Lecture 6

Directional derivatives, the gradient vector, local extrema (2nd derivative test), absolute extrema

- Noted that (as long as f is differentiable) that D_u f = grad(f) dot u. where grad f = <f_x, f_y> is the gradient vector.
- It turns out that grad(f) is a very useful quantity. Some important facts (1) The direction of grad(f) is the direction of the maximum rate of change of f. (2) || grad f || is the maximum rate of change of the function. (3) grad is orthogonal to the level curve of f
- Used (3) to find the tangent plane to a surface.
- Reviewed critical points and the 2nd derivative test to classify extrema in 1 variable.
- Looked at f(x,y) = x^2 + y^2 - cxy. Noted how the type of critical point at (0,0) changes from a local min to a saddle as c changes from less than 2 to greater than 2. Looked at computer generated surfaces for different values of c (see "minimum to saddle" in Code and Images). We understood that the xy term contributes most along the line x = y. And when c > 2 the surface is "pulled down" so much along the x=y direction that the surface becomes a saddle.
- Stated the second derivative test for a function of 2 variables. The quantity f_xx f_yy - (f_xy)^2 is a little mysterious at first. The bonus exercise in homework#4 will help you understand - it is essentially a proof of the 2nd derivative test. The related later discussion of Taylor series/polynomials will help also.
- Did an example of finding and classifying local extrema
- Reviewed absolute extrema in 1 variable
- ~~Introduced closed and bounded domains and stated than on such a domain a contunuous function of 2 variables attaims its extrema.~~
- ~~Noted that finding absolute extrema breaks down into looking at the interior and the boundary. Finding extrema on the boundary is a type of constrained optimization problem. Next class we will learn a method for dealing with this - the method of Lagrange multipliers.~~

Where to find this material

- Adams_and_Essex 12.7, 13.1
- Corral, 2.4, 2.5
- Guichard, 14.5, 14.7
- Active Calculus. 10.6,10.7

Directional derivative (2)

Now let's look at the formula again

$$D_{\vec{u}} f(P) = a \frac{\partial f}{\partial x}(P) + b \frac{\partial f}{\partial y}(P) \quad \left(\begin{array}{l} \vec{u} = \langle a, b \rangle \\ \|\vec{u}\| = 1 \end{array} \right)$$

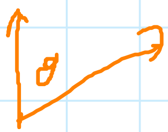
$$= \langle a, b \rangle \cdot \underbrace{\left\langle \frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P) \right\rangle}$$

$\vec{\nabla} f$ = Gradient of f

$$= \vec{u} \cdot \vec{\nabla} f(P)$$

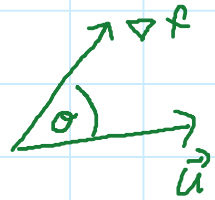
Note. This is true provided f is differentiable.

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$



Question: At a given point P, in which direction does $f(x, y)$ increase most raipdly?

(which $\vec{u}$ makes $D_{\vec{u}} f$ maximal)



$\boxed{\text{Answer}}$ $D_{\vec{u}} f = \vec{u} \cdot \vec{\nabla} f$

$$= \|\vec{u}\| \, \|\vec{\nabla} f\| \cos\theta$$

$$= \|\vec{\nabla} f\| \cos\theta$$

$\cos\theta$ has a max of 1 at $\theta = 0$

So, max of $D_{\vec{u}} f = \|\vec{\nabla} f\|$

and this occurs when $\vec{u}$ is parallel to $\vec{\nabla} f$

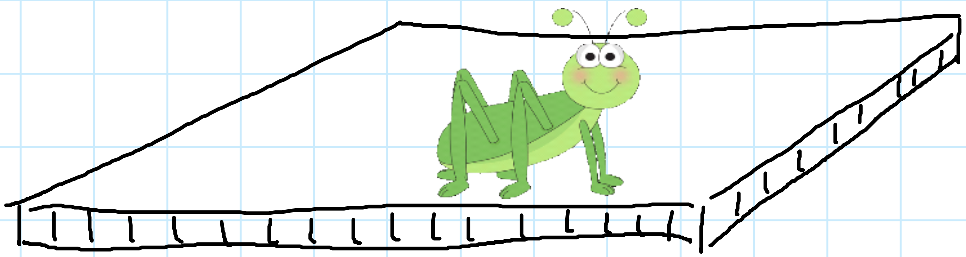
(that is, $\vec{u} = \dfrac{\vec{\nabla} f}{\|\vec{\nabla} f\|}$)

Note Minimum occurs when $\theta = \pi$.

