Lecture 10

Topics: Double integrals - polar coordinates, general changes of variables, center of mass

- Introduced polar coordinates. Showed geometrically that dA = r dr d theta

- Examples of double integrals in polar coordinates, including
  - Showed where the normlaization factor of sqrt(pi) comes from in the definition of the "normal distribution" in the following way. We computed the integral from -infity to -infinity of exp(-x^2) by relating it ot the double integral over the whole xy-plane of exp(-x^2 -y^2). This later calculation can be done in polar coordinates but is impossible to do directly in cartesian coordinates.

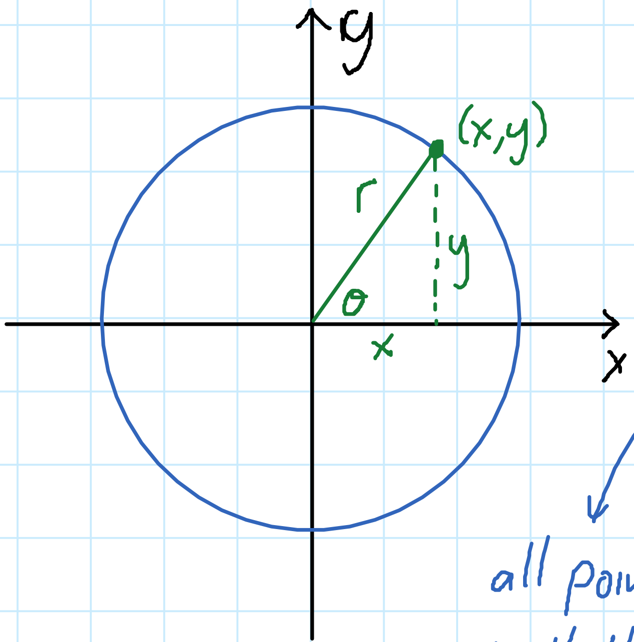
- Disucssed changes of variable in general and that the change in area of a small piece is given by multiplication by the absolute value of the determinant of the Jacobian matrix. This was obtained from the area of a parallelogram and computing its area using the cross product.

- Introduced the concept of center of mass. In the one variable case, starting with the case of discrete masses and using a Riemann sum, we derived the integral formula for center of mass. Derived the center of mass formuila in 2D. An example will be given next class.

Where to find this material:

- Adams and Essex 14.4, 14.7
- Corral, 3.5, 3.6
- Guichard, 15.2, 15.3 15.7
- Active Calculus. 11.4, 11.5, 11.9

Polar coordinates



$x = r\cos\theta$

$y = r\sin\theta$

$$\begin{cases} 0 \le r < \infty \\ 0 \le \theta < 2\pi \end{cases}$$

all points can be described with this domain, but we can also consider

$-\infty < r < \infty$

$-\infty < \theta < \infty$

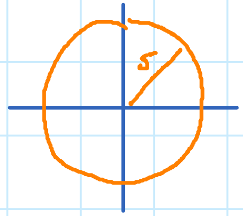
negative r is the reflection through the origin

