Figure 2.12

The other two special features are the *node* and the *bitangent* (Figure 2.12). Clearly these are dual to one another, the node being a point through which the pointwise curve passes twice, and the bitangent being a line through which the linewise curve passes twice.

3. Collineation

3.1 Desargues' triangle theorem

Let us suppose that we have two triangles *ABC* and *A'B'C'* situated more or less like those in Figure 3.1.

We notice that the common lines of corresponding points *AA'*, *BB'* and *CC'* form a little triangle, while the common points of pairs of corresponding sides *aa'*, *bb'* and *cc'* form a long thin triangle. By changing the two triangles very slightly, one could arrange that the little triangle of three lines *AA'*, *BB'* and *CC'* shrinks into a point; one would then find that the long thin triangle *aa' bb' cc'* melts into one line (Figure 3.2).

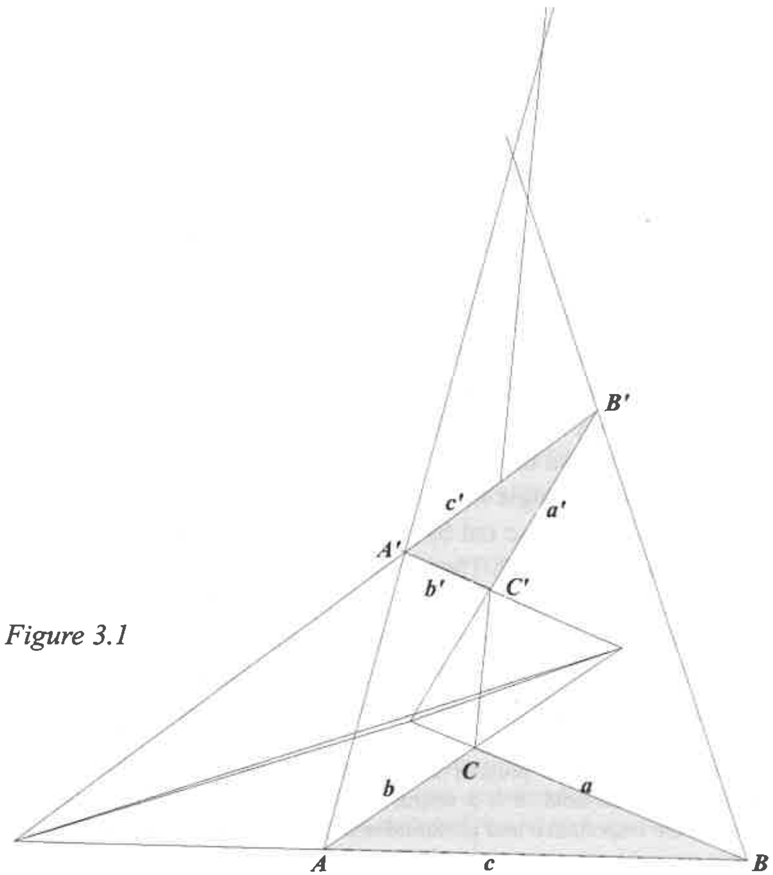

Figure 3.1

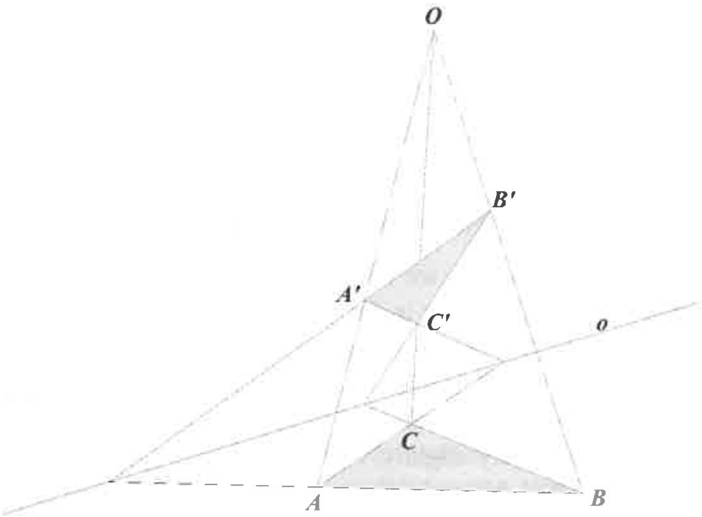

Figure 3.2

This is *Desargues' Triangle Theorem*, which may be stated thus:

If two triangles are such that the lines common to corresponding points meet in a point, then the points common to corresponding lines lie on a line.

It is interesting to note that this theorem cannot be proved from the propositions of incidence within the plane. However, as soon as we see the theorem in the context of three dimensions its truth becomes readily accessible to the imagination.*

Consider Figure 3.2. Imagine a trihedron with its apex at point O. This will be lines AA', BB' and CC', imagining, let us say, CC' the centre one to be little behind the other two. This trihedron is cut by a plane and its section is the triangle ABC.

Now the trihedron is also cut by another plane and on that plane its section is the triangle $A'B'C'$. These two triangles are now, as pictured in perspective, in the Desargues' relation — that is, the common lines of corresponding pairs of points all meet at a point, O.

Now consider the lines AB and $A'B'$. We must first assure ourselves that they do in fact meet, for now that we are working in space, two

* It is possible to 'invent' a 'projective' plane geometry in which the Desargues proposition does not hold. It is a weird, artificial geometry. But its existence emphasizes the importance and profoundness of the theorem. Also it suggests that plane projective geometry can hardly exist without a third dimension.

lines do not necessarily meet — they may be skew. However we see that these two lines have a plane in common, the front plane of the trihedron. Therefore, by the propositions of incidence they must also have a point in common. Similarly we can assure ourselves that BC and $B'C'$, as well as CA and $C'A'$, also meet; they do not pass one behind the other only appearing to meet in our figure.