So $\vec{u} = -\vec{\nabla} f / \|\vec{\nabla} f\|$ and

$$D_{\vec{u}} f = -\|\nabla f\|$$

Gradient vector example



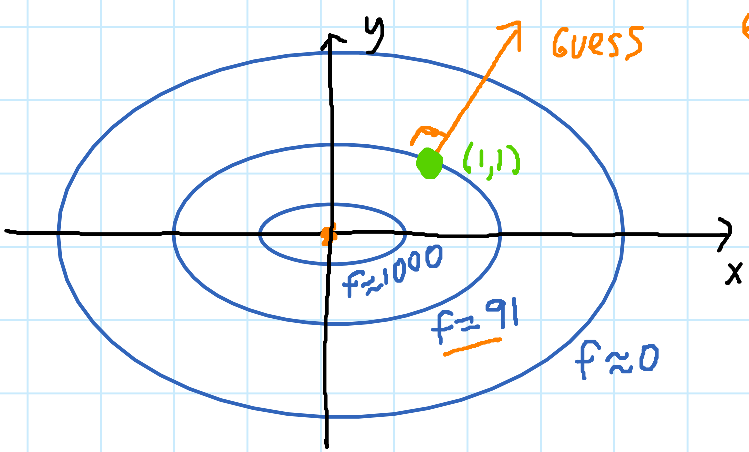
A happy bug accidentally lands on a hot grill plate with

surface temperature given by $T(x,y) = 5000e^{-(x^2+3y^2)}$ unrealistic

The bug has landed at the point $(1,1)$.

Question: In which direction should the bug start walking in order to cool its feet most rapidly.

Intuition The level curves are ellipses



Calculations ($\text{let } c = 5000$)

$\frac{\partial T}{\partial x} = -2xc\, e^{''}$, $\frac{\partial T}{\partial x}(1,1) = -2ce^{''}$

$\frac{\partial T}{\partial y} = -6yc\, e^{''}$, $\frac{\partial T}{\partial y}(1,1) = -6ce^{''}$

$$\vec{\nabla} T(1,1) = ce^{-4}\langle -2,-6 \rangle$$
$$= -2ce^{-4}\langle 1,3 \rangle$$

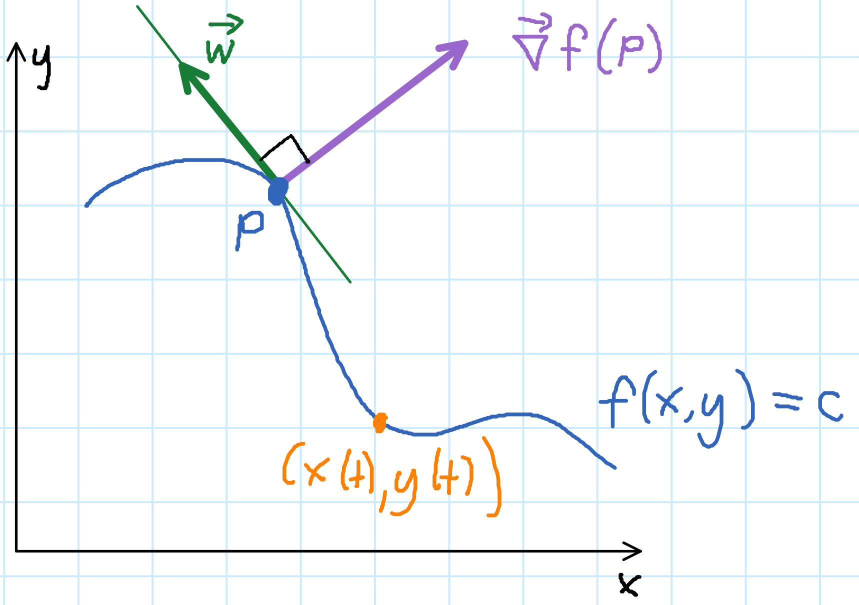
Direction of max increase is $\langle -1,-3 \rangle$