<u>Note</u> $\tan\theta = y/x$

$x^2 + y^2 = r^2$

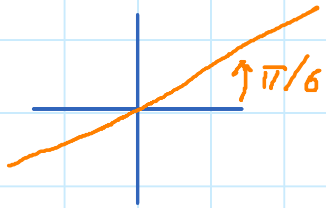
Curves in polar coordinates

① $r = 5$

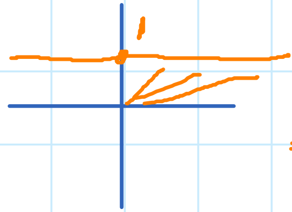


$x^2 + y^2 = 5^2$

② $\theta = \pi/6$



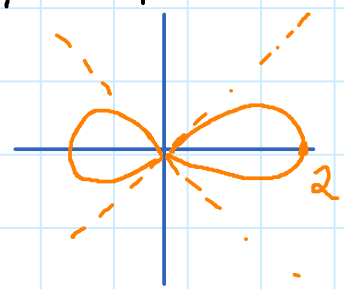
③ Line $y = 1$



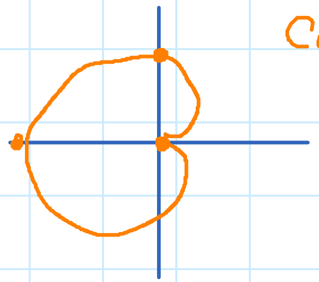
$1 = r\sin\theta$

$\Rightarrow r = 1/\sin\theta$

④ $r^2 = 4\cos 2\theta$



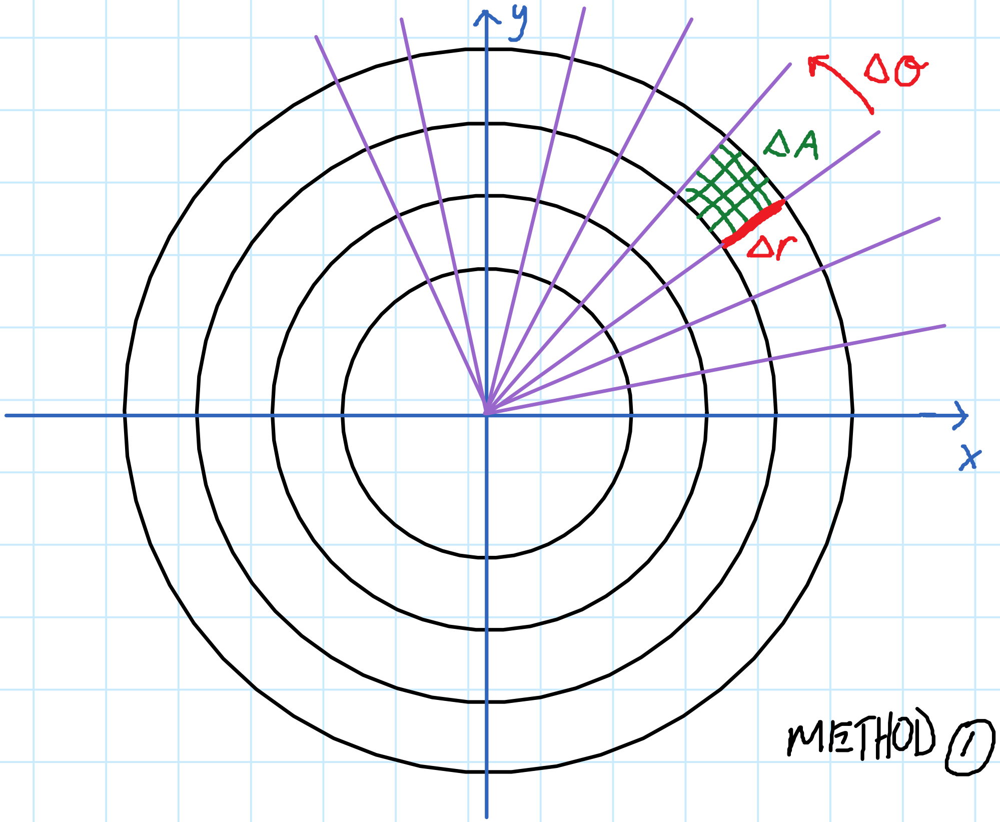
⑤ $r = 1 - \cos\theta$

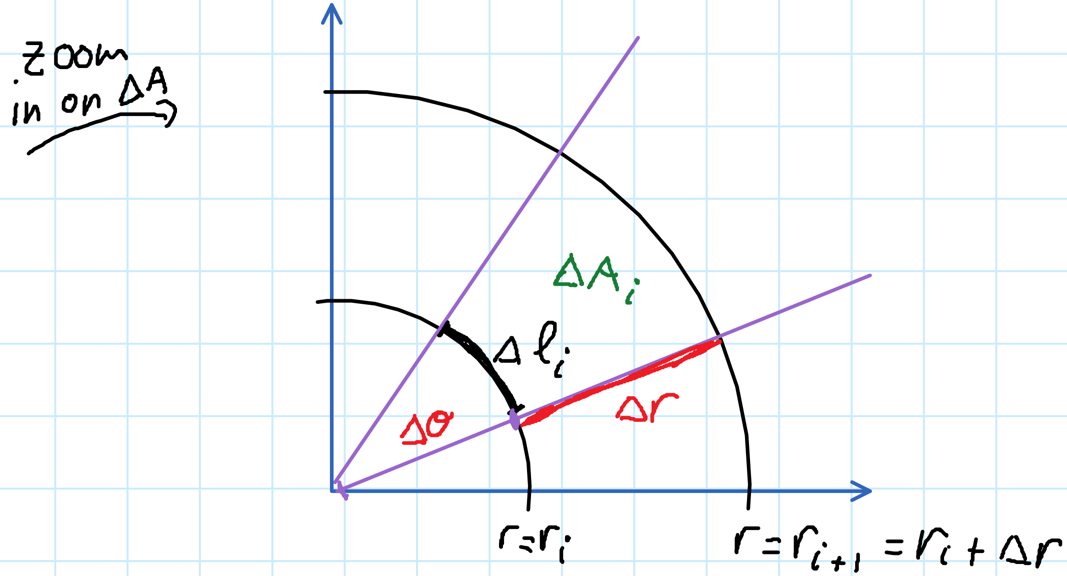


cardioid

not expected to know (memorize)

Area in polar coordinates





Aim: Find the first order approximation for ΔA in terms of $r_i, \Delta r, \Delta\theta$

Fix Δr and $\Delta\theta$

Note that ΔA changes with r

METHOD ① $\Delta l_i = \left(\frac{\Delta\theta}{2\pi}\right)(2\pi r_i) = r_i \Delta\theta$

For $\Delta\theta$ and Δr small, the region is approximately a rectangle, so $\Delta A_i \approx \underbrace{r_i \Delta\theta}_{=\Delta l_i} \Delta r$

METHOD ② $\Delta A_i = \frac{\Delta\theta}{2\pi}\left[\pi(r_i+\Delta r)^2 - \pi r_i^2\right]$

$$= r_i \Delta r \Delta\theta + \frac{1}{2}\Delta\theta \Delta r^2$$

$$\approx r_i \Delta r \Delta\theta$$

For integration "$dA = r\,dr\,d\theta$"

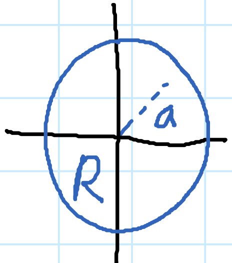
(See HW 5, Q2
$r = |\text{Jacobian}|$)

Examples

① Area of a disk of radius a

