MEC-E8003

BEAM, PLATE AND SHELL MODELS 2021

LEARNING OUTCOMES

Assumptions, equations and analytical solutions of linearly elastic beam and plate models in flat and curved geometries. After the course, student

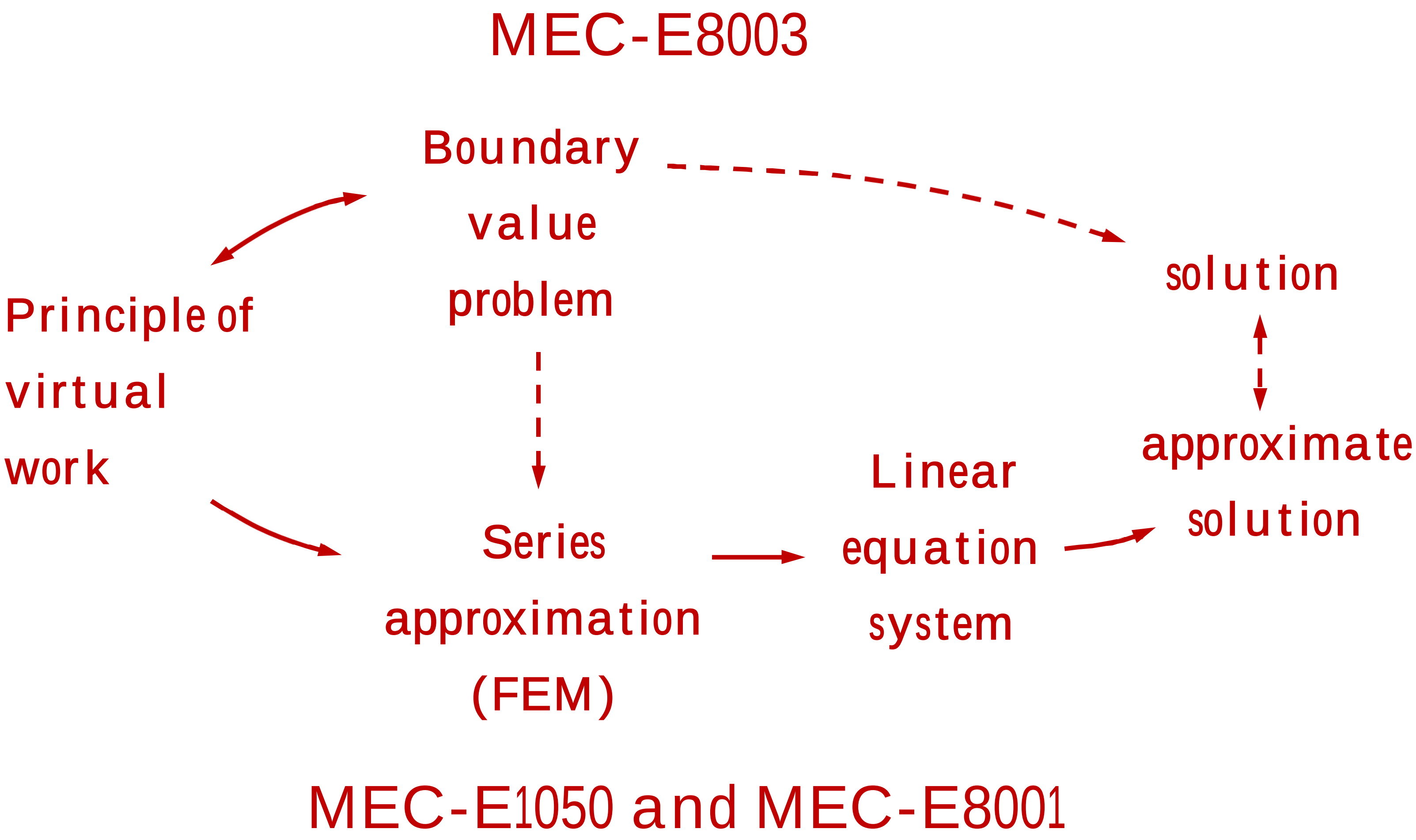
(1) is able to represent the quantities and operators of continuum mechanics in different coordinate systems,

(2) knows the assumptions of the beam and plate models and derivation of the beam and plate equations out of the principle of virtual work,

(3) is able to write the equations in coordinate systems for flat and curved geometries and solve the equations for the displacement in simple cases.

Assessment: Modelling, lecture and home assignments (1/5) and final exam of four (4) problems (4/5).

THE PLAN


MEC-E8003
Boundary value problem
Principle of virtual work
solution
approximate solution
Linear equation system
Series approximation (FEM)
MEC-E1050 and MEC-E8001

1 INTRODUCTION

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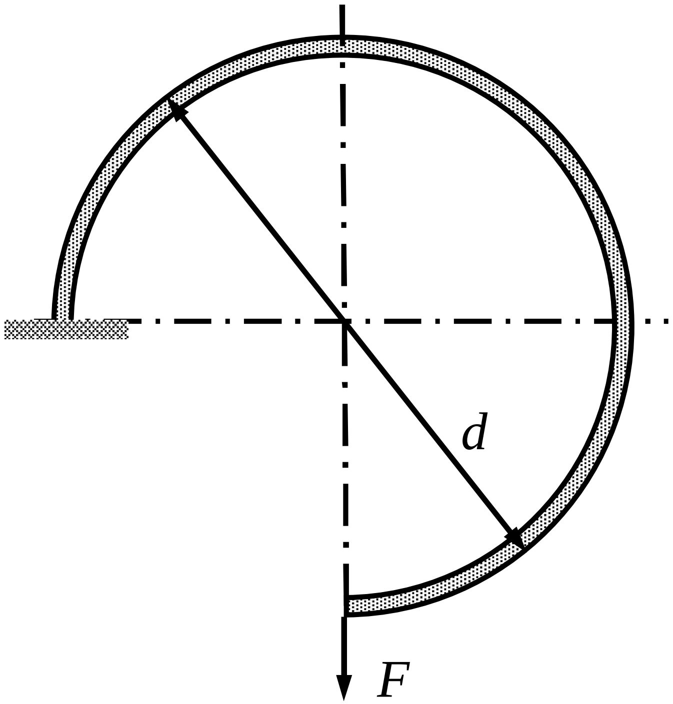
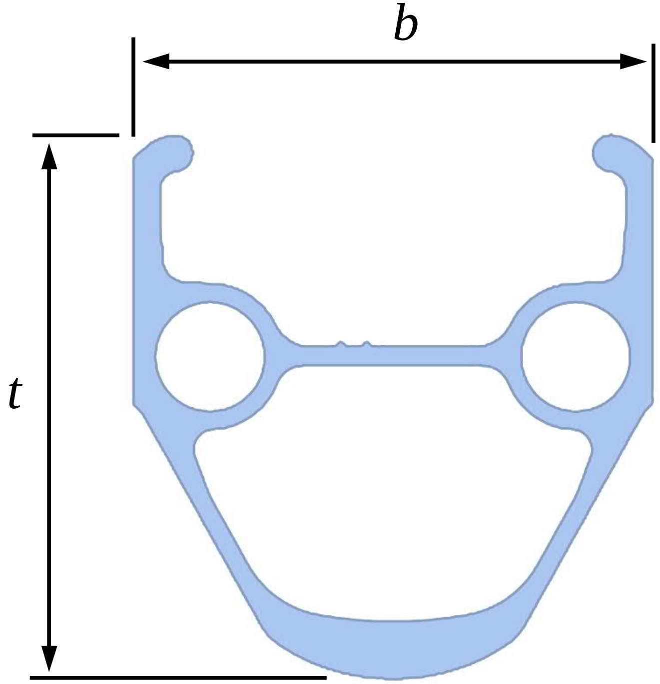
BICYCLE WHEEL RIM RIGIDITY