Direction of max decrease is $\langle 1,3 \rangle$

Looks about right

Gradient vector (2)

The gradient vector and level curves



$\vec{w}$ is tangent to the curve at P

Therefore (intuitively) $D_{\vec{w}} f(P) = 0$

implies

FACT: $\nabla f(P)$ is orthogonal to the level curve of f at P.

tangent of the

True for level curves and surfaces (same proof)

Proof

- Let $\vec{r}(t) = \langle x(t), y(t) \rangle$ be a parametrization of the curve
- So $f(x(t), y(t)) = c$
- $\frac{d}{dt} f(x(t), y(t)) = \frac{dc}{dt} = 0$
- By the chain rule

$$\frac{df(x(t),y(t))}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

$$= \vec{\nabla} f \cdot \vec{r}\,'(t)$$

$$= 0$$

!!!

- So $\vec{\nabla} f$ is orthogonal to the tangent

Gradient vector (3)

The fact that the gradient is orthogonal to the level surface can be used to find tangent planes

Here is the example from lecture #4.

Find the tangent plane to the surface $z = 6 - x^2 - y^2$ at the point $(1,2,1)$

<u>Solution</u> Write the surface as a level surface

$$z = 6 - x^2 - y^2 \iff x^2 + y^2 + z = 6$$

Let $F(x,y,z) = x^2 + y^2 + z$

So the surface is $F(x,y,z) = 6$

Need ① Point ✓ Given $(1,2,1)$

② normal ? $= \nabla F(1,2,1)$

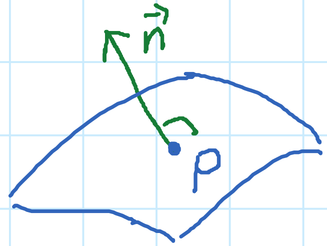
$$\vec{\nabla} F = \langle F_x, F_y, F_z \rangle = \langle 2x, 2y, 1 \rangle$$

$$\vec{\nabla} F(1,2,1) = \langle 2, 4, 1 \rangle$$

<u>Plane</u>: $2(x-1) + 4(y-2) + 1(z-1) = 0$

$$2x + 4y + z = 11 \quad ✓$$

Note: We could have used this idea to derive the general tangent plane equation **much** easier than what we did in lecture #4. However, what we did in lecture #4 is conceptually important and useful for other purposes.



$P(x_0, y_0, z_0)$

$z = f(x,y)$

Then $-f(x,y) + z = 0$

Let $F(x,y,z) = -f(x,y) + z = 0$

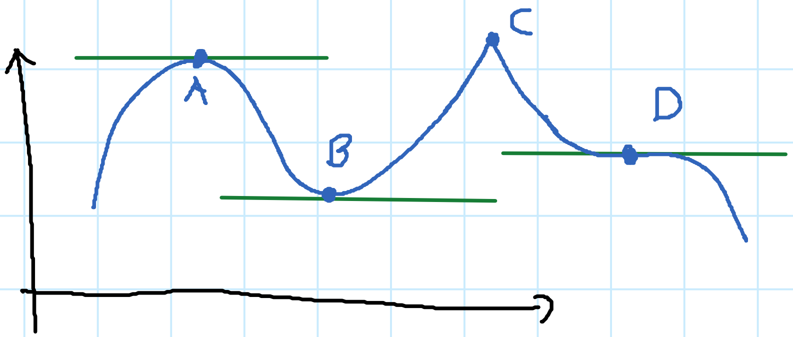
$$\text{normal} = \vec{n} = \nabla F(P)$$

$$= \left\langle -\frac{\partial f}{\partial x}(P), -\frac{\partial f}{\partial y}(P), 1 \right\rangle$$

as before !!!