$\lim \sum 1 \Delta A_i$

$\quad \parallel$

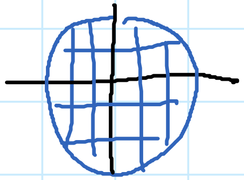


$$\text{Area} = \iint_R 1 \underbrace{dA}_{r\,dr\,d\theta}$$

In Cartesian coords

$R: -a \le x \le a$

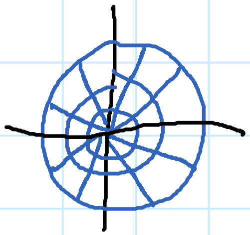
$-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$



In polar coords

$R: 0 \le \theta \le 2\pi$

$\quad 0 \le r \le a$



$$= \int_0^{2\pi}\int_0^a 1 \; r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}r^2 \Big|_0^a \, d\theta$$

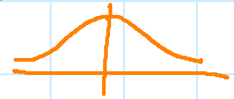
$$= \int_0^{2\pi} \frac{1}{2}a^2 \, d\theta$$

$$= \frac{1}{2}a^2 \; \theta \Big|_0^{2\pi}$$

$$= \frac{1}{2}a^2 \times 2\pi$$

$$= \pi a^2 \quad \text{as expected}$$

② $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$ ← why? $y = e^{-x^2}$



$I \;\; \parallel$

<u>Note</u> $\int e^{-x^2}dx$ can not be expressed in terms of elementary functions

<u>TRICK!!</u>

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA \qquad e^{A+B} = e^A e^B$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dy\right)dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2}\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)dx$$

$$= \underbrace{\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)}_{= I}\underbrace{\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)}_{= I}$$

$$= I^2$$

Examples (2)