STRUCTURE IDEALIZATION





CURVED BEAM MODELLING

Assuming a planar beam, clamping and the center of the rim on the same horizontal line, and $L = 3\pi R/2$ (curvilinear $xy-$coordinate system, x along the rim at the area centroid, x directed to the center of circle):

Equilibrium: $\frac{dN}{dx} - \frac{1}{R}Q = 0$, $\frac{dQ}{dx} + \frac{1}{R}N = 0$, $\frac{dM}{dx} + Q = 0$,

Constitutive: $N = EA(\frac{du}{dx} - \frac{1}{R}v)$, $Q = GA(\frac{dv}{dx} + \frac{1}{R}u - \psi)$, $M = EI\frac{d\psi}{dx}$

BC:s at the free end: $N(L) = 0$, $Q(L) + F = 0$, $M(L) = 0$

BC:s at the clamped end: $u(0) = 0$, $v(0) = 0$, $\psi(0) = 0$

In a statically determined case, it is possible to solve for the stress resultants first. Elimination is used in the connected first order equations to end up with second order non-connected equations:

Resultants: $Q(x) = F\sin(\frac{x}{R}), \quad N(x) = -F\cos(\frac{x}{R}), \quad M(x) = FR\cos(\frac{x}{R})$

Displacements: $\psi(x) = \frac{FR^2}{EI}\sin(\frac{x}{R}), \quad v(x) = \frac{F}{2}(\frac{R^2}{EI} + \frac{1}{GA} + \frac{1}{EA})x\sin(\frac{x}{R}),$

$$u(x) = -\frac{FR}{2}[\frac{x}{R}(\frac{R^2}{EI} + \frac{1}{GA} + \frac{1}{EA})\cos(\frac{x}{R}) - (\frac{R^2}{EI} + \frac{1}{GA} - \frac{1}{EA})\sin(\frac{x}{R})]$$

Rigidity: $k = -\frac{F}{v(L)} = \frac{4}{3\pi R}\frac{1}{\frac{R^2}{EI} + \frac{1}{AE} + \frac{1}{AG}} \approx \frac{4EI}{3\pi R^3}$ 🡸

MODELLING IN MEC-E8003

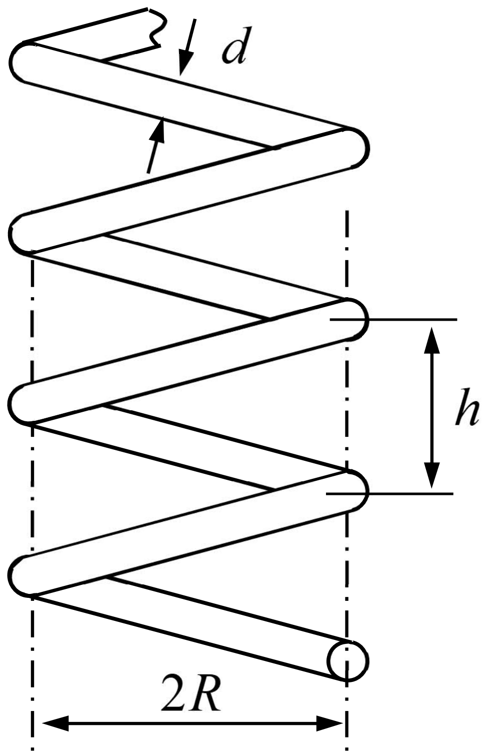
- ☐ **Crop:** Decide the boundary of a structure. Interaction with surroundings need to be described in terms of known forces, moments, displacements, and rotations. All uncertainties with this respect bring uncertainty to the model too.

- ☐ **Idealize and parameterize:** Simplify the geometry. Ignoring the details not likely to affect the outcome may simplify the analysis a lot. Assign symbols to geometric and material parameter of the idealized structure.

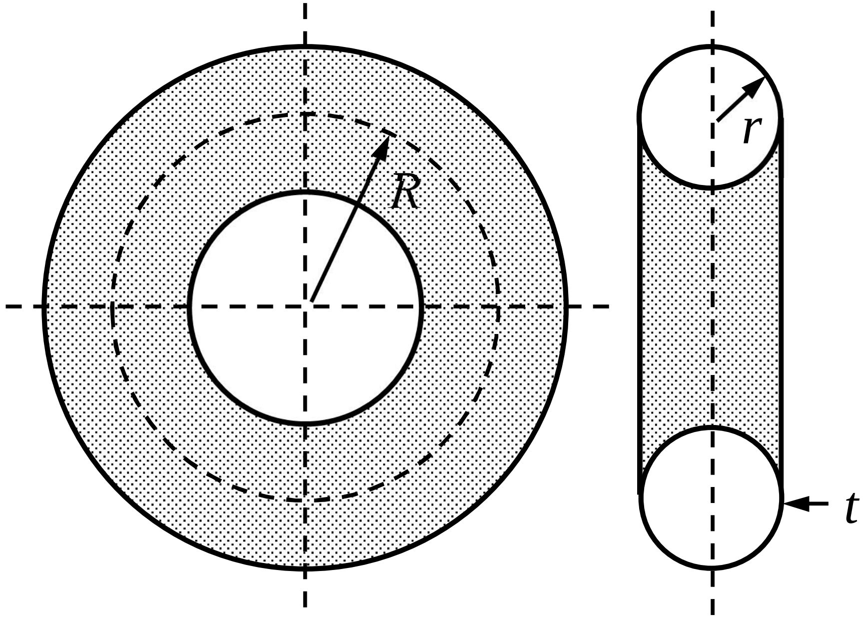
- ☐ **Model:** Write the equilibrium equations, constitutive equations, and boundary conditions of the plate or beam structure. 🡐

- ☐ **Solve:** Use an analytical or approximate method and hand calculations or Mathematica to find the solution. 🡐