But, again by the propositions of incidence, any two distinct planes meet in a line. Therefore the meeting points of corresponding sides must all lie on a line, o, the common line of the two planes which contain those sides.

We see that Desargues' theorem is in fact just a two dimensional picture of a three dimensional truth which is almost self-evident.*

3.2 Shadow drawing

From this we can get a useful method of shadow drawing in perspective. We can imagine that point O is a source of light. Then the triangle ABC becomes the shadow of triangle $A'B'C'$. We see that every line connecting a point with its shadow passes through one point (O, the source of light) and that each line and its shadow, produced if necessary, will meet in a point which lies on one particular line (the common line, o, of the two planes).

As an example, draw a horizontal line l to represent the meet of the horizontal plane (the ground) with a vertical plane. High up in the figure place a point O, to represent the sun and in the vertical plane draw any object you wish.

Now select some point of the object, A, and draw the line OA. As soon as you fixed point O you sacrificed a part of your freedom, for the shadow A' of A must lie somewhere along the line OA. You are still free to decide where, so now place A' in a convenient position. It is helpful in drawing this figure:

1 to have the object, whose shadow you are seeking, slightly above line l,
2 to make the object a simple two-dimensional form — say a cross, or five-pointed star, or some other simple form
3 to place point A at the very top of the object, and

* To complete the proof one has to show that there exists a space configuration which is perspective with the plane one, or equivalently, one has to find a triangle outside the plane which is perspective with both ones in the plane. This, however, is straightforward.

4 to place its shadow, A', on the other side of line l, low down on the page along the line OA.

Having done so you have no more freedom left; the whole phenomenon, in relation to your position as viewer, is now fixed.

Next take any other point of the object, B. Draw OB, and, of course, B' must lie somewhere along this line. Now draw line AB, it meets line l in P. B' must also be on PA'. So the meeting point of PA' and OB is B', which is the shadow of B. In the same way all the rest of the shadow may be constructed.

Notice that having found B' in this way, you may proceed to find the point C' corresponding to some C, by the same process. But you can also find point C' by using points B and B' in the same way that you used A and A' and every time the same point C' will result. One finds here a wonderful spatial anastamosis. In doing a large figure one often needs, for the sake of convenience, to change the pair of points which one is using.

Exercise 3a

Repeat the above construction using a circle for the object in the vertical plane. Obviously the light streaming through this circle forms a cone and this is caught on the second plane as a conic section.

3.3 Collineation

The construction of the previous section is something more far-reaching than just a method of drawing shadows. Let us forget the three dimensional aspect for a moment and look at it as just a plane figure. We find that we have here a *projective transformation* of the plane within itself. We may say that we transform the plane into itself, so that A transforms into A', B into B', etc. We see that the transformation is one-to-one, that is to say, each point transforms uniquely into one point only. *Projective* means that incidence is kept: if some point X is on a line y before the transformation, after the transformation the image point X' must be on the image line y'.

A projective transformation which turns points into points, lines into lines and, in space, planes into planes, one to one, is in general known as a *collineation*. Clearly the order and class of any curve remains unchanged in any collineation.*

* The other type of projective transformation is *correlation* or *polarity*, which turns points into planes and vice versa, and lines into lines; or, in the plane, which turns points into lines and vice versa.

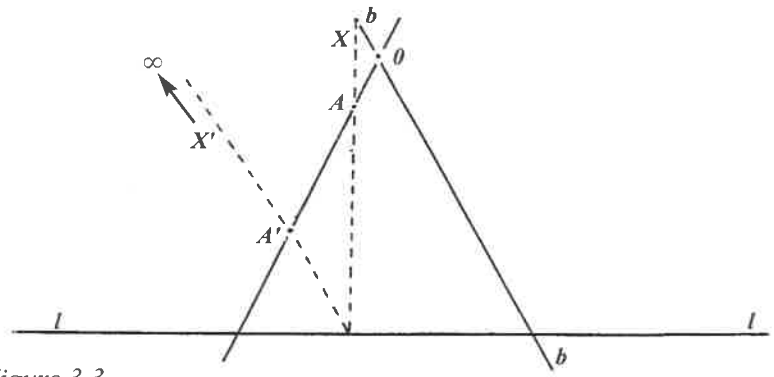

Figure 3.3

Notice that there is a line of self-corresponding points, line l, called the *axis* of the collineation, and a point of *self-corresponding* lines, point O, its *centre*. But the points of these self-corresponding lines are not themselves self-corresponding; each transforms into another point of the same line, excepting of course, point O and the common point with line l. Similarly the lines of the self-corresponding points of line l are not themselves self-corresponding, but each transforms into another line of the same point, excepting of course, line l and the common line with point O. It can be seen that the functions of point O and of line l are in every way dual to one another. The transformation with which we are dealing here is known as *homology*, and it is a special case of a collineation, viz. a collineation with a line of self-corresponding points and a point of self-corresponding lines.

Let us establish a homology, as in Figure 3.3, the self-corresponding elements being point O and line l.

Note that it is not necessary for A' to be on the other side of line l from A. Consider any other line b, through point O. We wish to find the point X of this line which will transform into the point at infinity X' of the line. We draw the line $X'A'$ parallel to b, and from where this meets line l we draw a line through point A to meet b in X. Now for all the other lines through O we can find their points X which will transform into their points at infinity. If the infinite points of a plane really lie on a straight line, these points X ought to be found to be collinear. We can say that their line (let us call it x) transforms into the line at infinity. But a line and that into which it transforms are bound to meet on line l. In other words, lines x and l should meet at infinity, i e. they should be parallel.