function and it is assumed that this grows at a constant rate a
- Then one can define *effective labour* as follows

$$\begin{aligned}&\text{effective labour } (AL)\\&= \text{the productivity of labour } (A) \times \text{physical labour } (L).\end{aligned}$$

- The rate of growth of AL is $a + n$

Modifications to the basic Solow model

- Redefine y and k as ratios of output and capital relative to effective labour

$$k = \frac{K}{AL} \quad \text{and} \quad y = \frac{Y}{AL}$$

- The production function wrt capital per effective labour is

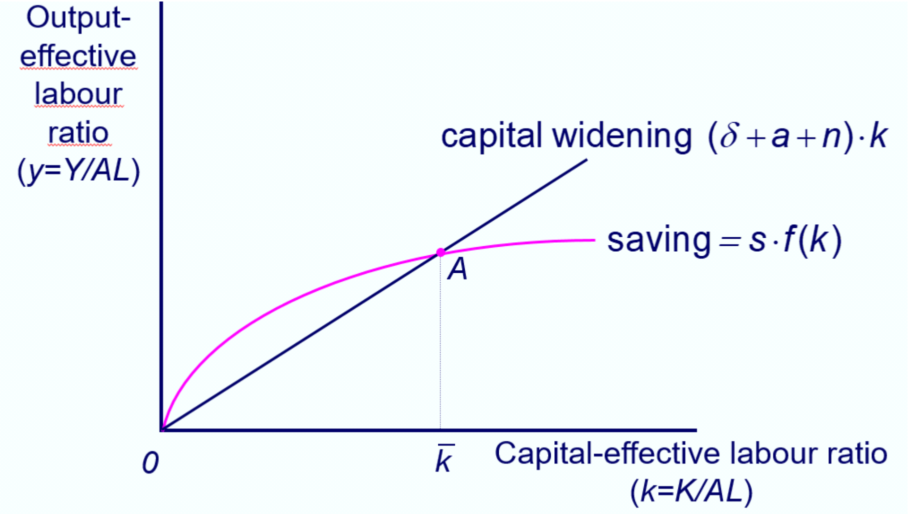
$$y = f(k) = k^{\alpha}$$

and the capital per effective labour ratio evolves almost exactly as before

$$\Delta k = sf(k) - (n + a + \delta) k$$

- To keep the capital effective labour ratio constant, the capital stock must also rise to keep up with workers' enhanced effectiveness

The Steady State with Population Growth and Technological Progress



Growth Rates Along the Steady State

- While output per effective worker and capital per effective worker are constant (zero growth rate) in the steady state, we need to keep in mind that these are constant ratios.
- The common denominator AL is growing approximately at a rate of $a + n$.
- Hence the numerators Y and K will be growing at $a + n$ in the steady state as well.

Growth Rates Along the Steady State