1.1 BEAMS AND PLATES

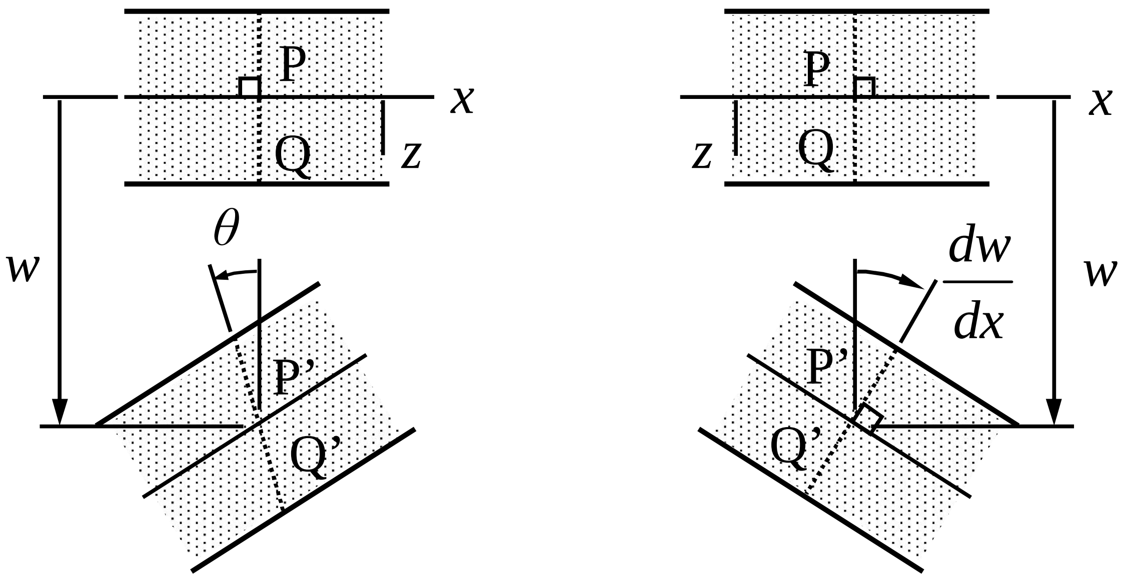


Thin body in two dimensions



Thin body in one dimension.

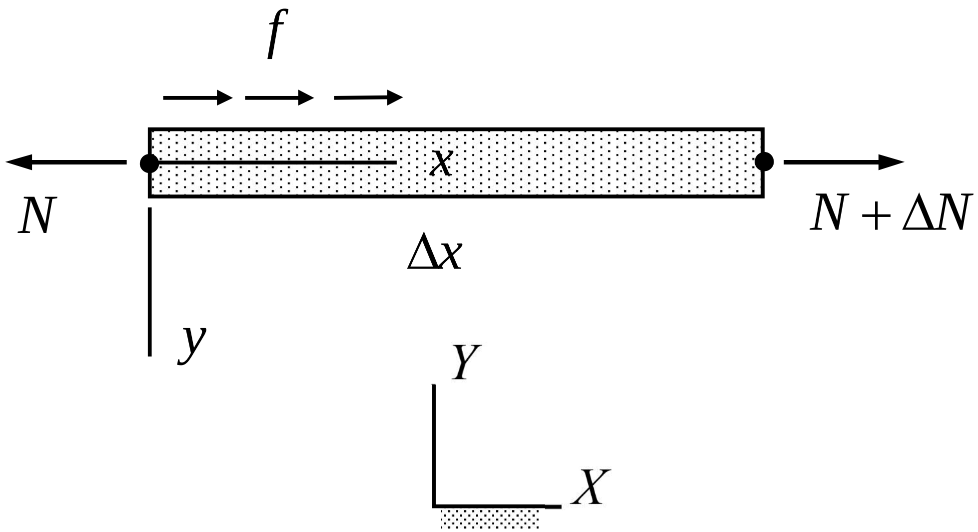
BEAM MODEL



Timoshenko **Bernoulli**

The primary unknowns are $u(x)$, $v(x)$, $w(x)$, $\phi(x)$, $\theta(x)$, $\psi(x)$. Normal planes to the (material) axis of beam remain planes (Timoshenko) and normal to the axis (Bernoulli) in deformation. In short, for points P and Q of a cross-section $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}}$. Stress components $\sigma_{yy} \ll \sigma_{xx}$ and $\sigma_{zz} \ll \sigma_{xx}$.

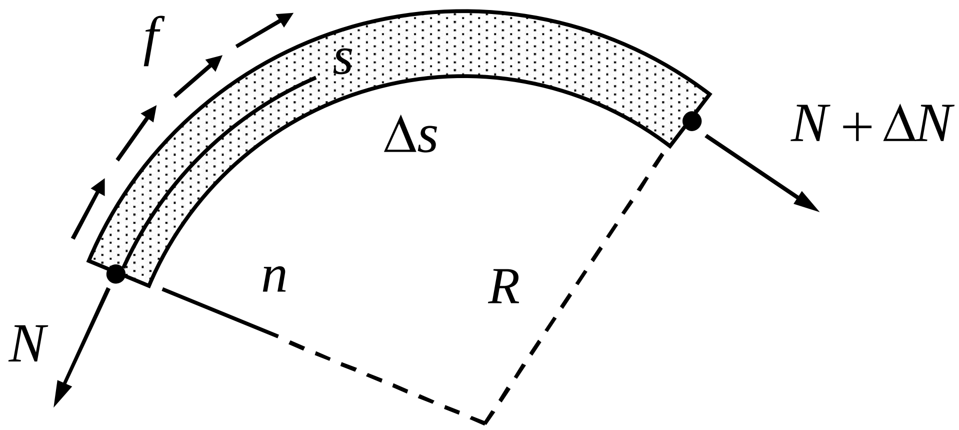
CURVATURE EFFECT



$$\vec{N} = N\vec{i}$$

$$\vec{f} = f_x\vec{i} + f_y\vec{j}$$

$$\frac{d\vec{N}}{dx} + \vec{f} = 0$$



$$\vec{N} = N\vec{e}_s$$

$$\vec{f} = f_s\vec{e}_s + f_n\vec{e}_n$$

$$\frac{d\vec{N}}{ds} + \vec{f} = 0$$

The basis vectors of the Cartesian material $(x, y, z)-$coordinate system are constants

$$\frac{d\vec{N}}{dx} + \vec{f} = \frac{d(N\vec{i})}{dx} + f_x\vec{i} + f_y\vec{j} = (\frac{dN}{dx} + f_x)\vec{i} + f_y\vec{j} = 0 \quad \Leftrightarrow$$

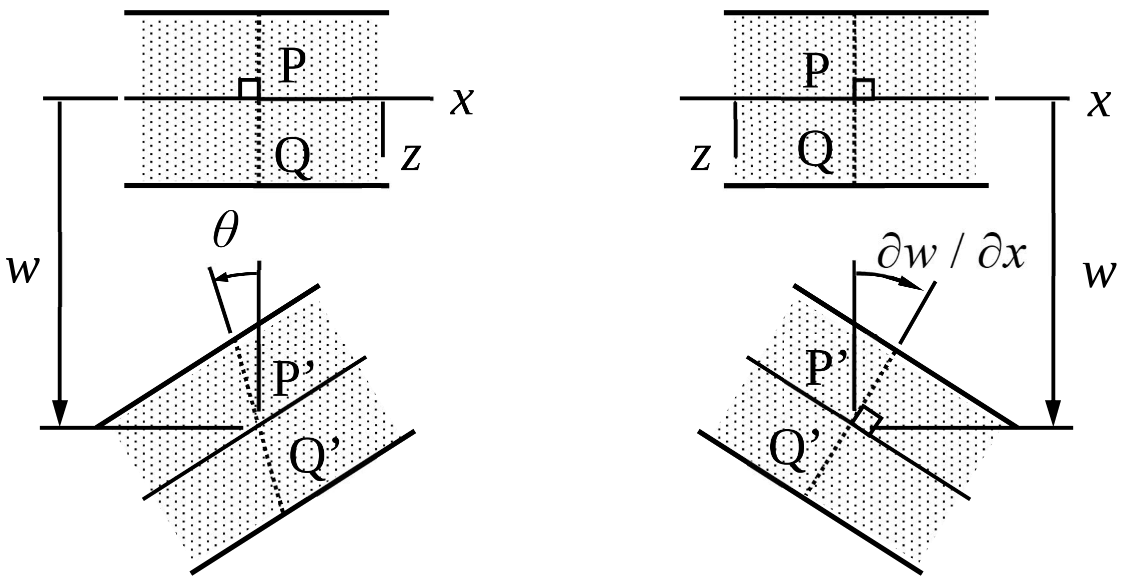
$$\frac{dN}{dx} + f_x = 0 \quad \text{and} \quad f_y = 0. \quad \leftarrow$$