$x = r\cos\theta$, $y = r\sin\theta$

We now compute $\iint_{\mathbb{R}} e^{-x^2-y^2}\,dA$ using polar coords

$\mathbb{R}^2$: $0 \leq \theta \leq 2\pi$, $0 \leq r < \infty$; $x^2+y^2=r^2$

So, $$\iint_{\mathbb{R}} e^{-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

(substitute $u = -r^2$)

$$= 2\pi \lim_{b\to\infty} -\frac{1}{2} e^{-r^2}\Big|_0^b$$

$$= \pi\left(1 - \lim_{b\to\infty} e^{-b^2}\right)$$

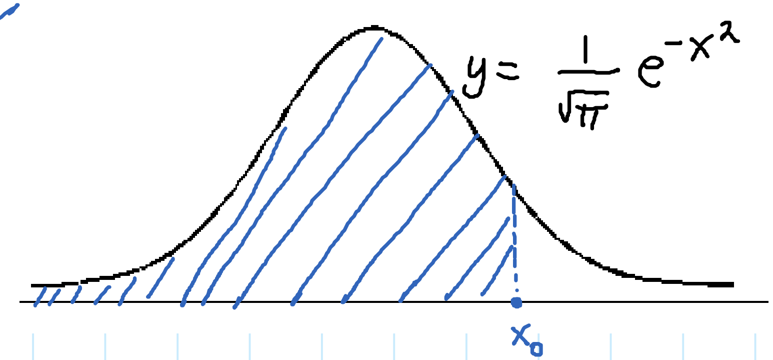
$$= \pi$$

$$\therefore \boxed{\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}}$$

Note ① This is why there is a $\frac{1}{\sqrt{\pi}}$ in the definition of the "normal distribution" $\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2}\,dx = 1$

② We can not evaluate $\int_{-\infty}^{x_0} e^{-x^2}\,dx$

Aside



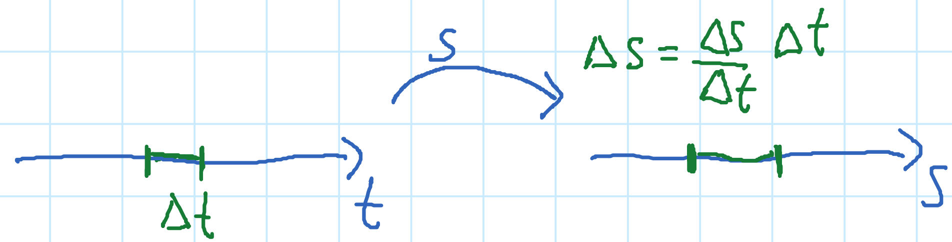
In general, the normal distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx$$

Recall Change of variables (substitution)

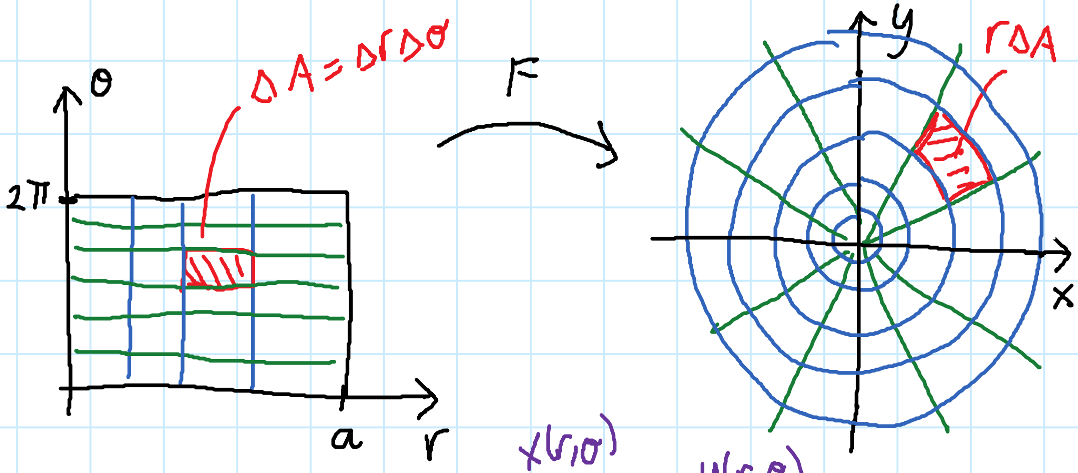
$$\int \sin(t^2)\,2t\,dt = \int \sin(s)\,ds$$