Local extrema (min / max)

1 variable review



A, B, D and C are CRITICAL POINTS

$\underbrace{A, B, D}_{f'=0}$ and $\underbrace{C}_{f' \text{ does not exist}}$

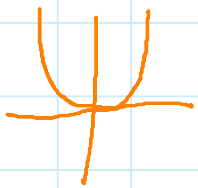
$f'=0$ f' does not exist

$f''(A) < 0 \Rightarrow$ local max

$f''(B) > 0 \Rightarrow$ local min

$f''(D) = 0$ no info (could be a min, max or inflection point)

2nd derivative test



Eg. $f(x) = x^4$

$f'(0) = 0$

$f''(0) = 0$

LOCAL MIN

New

Local extrema in 2 variables

Recall that we have 4 2^{nd}-derivatives

$f_{xx}, \; f_{yy}, \; f_{xy} = f_{yx}$

First, let's do a series of examples before we state the 2nd derivative test.

Find and classify the critial points of the function $f(x, y) = x^2 + y^2 - cxy$ for $c = 0, 1, 2, 4$.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x - cy = 0 \\ \frac{\partial f}{\partial y} &= 2y - cx = 0 \end{aligned}\right\} \xrightarrow{\cdot 2} 4x - c(cx) = 0$$

$\Rightarrow x = 0$ or $c = \pm 2$

Case $c \neq \pm 2$ Only 1 critical point at $(0,0)$

Case $c = \pm 2$ Every point on the line $y = \pm x$ is a critical point

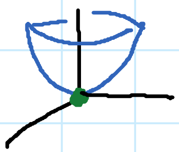
Example (continued)

Find and classify the critial points of the function
$f(x, y) = x^2 + y^2 - cxy$

$f_x = 2x - cy$, $f_y = 2y - cx$, $f_{xx} = 2, f_{yy} = 2, f_{xy} = -c$

① $c = 0$. Critical point at $(0,0)$

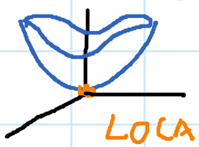
$f = x^2 + y^2$



$f_{xx} > 0$, $f_{yy} > 0$, $f_{xy} = 0$

LOCAL MIN

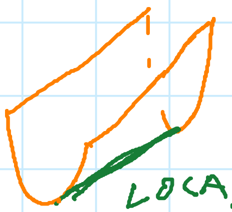
② $c = 1$ from software



$f_{xx} > 0$, $f_{yy} > 0$, $f_{xy} = -1$

LOCAL min

③ $c = 2$. $f = x^2 + y^2 - 2xy = (x-y)^2$



LOCAL min

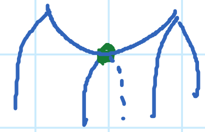
$f_{xx} > 0$, $f_{yy} > 0$, $f_{xy} = -2$

$x = y \Rightarrow f = 0$

$x = -y \Rightarrow f = 4x^2$

④ $c = 4$. $f = x^2 + y^2 - 4xy$

SADDLE

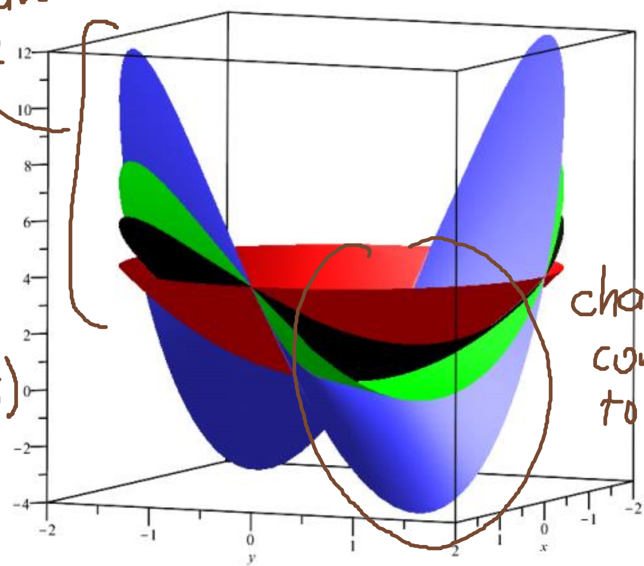


$x = y = t$ $f = -2t^2$

$x = -y = s$ $f = 6s^2$

$f_{xx} > 0$, $f_{yy} > 0$, $f_{xy} = -4$

all concave up in the $y = -x$ direction (and in the x and y directions)