The basis vectors of the curvilinear material (s, n, b) coordinate system are *not* constants

$$\frac{d\vec{N}}{ds} + \vec{f} = \frac{d(N\vec{e}_s)}{ds} + f_s\vec{e}_s + f_n\vec{e}_n = (\frac{dN}{ds} + f_s)\vec{e}_s + (\frac{N}{R} + f_n)\vec{e}_n = 0 \quad \Leftrightarrow$$

$$\frac{dN}{ds} + f_s = 0 \quad \text{and} \quad \frac{N}{R} + f_n = 0\ . \quad \leftarrow$$

PLATE MODEL

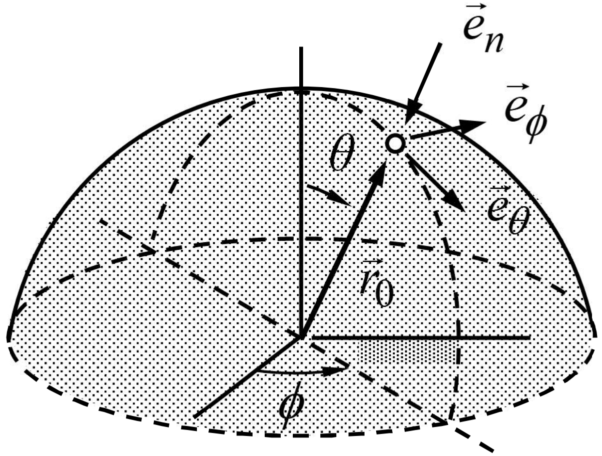


Reissner-Mindlin **Kirchhoff**

The primary unknowns are $u(x,y)$, $v(x,y)$, $w(x,y)$, $\phi(x,y)$, $\theta(x,y)$, $\psi(x,y)$. Line segments perpendicular to the mid/reference-plane remain straight in deformation (Reissner-Mindlin) and perpendicular to the mid-plane (Kirchhoff). In short, for points P and Q of a line segment $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}}$. Stress components $\sigma_{zz} \ll \sigma_{yy}$, $\sigma_{zz} \ll \sigma_{xx}$.

CURVATURE EFFECT

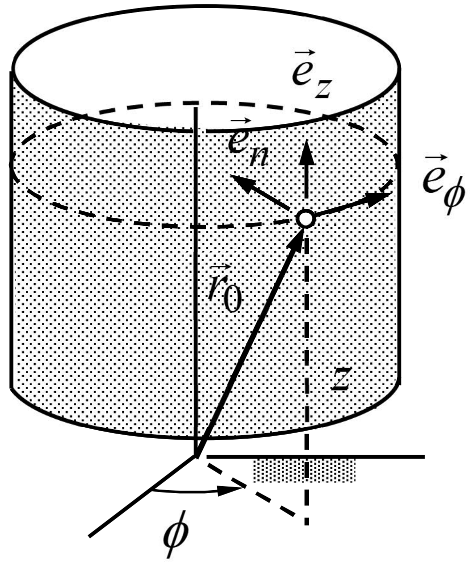
Sphere subjected to internal pressure:



$$N_{\phi\phi} = \frac{1}{2} pR \quad \text{and} \quad N_{\theta\theta} = \frac{1}{2} pR \quad \Rightarrow$$

$$\overleftrightarrow{N} = \frac{1}{2} pR(\vec{e}_\phi \vec{e}_\phi + \vec{e}_\theta \vec{e}_\theta) = \frac{1}{2} pR\overleftrightarrow{I} \quad \text{(isotropic stress)}$$

Long cylinder subjected to internal pressure:

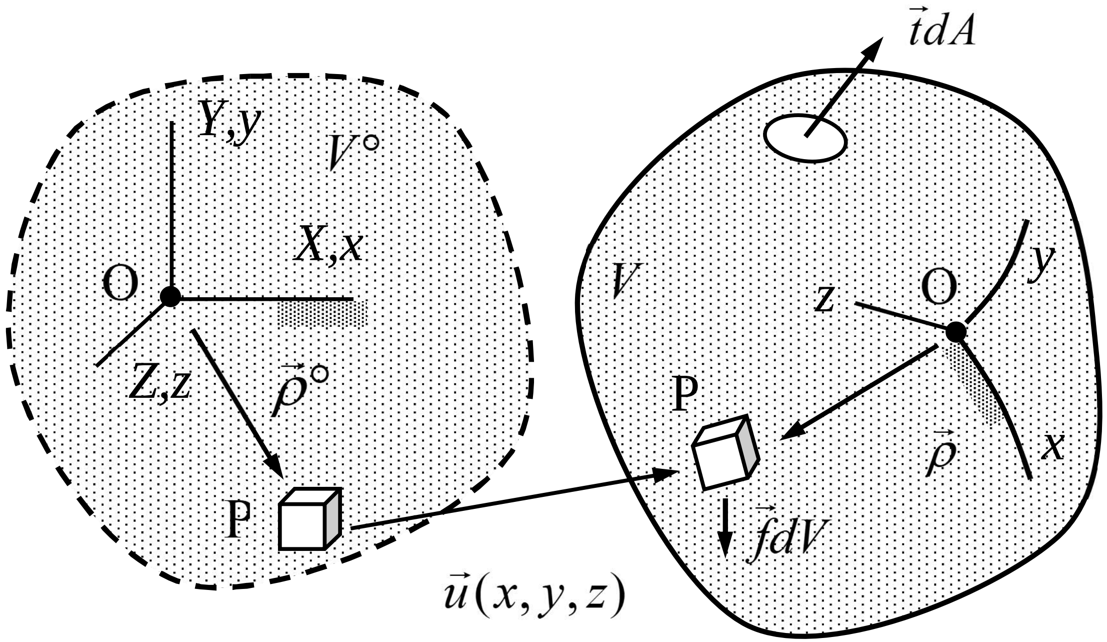


$$N_{zz} = \frac{1}{2} pR \quad \text{and} \quad N_{\phi\phi} = pR \quad \Rightarrow$$

"radius of curvature"

$$\overleftrightarrow{N} = \frac{1}{2} pR(\vec{e}_z \vec{e}_z + 2\vec{e}_\phi \vec{e}_\phi)$$

SOLID MODEL



The primary unknowns are $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$, ($\phi(x,y,z)$, $\theta(x,y,z)$, $\psi(x,y,z)$). Material elements may translate, rotate, and deform. In short, for points P and Q of an element $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}} + \vec{\rho}_{\mathrm{PQ}} \cdot \vec{\vec{\varepsilon}}_{\mathrm{P}}$. Displacement follows from stress-strain relationship (generalized Hooke's law) and equilibrium of material elements.