$s = s(t) = t^2$, $\frac{ds}{dt} = 2t$ or " $ds = 2t\,dt$ "



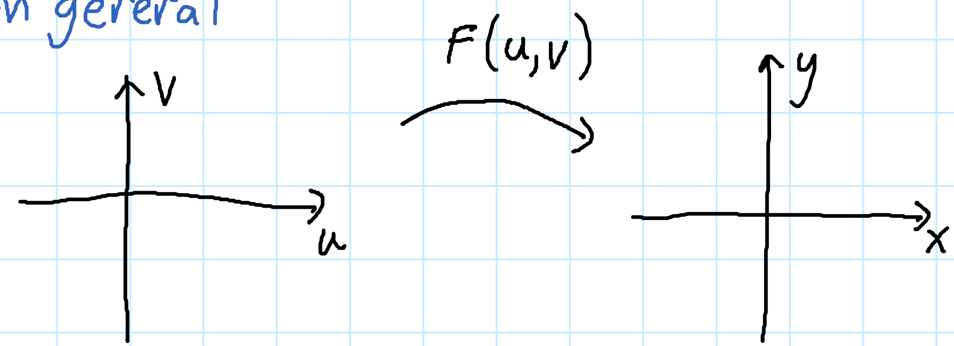
General changes of variable

Let's look at polar coordinates again as a function $\mathbb{R}^2 \to \mathbb{R}^2$



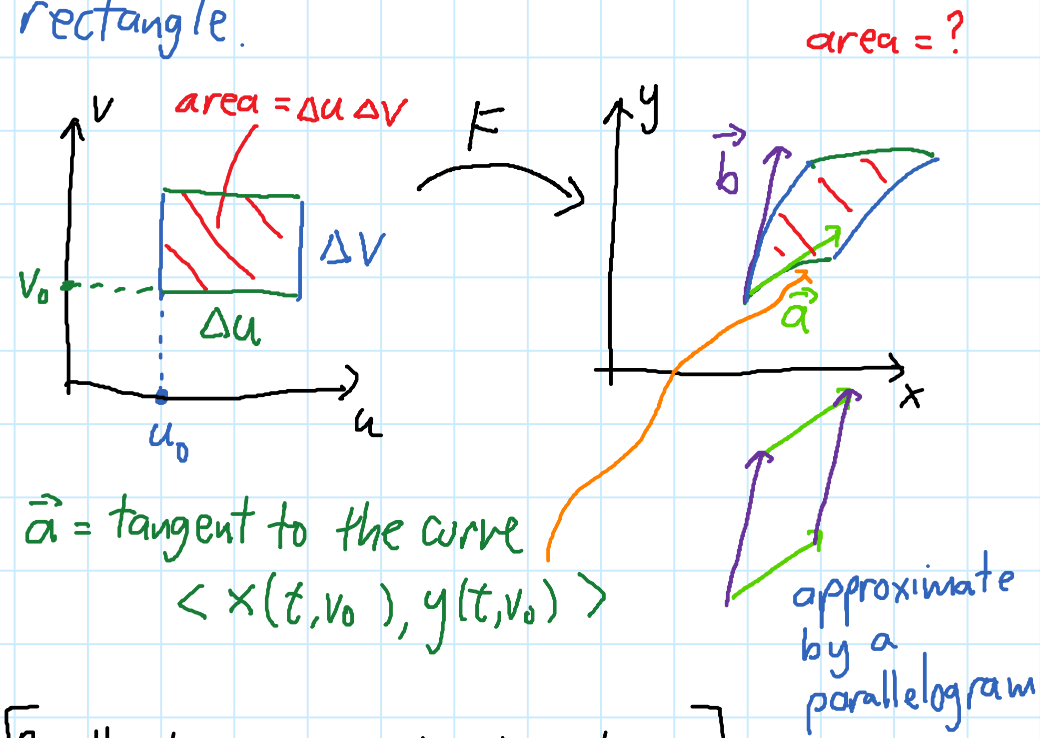
$$F(r,\theta) = (\underbrace{r\cos(\theta)}_{=x(r,\theta)},\ \underbrace{r\sin\theta}_{=y(r,\theta)})$$