changes from concave up to concave down in the $y = x$ direction

Think! In all cases $f_{xx} = 2 = f_{yy}$

$f_{xy} = -c$ is measuring the "size" of the xy term

Pulls the surface down in the $x = y$ direction

2nd derivative test

$f_x = 0 = f_y$ or at least one is undefined

Let $P = (a,b)$ be a critical point of a function $f(x,y)$

Let $D = \begin{vmatrix} fxx & fxy \\ fyx & fyy \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - f_{xy}^2$

(the Hessian matrix of f)

1. $D(a,b) < 0$ ⇒ saddle point
2. $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ ⇒ local min
3. $D(a,b) > 0$ and $f_{xx}(a,b) < 0$ ⇒ local max

Notes

(a) If $D=0$ then "NO INFO"

(b) If $f_{xx} > 0$ and $f_{yy} < 0$ then $D < 0$

(c) Because of (b), if $D > 0$ then $f_{xx} \underset{<0}{>0}$ if and only if $f_{yy} \underset{<0}{>0}$

So in 2. and 3. the condition on f_{xx} could be replaced with f_{yy}

Back to the example on the previous page

Find and classify the critial points of the function $f(x,y) = x^2 + y^2 - cxy$

$f_x = 2x - cy$, $f_y = 2y - cx$, $f_{xx} = 2, f_{yy} = 2, f_{xy} = -c$

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 4 - c^2$

At the critical point $P = (0,0)$

$D(0,0) = 4 - c^2$

① $c = 0$, $D = 4 > 0$, $f_{xx} > 0$ LOCAL MIN

② $c = 1$, $D = 4 - 1 > 0$, $f_{xx} > 0$ LOCAL MIN

③ $c = 2$, $D = 0$ NO INFO (But we figured directly it is a local min)

④ $c = 4$, $D = 4 - 4^2 < 0$ SADDLE

Local extrema example

Find and classify the local extrema of
$f(x,y) = x^3 - 3xy + y^3$

Solution:

STEP 1 Find all the critical points

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 - 3y = 0 \\ \frac{\partial f}{\partial y} &= -3x + 3y^2 = 0 \end{aligned}\right\}$$

Nonlinear system of equations. Could have any # of solutions.