Let us consider the displacement of a small material element centered at point P. As the material element is assumed to be small, first two terms of the Taylor series represent the displacement inside the material element

$$\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\rho}_{\mathrm{PQ}} \cdot (\nabla \vec{u})_{\mathrm{P}},$$

where the relative position vector $\vec{\rho}_{\mathrm{PQ}} = \vec{r}_{\mathrm{Q}} - \vec{r}_{\mathrm{P}}$. Division of the displacement gradient into its anti-symmetric and symmetric parts $(\nabla \vec{u})_{\mathrm{P}} = \overleftrightarrow{\theta}_{\mathrm{P}} + \overleftrightarrow{\varepsilon}_{\mathrm{P}}$ and using the concept of an associated vector $\vec{\theta}$ to an antisymmetric tensor $\overleftrightarrow{\theta}$, gives

$$\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}} + \vec{\rho}_{\mathrm{PQ}} \cdot \overleftrightarrow{\varepsilon}_{\mathrm{P}}.$$

The first term on the right-hand side describes translation, the second term small rigid body rotation, and the last term deformation (shape distortion) when the rotation part is small. Stress acting on material elements depend on shape distortion $\overleftrightarrow{\varepsilon}_{\mathrm{P}}$.

1.2 FIRST PRINCIPLES

Balance of mass Mass of a fixed set of particles, called as a body, is constant. 🡐

Balance of linear momentum The rate of change of linear momentum of a body equals the external force resultant acting on the material volume. 🡐

Balance of angular momentum The rate of change of angular momentum of a body equals the external moment resultant acting on the material volume. 🡐

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

PRINCIPLE OF VIRTUAL WORK

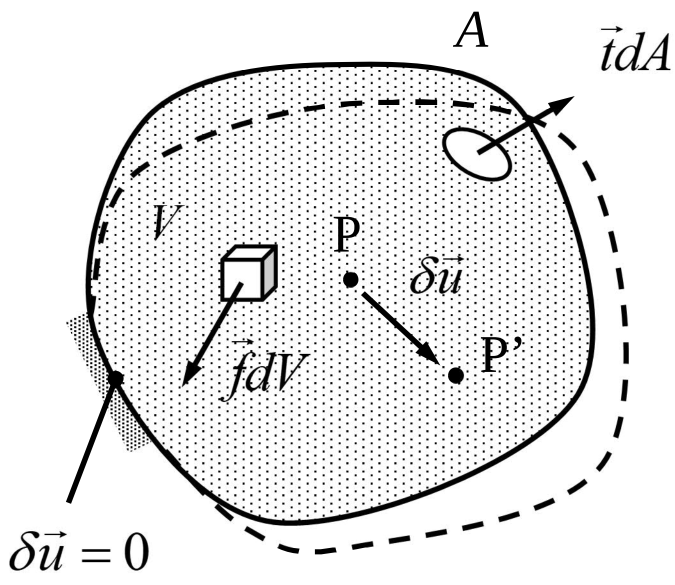
Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta\vec{u} \in U$ is just one representation of the balance laws of continuum mechanics. It is important due to its wide applicability and physical meanings of the terms.

virtual work density

$$\delta W^{\text{int}} = \int_\Omega \delta w_V^{\text{int}} dV = -\int_\Omega (\delta\vec{\vec{\varepsilon}}_{\text{c}} : \vec{\vec{\sigma}}) dV$$

$$\delta W_\Omega^{\text{ext}} = \int_\Omega \delta w_V^{\text{ext}} dV = \int_\Omega (\vec{f} \cdot \delta\vec{u}) dV$$

$$\delta W_{\partial\Omega}^{\text{ext}} = \int_{\partial\Omega} \delta w_A^{\text{ext}} dA = \int_{\partial\Omega} (\vec{t} \cdot \delta\vec{u}) dA$$



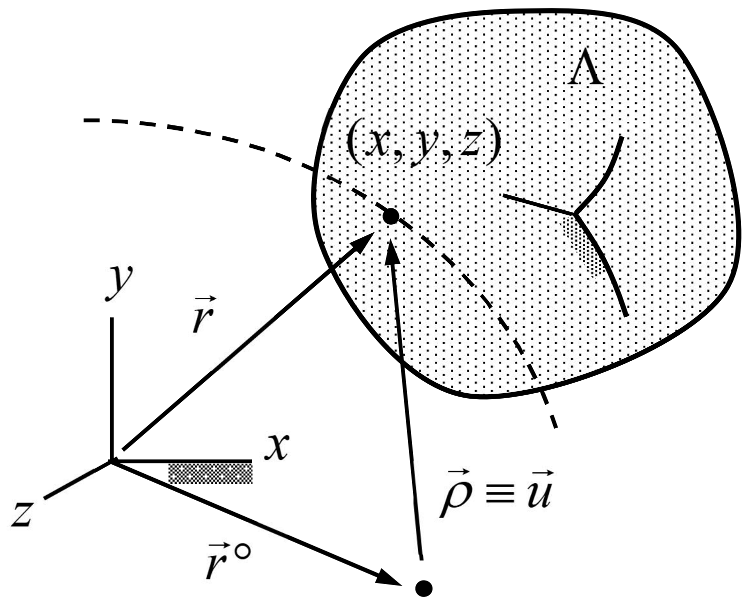
The details of the expressions vary case by case, but the principle itself does not!

DISPLACEMENT

In continuum mechanics with solids, one considers the motion of particles of a body relative to the initial position at $t=0$. Displacement $\vec{u}(x,y,z,t)$ is relative position vector of a particle identified by its material coordinates (x,y,z) so $\vec{u}(x,y,z,0)=0$

Non-stationary $\vec{r}=\begin{Bmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{Bmatrix}^{\mathrm{T}}(\begin{Bmatrix}x\\y\\z\end{Bmatrix}+\begin{Bmatrix}u_x(x,y,z,t)\\u_y(x,y,z,t)\\u_z(x,y,z,t)\end{Bmatrix})$