In general



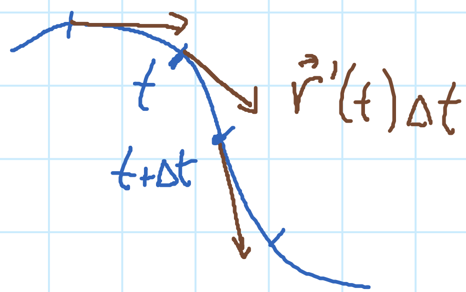
$$F(u,v) = (x(u,v),\ y(u,v))$$

Now look at what happens to a small rectangle.



$\vec{a}$ = tangent to the curve $\langle x(t,v_0), y(t,v_0)\rangle$

Recall the arc length derivation



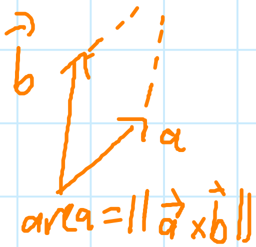
$$\vec{a} = \left\langle \frac{\partial x}{\partial u}(u_0,v_0),\ \frac{\partial y}{\partial u}(u_0,v_0) \right\rangle \Delta u$$

Change of variables continued

$$\vec{a} = \left\langle \frac{\partial x}{\partial u}(u_0,v_0),\ \frac{\partial y}{\partial u}(u_0,v_0),\ 0 \right\rangle \Delta u$$

$$\vec{b} = \left\langle \frac{\partial x}{\partial v}(u_0,v_0),\ \frac{\partial y}{\partial v}(u_0,v_0),\ 0 \right\rangle \Delta v$$

Area $\approx$ Area of the parallelogram



$$= \|\vec{a}\times\vec{b}\|$$

$$= \begin{vmatrix} i & j & k \\ x_u(u_0,v_0) & y_u(u_0,v_0) & 0 \\ x_v(u_0,v_0) & y_v(u_0,v_0) & 0 \end{vmatrix} \Delta u \Delta v$$

$$= \|\langle 0, 0, x_u y_v - x_v y_u \rangle\| \Delta u \Delta v$$

$$= |x_u y_v - x_v y_u| \Delta u \Delta v$$

$$= \left|\begin{matrix} x_u(u_0,v_0) & y_u(u_0,v_0) \\ x_v(u_0,v_0) & y_v(u_0,v_0) \end{matrix}\right| \Delta u \Delta v$$

$$= |J_F(u_0,v_0)| \Delta u \Delta v$$

Conclusion

$$F(u,v) = (x(u,v), y(u,v)) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$\Delta u \Delta v$

$|J|\Delta u \Delta v$

$$\iint_R h(x,y)\, \underset{\displaystyle dxdy}{dA}$$

$$= \iint_{R'} h(x(u,v), y(u,v))\, |J_F(u,v)|\, dA$$

$$|J_F(u,v)| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|, \qquad dA = du\,dv$$

Example Polar coords. $F(r,\theta) = (r\cos\theta, r\sin\theta)$, $x = r\cos\theta$, $y = r\sin\theta$

$$|J_F(r,\theta)| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

More of the change of variables formula (NOT COVERED IN LECTURE)

