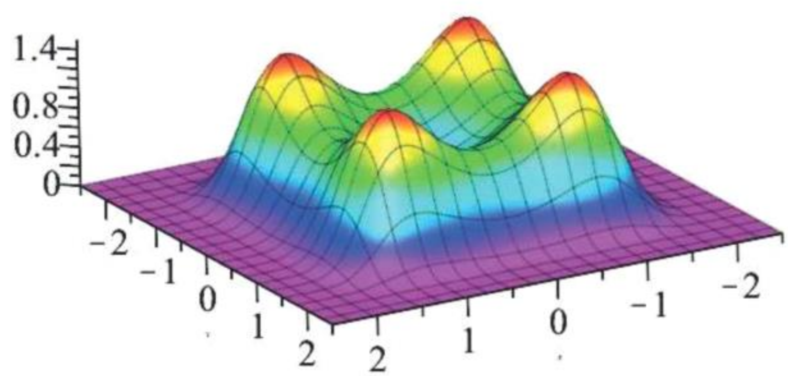
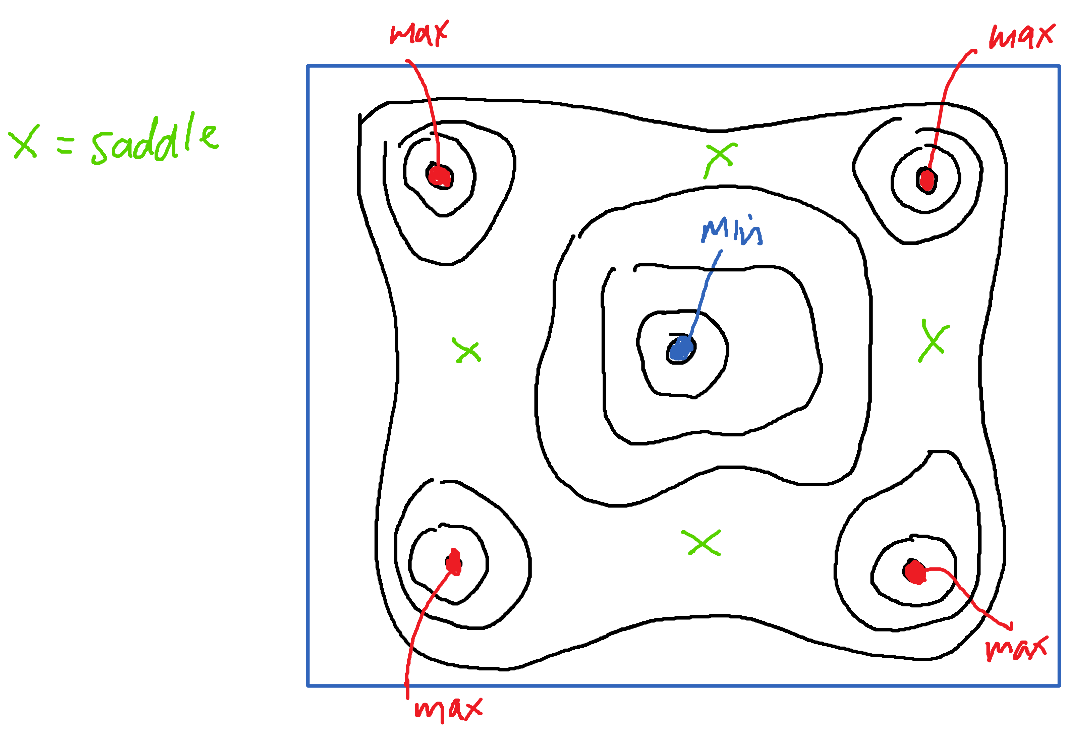
MS-A0211, Midterm Exam Solutions 03/02/2021

1. **[4 pts]** Sketch a contour plot (ie level curves) of the following surface. Clearly indicate on your plot the locations of local minima, local maxima and saddle points.





2. **[4 pts]** Determine if the following limit exists: $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+4y^6}$

Along the line $y=0$: $\lim_{x\to 0} \frac{0}{x^2} = 0$

Along the curve $x=y^3$: $\lim_{x\to 0} \frac{x^3 x^3}{(x^3)^2+4x^6}$

$= \lim_{x\to 0} \frac{1}{5} = \frac{1}{5} \neq 0$

We obtained different limits along different paths approaching (0,0), therefore the limit does not exist.

3. **[4 pts]** Find the tangent plane to the surface $z = \ln(xy)$ when $x = 1$ and $y = 1$.

$z = \ln(xy) = \ln(x) + \ln(y)$

$\frac{\partial z}{\partial x} = \frac{1}{x}$; $\frac{\partial z}{\partial x}(1,1) = 1$

$\frac{\partial z}{\partial y} = \frac{1}{y}$, $\frac{\partial z}{\partial y}(1,1) = 1$

Tangent plane ; $z = \ln(1\cdot 1) + 1(x-1) + 1(y-1)$

$\Rightarrow z = 0 + x - 1 + y - 1$

$\Rightarrow z = x + y - 2$

Note: The version of the exam with the typo $x=0, y=1$ has no tangent plane as $(0,1)$ is not in the domain of $\ln(xy)$

4. **[8 pts]** Consider the function

$$f(x,y) = \frac{\sqrt{1-x^2-y^2}}{x^2}$$

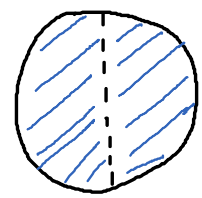
and let D be its domain.