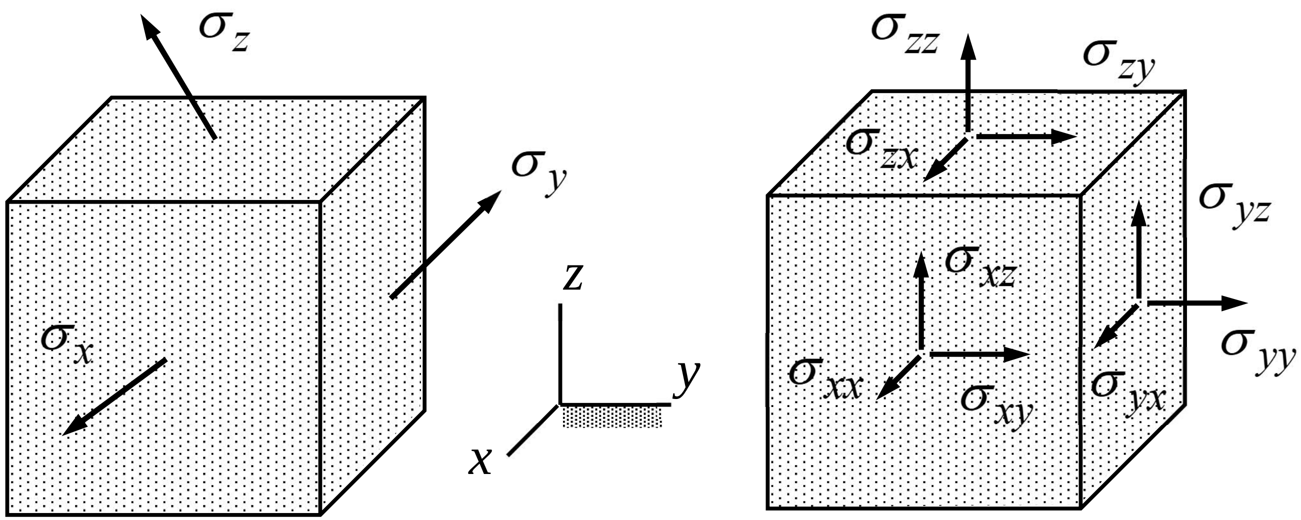
Stationary $\vec{r}=\begin{Bmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{Bmatrix}^{\mathrm{T}}(\begin{Bmatrix}x\\y\\z\end{Bmatrix}+\begin{Bmatrix}u_x(x,y,z)\\u_y(x,y,z)\\u_z(x,y,z)\end{Bmatrix})$



In stationary case, one considers only the initial and the final positions of the particles.

TRACTION AND STRESS

In continuum mechanics, traction $\vec{\sigma} = \Delta\vec{F} / \Delta A$ (a vector) describes the surface force between material elements of a body. Cauchy stress $\overleftrightarrow{\sigma}$ describes the surface forces acting on all edges of a material element. Traction and stress are related by $\vec{\sigma} = \vec{n} \cdot \overleftrightarrow{\sigma}$.



The first index of a stress component refers to the direction of the surface normal and the second that of the force component.

The representation of the traction vectors acting on the three edges of a material element in the $(\vec{i},\vec{j},\vec{k})-$basis (directions are opposite on the opposite edges)

$$\begin{Bmatrix}\vec{\sigma}_x\\ \vec{\sigma}_y\\ \vec{\sigma}_z\end{Bmatrix}=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix} \Rightarrow \vec{\vec{\sigma}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{\sigma}_x\\ \vec{\sigma}_y\\ \vec{\sigma}_z\end{Bmatrix}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}.$$

Stress is a tensor quantity (vector of vectors) which represents the surface forces acting on all the surfaces of the material element simultaneously. In terms of the unit outward normal $\vec{n}$ of an edge, traction acting on the edge is given by $\vec{\sigma}=\vec{n}\cdot\vec{\vec{\sigma}}$ and the force the force acting on the material element through the edge

$$d\vec{F}=\vec{\sigma}dA=\vec{n}\cdot\vec{\vec{\sigma}}dA=(dA\vec{n})\cdot\vec{\vec{\sigma}}=d\vec{A}\cdot\vec{\vec{\sigma}}.$$

LINEAR STRAIN

Linear strain measure $\vec{\vec{\varepsilon}} = [\nabla\vec{u} + (\nabla\vec{u})_c]/2$ describes shape deformation of material elements. The components of the (invariant) tensor quantity depends on the selection of the coordinate system. In a Cartesian $(x, y, z)-$coordinate system

$$\vec{\vec{\varepsilon}} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_c] = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}.$$

Normal components: $\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}, \ \varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}, \ \varepsilon_{zz} = \dfrac{\partial u_z}{\partial z}$

Shear components: $\varepsilon_{xy} = \dfrac{1}{2}(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y}), \ \varepsilon_{yz} = \dfrac{1}{2}(\dfrac{\partial u_z}{\partial y} + \dfrac{\partial u_y}{\partial z}), \ \varepsilon_{zx} = \dfrac{1}{2}(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z})$

Let us consider the displacement of a small material element centered at $\vec{r}_0$. As the material element is assumed to be small, first two terms of the Taylor series represent the displacement inside the material element

$$\vec{u}(\vec{r}) = \vec{u}_0 + \vec{\rho} \cdot (\nabla \vec{u})_0,$$

where the relative position vector $\vec{\rho} = \vec{r} - \vec{r}_0$. Division of the displacement gradient into its anti-symmetric and symmetric parts with notation $\overleftrightarrow{\varepsilon} = (\nabla \vec{u})_{\mathrm{s}}$, $\overleftrightarrow{\theta} = (\nabla \vec{u})_{\mathrm{u}}$ and using the concept of an associated vector $\vec{\theta}$ to an antisymmetric tensor $\overleftrightarrow{\theta}$, gives

$$\vec{u}(\vec{r}) = \vec{u}_0 + \vec{\rho} \cdot \overleftrightarrow{\theta}_0 + \vec{\rho} \cdot \overleftrightarrow{\varepsilon}_0 = \vec{u}_0 + \vec{\theta}_0 \times \vec{\rho} + \vec{\rho} \cdot \overleftrightarrow{\varepsilon}_0.$$

The first term on the right-hand side describes translation, the second term small rigid body rotation, and the last term deformation (shape distortion) when the rotation part is small.

1.3 VECTORS AND TENSORS

The quantities in mechanics can be classified into scalars a, vectors $\vec{a}$ and multi-vectors (vectors of vectors) $\overleftrightarrow{a}$ called also as tensors of ranks 0,1, and 2.

basis vector singlet

Vector $$\vec{a}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}a_x\\ a_y\\ a_z\end{Bmatrix}=\begin{Bmatrix}a_x\\ a_y\\ a_z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}\quad \text{(rank 1 tensor)}$$

basis vector doublet

Tensor $$\overleftrightarrow{a}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}a_{xx} & a_{xy} & a_{xz}\\ a_{yx} & a_{yy} & a_{yz}\\ a_{zx} & a_{zy} & a_{zz}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=a_{xx}\vec{i}\vec{i}+a_{xy}\vec{i}\vec{j}+\ldots+a_{zz}\vec{k}\vec{k}\quad \text{(rank 2 tensor)}$$

Also, rank 4 tensors are needed. Their representations require basis vector quadruplets and 4 indices in the components.