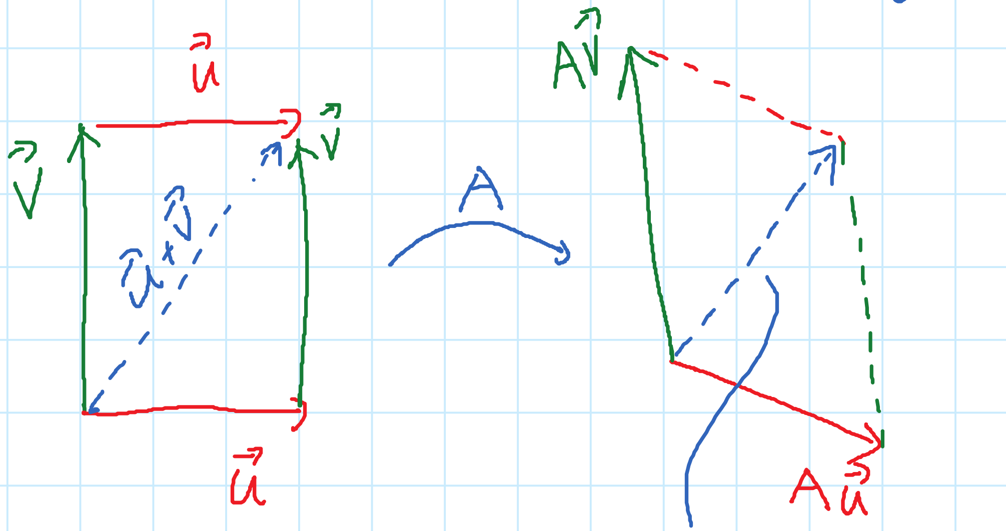
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\vec{x} \longmapsto A\vec{x}$$

This is a linear transformation:

① $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

② $A(c\vec{x}) = cA\vec{x}$

How does A transform a rectangle?



$$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$$

FACT (check using the cross product as before)

Area is changed by a factor of $\det(A) = |A|$

Given $F : \mathbb{R}^2 \to \mathbb{R}^2$

then the linear aproximation at (u_0, v_0) is

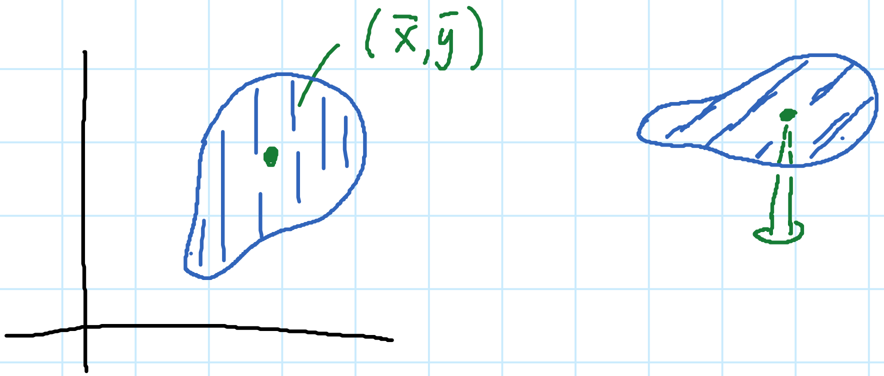
$J_F(u_0, v_0) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (JACOBIAN)

It changes area by a factor $\det(J)$

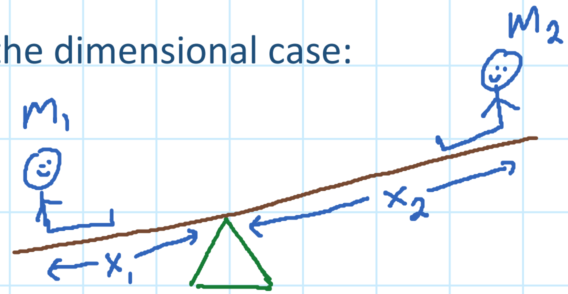
Note: this is true in any dimension

Center of mass

Aim: Find a formula for the center of mass of 2-dimensional regions (lamina) with density $\rho(x, y)$ — units. MASS/AREA



First, the dimensional case:



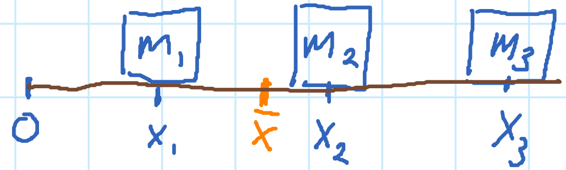
Balance occurs when

$$m_1 x_1 = m_2 x_2$$

"TORQUE"

"1st Moment"

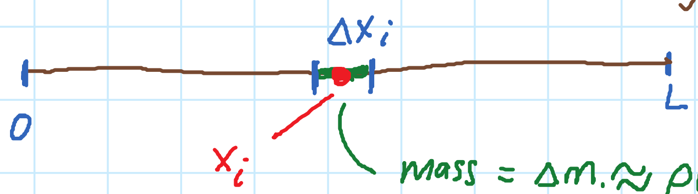
Discrete masses



$$\text{Center of mass} = \bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