Solve. $y = x^2$, $x = y^2$ $\Rightarrow y = y^4 \Rightarrow y^4 - y = 0$

$\Rightarrow y(y^3 - 1) = 0$

$y = 0, y = 1$

$\Rightarrow \cancel{1 = y^3}$

$\Rightarrow \cancel{y = 1}$

Case $y = 0$. Then $x = 0$

Case $y = 1$, Then $x = 1$

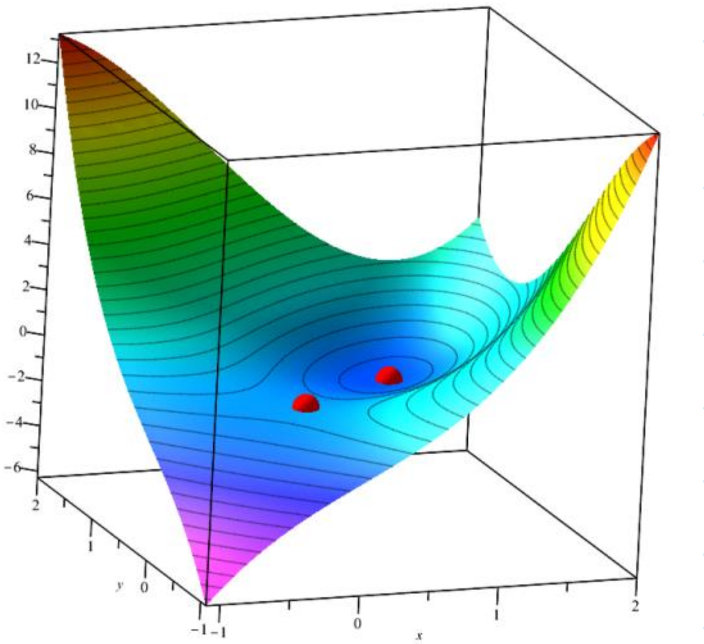
Critical points are $(0,0)$ and $(1,1)$

STEP 2 Apply the 2^{nd} derivative test

$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3$

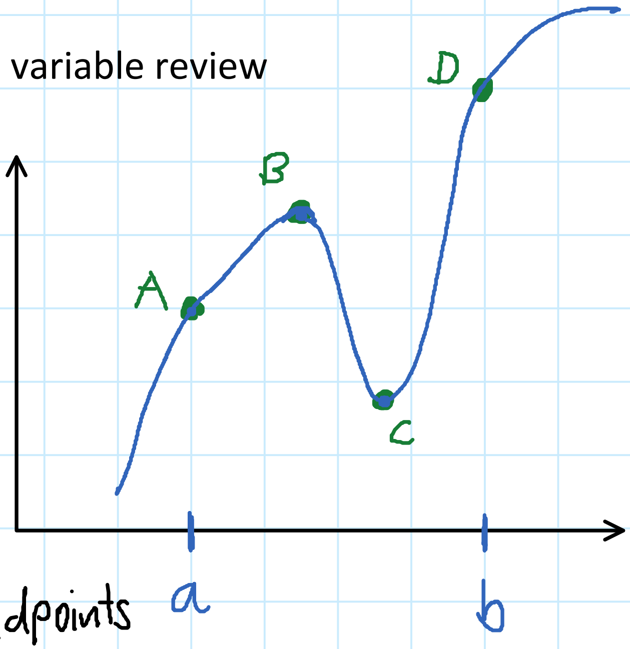
At $(0,0)$: $D(0,0) = f_{xx}(0,0)\, f_{yy}(0,0) - 9$
$= 0 - 9$
< 0 SADDLE

At $(1,1)$: $f_{xx}(1,1) = 6 > 0$
$D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - 9$
$= 6 \cdot 6 - 9$
> 0 LOCAL MIN



Absolute extrema

1 variable review



Find the absolute extrema of the continuous function $f(x)$ on the closed and bounded interval $[a,b]$

endpoints

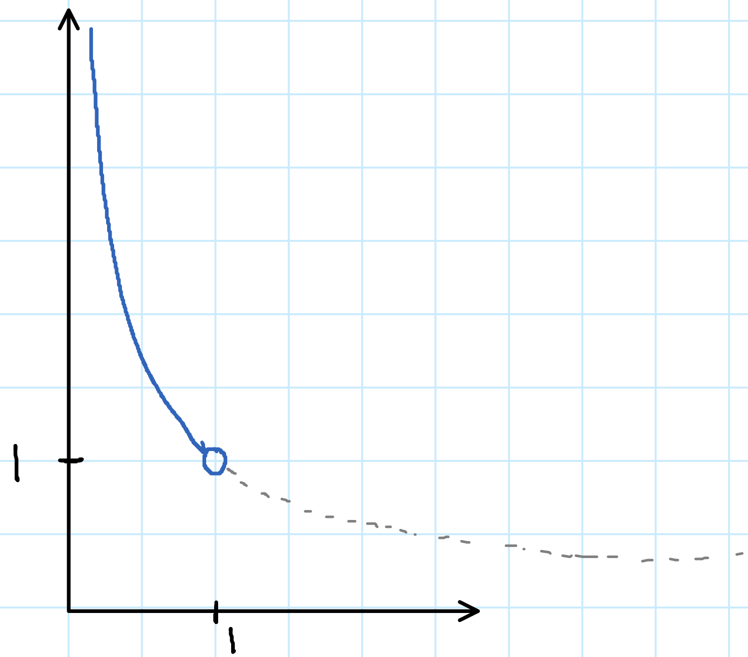
A, B, C, D are the possible locations

local extrema

Absolute minimum is at C

Absolute maximum is at D

Note $f(x) = \frac{1}{x}$ is defined and is continuous on the open interval $(0,1)$



But $f(x)$ does not have an absolute min or max on this interval

Note: the range of f is $(1,\infty)$