TENSOR COMPONENTS

The multipliers of the basis vector singlets, doublets, etc. of a tensor are called as the components. The components of the first and second order tensors can be represented as column { } and square [ ] matrices:

Vector $$\vec{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \text{ of components } \{u\} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}$$

index 1 → row

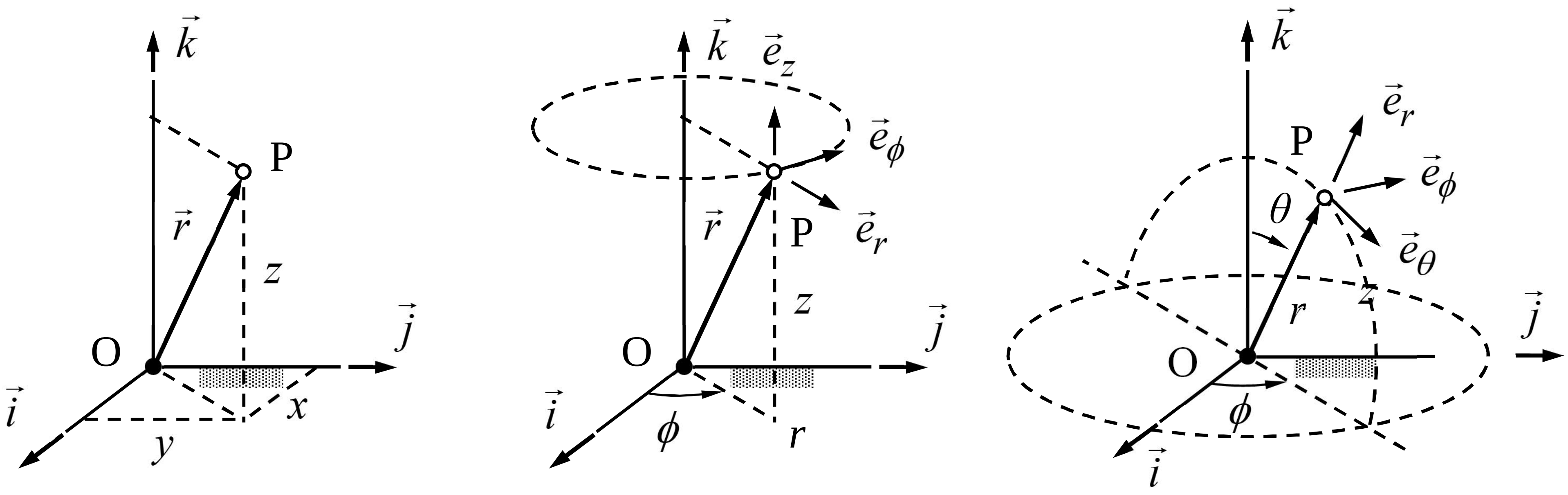
index 2 → column

Tensor $$\vec{\vec{\sigma}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \text{ of components } [\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Notice. A column matrix is often called as the vector. Here, vector is a tensor of rank 1.

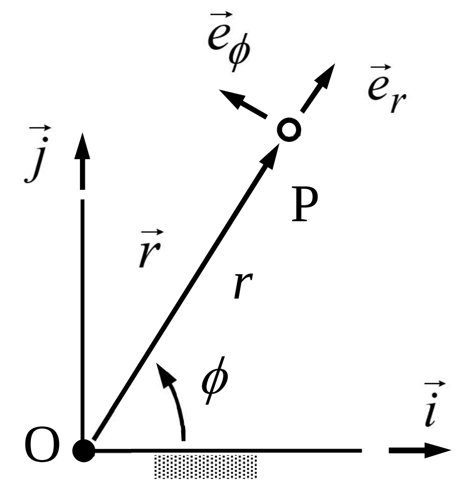
INVARIANCE

Tensor quantities are invariant with respect to coordinate system. Representation change from one coordinate system to another but the tensor does not. Rectilinear-orthonormal (Cartesian) and curvilinear-orthonormal coordinate systems are common choices for the tensor representations.



Change from one coordinate system to another requires the relationship between the basis vectors. Considering $\overleftrightarrow{a}$ of a planar case and using the relationship between the basis vectors of the Cartesian and polar coordinate systems shown $(\mathrm{c} \sim \cos,\ \mathrm{s} \sim \sin)$

$$\overleftrightarrow{a} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \text{ and } \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi \\ \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \Leftrightarrow$$



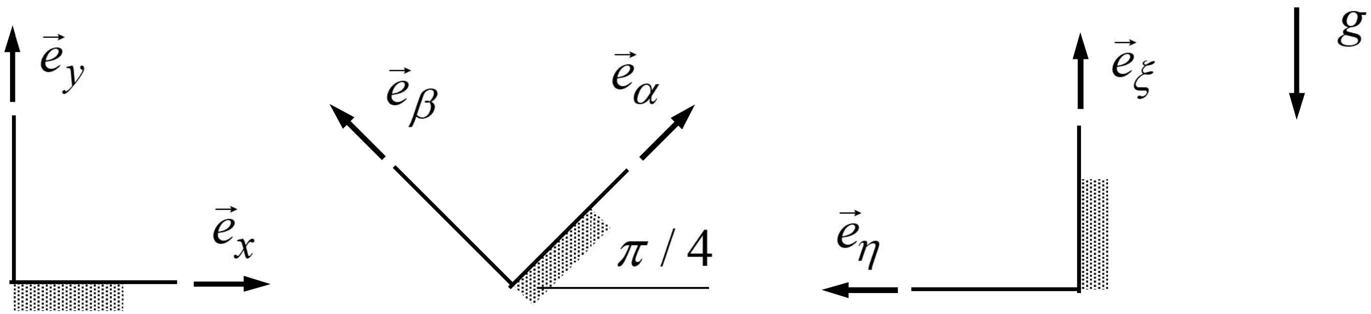
$$\overleftrightarrow{a} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathrm{c}\phi & \mathrm{s}\phi \\ -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi \\ \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \Leftrightarrow$$

$$\overleftrightarrow{a} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{rr} & a_{r\phi} \\ a_{\phi r} & a_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \text{ where } \begin{bmatrix} a_{rr} & a_{r\phi} \\ a_{\phi r} & a_{\phi\phi} \end{bmatrix} = \begin{bmatrix} \mathrm{c}\phi & \mathrm{s}\phi \\ -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi \\ \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix}.$$

EXAMPLE Acceleration by gravity $\vec{g}$ can be represented in any of the coordinate systems of the figure starting with the known representation in one of the systems. Starting with the representation $\vec{g} = -g\vec{e}_y$ and using the relationship between the basis vectors

$$\vec{g} = -g\vec{e}_y = -g(\vec{e}_\alpha + \vec{e}_\beta)/\sqrt{2} = -g\vec{e}_\xi .$$

All these give the same direction and magnitude for the acceleration by gravity.

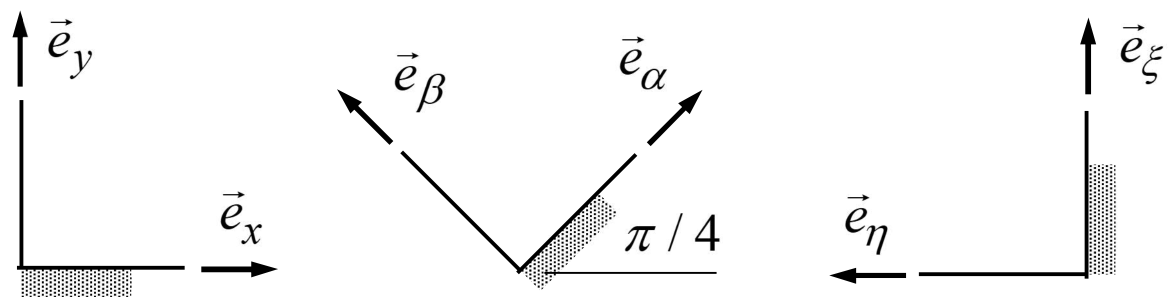


$$\vec{e}_y = \frac{1}{\sqrt{2}}(\vec{e}_\alpha + \vec{e}_\beta) \qquad \vec{e}_y = \vec{e}_\xi$$

EXAMPLE Second order tensor $\vec{\vec{a}}$ can be represented in any coordinate systems of the figure starting with the known representation $\vec{\vec{a}} = a\vec{e}_y\vec{e}_y$ in the Cartesian system. Using the relationship between the basis vectors

$$\vec{\vec{a}} = a\vec{e}_y\vec{e}_y = \frac{a}{2}(\vec{e}_\alpha\vec{e}_\alpha + \vec{e}_\alpha\vec{e}_\beta + \vec{e}_\beta\vec{e}_\alpha + \vec{e}_\beta\vec{e}_\beta) = a\vec{e}_\xi\vec{e}_\xi \,.$$

Graphical representation of a rank 2 tensor is not as obvious as that of a vector.