Continuous mass distribution

density $\rho(x)$



$$\text{mass} = \Delta m_i \approx \rho(x_i)\Delta x_i$$

$$\text{Total mass} = M = \lim_{N \to \infty} \sum_{i=1}^{N} \rho(x_i)\Delta x_i$$

$$= \int_0^L \rho(x)\,dx$$

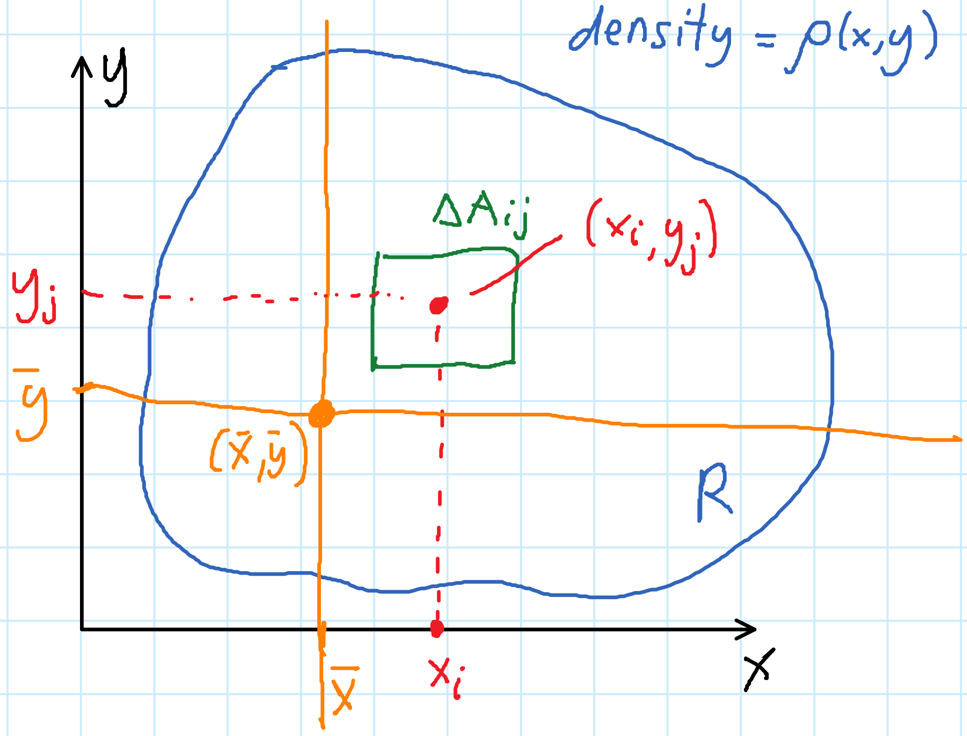
$$\text{Total moment} = \lim_{N \to \infty} \sum_{i=1}^{N} x_i \overbrace{\rho(x_i)\Delta x_i}^{\Delta m_i}$$

$$= \int_0^L x \underbrace{\rho(x)\,dx}_{dm}$$

$$\bar{x} = \frac{\int_0^L x\rho(x)\,dx}{M}$$

Center of mass (2)

density $= \rho(x,y)$



Δmass $= \Delta m_{ij} \approx \rho(x_i, y_j)\,\Delta A_{ij}$

Total mass $= M = \displaystyle\iint_R \rho(x,y)\,dA$

Δ moment about the x-axis $= \Delta m_{ij} \cdot y_j$

Total moment about the x-axis $= M_x = \displaystyle\iint_R y\, \overbrace{\rho(x,y)\,dA}^{dm}$

Total moment about the y-axis $= M_y = \displaystyle\iint_R x\, \rho(x,y)\,dA$

$\bar{x} = M_y / M$

$\bar{y} = M_x / M$

Center of mass $= (\bar{x}, \bar{y})$

Note. We do not discuss the moment of inertia. This is the 2^{nd} moment: $\text{dist}^2 \cdot \text{mass}$