(a) Find and sketch D.

$x \neq 0$

and $1 - x^2 - y^2 \geq 0$

$\Leftrightarrow \quad x^2 + y^2 \leq 1$



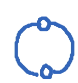
(b) Is the domain open, closed or neither?

Neither, as D contains the boundary circle but does not include the y-axis

(c) Does the function have an absolute minimum on D? If so, then find it. If not, explain why not.

① $f(x,y) \geq 0$ for all (x,y) in D, $x \neq 0$

② For any (a,b) on the circle, $a^2+b^2=1$, so $f(a,b)=0$ for $b \neq 0$

So f has a min of 0 on 

(d) Does the function have an absolute maximum on D? If so, then find it. If not, explain why not.

No. For example, let $y=0$, then

$$f(x,y) = f(x,0) = \frac{\sqrt{1-x^2}}{x^2}$$

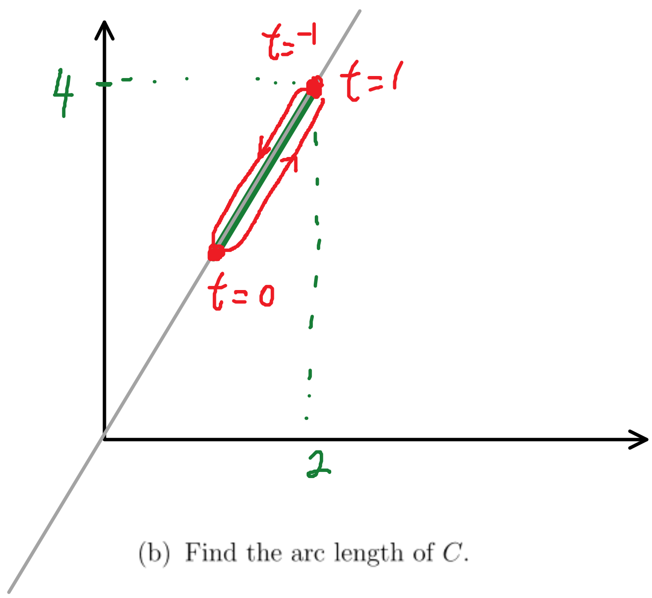
Note: $\displaystyle \lim_{x \to 0} \frac{\sqrt{1-x^2}}{x^2} = \infty$

So no absolute max

5. **[6 pts]** Let C be the curve with parametric equation $\mathbf{r}(t) = \langle 1+t^2, 2+2t^2\rangle$, for $-1 \le t \le 1$.

(a) What shape is the curve?

$$\left.\begin{array}{l} x = 1+t^2 \\ y = 2+2t^2 \end{array}\right\} \Rightarrow y = 2x$$



Note: $\vec{r}(-1) = \vec{r}(1) = \langle 2,4\rangle$

$\vec{r}(0) = \langle 1,2\rangle$

The curve is a segment of the straight line joining the points (1,2) and (2,4)

(b) Find the arc length of C.

By pythagorus the length of the line segment is $\sqrt{2^2+1^2} = \sqrt{5}$

Or, a much harder way is:

$\|\vec{r}\,'(t)\| = \|\langle 2t, 4t\rangle\| = \|2t\langle 1,2\rangle\| = 2|t|\sqrt{5}$

$$\text{Length} = \int_{\boxed{0}}^{\boxed{1}} \|\vec{r}\,'(t)\|\,dt = 2\sqrt{5}\int_0^1 |t|\,dt$$

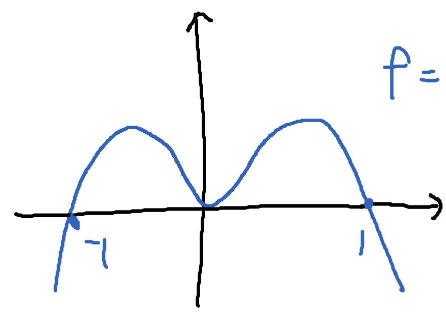
$$= 2\sqrt{5}\int_0^1 t\,dt$$

$$= \sqrt{5}$$

Because we only want to count the line segment once

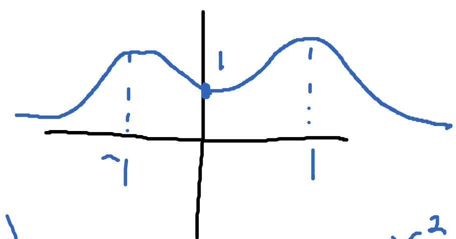
6. [**bonus 4 pts**] Propose a function $f(x,y)$ whose graph is the surface in question 1.

- Think first in 2D



$$f = -x^2(x-1)(x+1)$$

- The surface flattens to 0.



(exponentiating does not change the location of the local extrema)

$$f = e^{-x^2(x-1)(x+1)}$$

- One possible reasonable guess is

$$f(x,y) = e^{-x^2(x-1)(x+1)} \cdot e^{-y^2(y-1)(y+1)}$$

This produces a very similar surface

- The actual function is

$$f(x,y) = \frac{3}{2}\, e^{-\frac{1}{2}(x-1)^2(x+1)^2}\, e^{-\frac{1}{2}(y-1)^2(y+1)^2}$$