TENSOR PRODUCTS

In manipulation of an expression containing tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order matters). For simplicity of presentation, outer (tensor) products like $\vec{a}\otimes\vec{b}$ are denoted by $\vec{a}\vec{b}$ in MEC-E8003. Otherwise, the usual rules of vector algebra apply:

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z ,$$

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k} ,$$

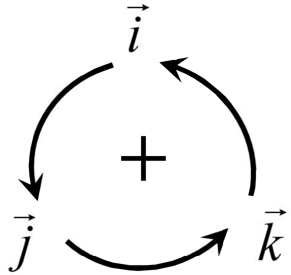
$$\vec{a}\vec{b} = a_x b_x \vec{i}\vec{i} + a_x b_y \vec{i}\vec{j} + a_x b_z \vec{i}\vec{k} + a_y b_x \vec{j}\vec{i} + a_y b_y \vec{j}\vec{j} + a_y b_z \vec{j}\vec{k} + a_z b_x \vec{k}\vec{i} + a_z b_y \vec{k}\vec{j} + a_z b_z \vec{k}\vec{k} .$$

Calculation with tensors is straightforward although the number of terms may make manipulations somewhat tedious.

As an example, manipulations needed to find the cross-product of two vectors in a Cartesian system (orthonormal and right-handed) consists of steps

$$\vec{a}\times\vec{b} = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k})\times(b_x\vec{i} + b_y\vec{j} + b_z\vec{k}) \quad \Leftrightarrow$$

$$\vec{a}\times\vec{b} = a_x b_x \vec{i}\times\vec{i} + a_x b_y \vec{i}\times\vec{j} + a_x b_z \vec{i}\times\vec{k} +$$



$$a_y b_x \vec{j}\times\vec{i} + a_y b_y \vec{j}\times\vec{j} + a_y b_z \vec{j}\times\vec{k} +$$

$$a_z b_x \vec{k}\times\vec{i} + a_z b_y \vec{k}\times\vec{j} + a_z b_z \vec{k}\times\vec{k} \quad \Rightarrow$$

$$\vec{a}\times\vec{b} = 0 + a_x b_y \vec{k} - a_x b_z \vec{j} - a_y b_x \vec{k} + 0 + a_y b_z \vec{i} + a_z b_x \vec{j} - a_z b_y \vec{i} + 0 \quad \Leftrightarrow$$

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k}. \quad \Leftarrow$$

The manipulations are often (but not always) easier when the components and basis vectors are arranged as matrices

$$\vec{a} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} \text{ and } \vec{b} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Rightarrow$$

$$\vec{a} \times \vec{b} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \times \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & \vec{k} & -\vec{j} \\ -\vec{k} & 0 & \vec{i} \\ \vec{j} & -\vec{i} & 0 \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} \quad \Leftrightarrow$$

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k}. \quad \Leftarrow$$

EXAMPLE The local forms of the balance laws of momentum and moment of momentum are $\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = 0$ and $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}}$ (conjugate tensor). Assuming a planar case and a Cartesian coordinate system so that

$$\nabla = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}, \quad \vec{f} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}, \text{ and } \vec{\vec{\sigma}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}$$

derive the component forms of the balance laws.

Answer $\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + f_x = 0$, $\dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + f_y = 0$ and $\sigma_{xy} = \sigma_{yx}$

In a Cartesian system, basis vectors are constants and one may transpose the gradient operator to get (transposing cannot be used with non-constant basis vectors! Why?)

$$\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\end{Bmatrix}^{\mathrm{T}}(\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}\cdot\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}})\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}+\begin{Bmatrix}f_x\\ f_y\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0$$

$$\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=(\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}+\begin{Bmatrix}f_x\\ f_y\end{Bmatrix}^{\mathrm{T}})\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0. \quad \Leftarrow$$

$$\vec{\vec{\sigma}}-\vec{\vec{\sigma}}_{\mathrm{c}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}-\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0 \quad \Leftrightarrow$$

$$\vec{\vec{\sigma}}-\vec{\vec{\sigma}}_{\mathrm{c}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}0 & \sigma_{xy}-\sigma_{yx}\\ \sigma_{yx}-\sigma_{xy} & 0\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0. \quad \Leftarrow$$

SOME DEFINITIONS AND IDENTITIES

Conjugate tensor $\overleftrightarrow{a}_{\mathrm{c}}$: $\overleftrightarrow{a} \cdot \vec{b} = \vec{b} \cdot \overleftrightarrow{a}_{\mathrm{c}} \quad \forall \vec{b}$

Second order identity tensor $\overleftrightarrow{I}$: $\overleftrightarrow{I} \cdot \overleftrightarrow{a} = \overleftrightarrow{a} \cdot \overleftrightarrow{I} = \overleftrightarrow{a} \quad \forall \overleftrightarrow{a}$

Fourth order identity tensor $\overset{\leftrightarrow}{\overset{\leftrightarrow}{I}}$: $\overset{\leftrightarrow}{\overset{\leftrightarrow}{I}} : \overleftrightarrow{a} = \overleftrightarrow{a} : \overset{\leftrightarrow}{\overset{\leftrightarrow}{I}} = \overleftrightarrow{a} \quad \forall \overleftrightarrow{a}$

Associated vector $\vec{a}$ **of an antisymmetric tensor** $\overleftrightarrow{a}$: $\vec{b} \cdot \overleftrightarrow{a} = \vec{a} \times \vec{b}$, when $\overleftrightarrow{a} = -\overleftrightarrow{a}_{\mathrm{c}}$

Scalar triple product $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

Vector triple product $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

Symmetric-antisymmetric double product $\overleftrightarrow{a} = -\overleftrightarrow{a}_{\mathrm{c}}$ and $\overleftrightarrow{b} = \overleftrightarrow{b}_{\mathrm{c}} \quad \Rightarrow \quad \overleftrightarrow{a} : \overleftrightarrow{b} = 0$

Symmetric-antisymmetric division $\overleftrightarrow{a} = \overleftrightarrow{a}_s + \overleftrightarrow{a}_u = \frac{1}{2}(\overleftrightarrow{a} + \overleftrightarrow{a}_{\mathrm{c}}) + \frac{1}{2}(\overleftrightarrow{a} - \overleftrightarrow{a}_{\mathrm{c}})$