nately through A and B. Then the straight lines connecting opposite vertices intersect at one point (the Brianchon point of the hexagon).

tices lie alternately on a and b. Then the points of intersection of opposite sides lie on one straight line (the Pascal line of the hexagon).

Evidently the figure corresponding to the last theorem of Pascal must be the dual of the configuration $(9_3)_1$. Now the dual figure of a configuration $(p_\gamma l_\pi)$ is always another configuration, and its symbol is $(l_\pi p_\gamma)$. The special configurations we have denoted by the symbol (p_γ), and they only, have as duals configurations with the same symbol. It is conceivable that the configuration of Pascal's theorem, i.e. the dual of $(9_3)_1$, might be one of the other two con-

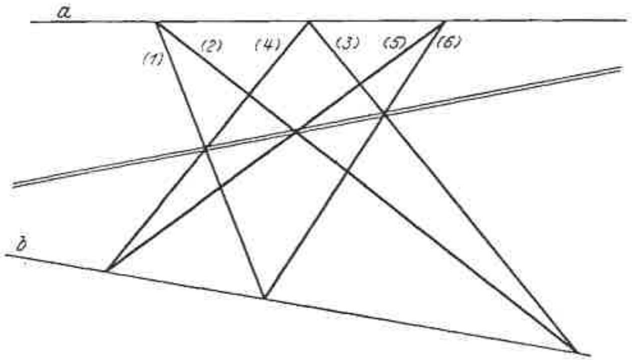

FIG. 131

figurations (9_3). It is found, however, that Pascal's theorem is also represented by the symbol $(9_3)_1$ (see Fig. 131). This is the reason why we have called the configuration the Brianchon-Pascal configuration from the very beginning. Thus $(9_3)_1$ is "dually invariant" or "self-dual." Just as the Brianchon point could be chosen arbitrarily, so we can also choose an arbitrary straight line of the configuration to serve as the Pascal line.

By using the ideal elements we can arrive at a special case of the last Pascal theorem which would not otherwise seem to have any connection with the original theorem. For, by moving the Pascal line to infinity we get the following theorem (Fig. 132): If the vertices of a hexagon lie alternately on two straight lines, and if two pairs of opposite sides are respectively parallel, then the third pair of opposite sides is also parallel.

This special case of Pascal's theorem is called Pappus' theorem.[2]

Having seen that $(9_3)_1$ is self-dual, it is easy for us to conclude that $(9_3)_2$ and $(9_3)_3$ must also be self-dual. For, the only other possibility would be that the figure obtained from $(9_3)_2$ by applying the duality principle is $(9_3)_3$. But since $(9_3)_2$ is a regular con-

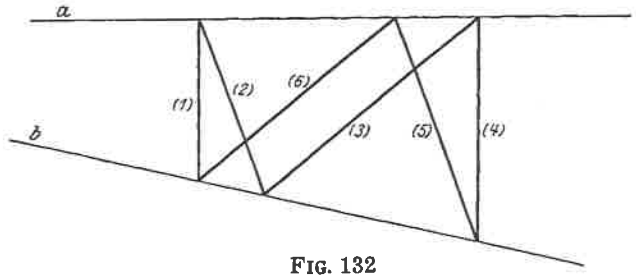

FIG. 132

figuration and $(9_3)_3$ is not, neither of these figures can be the dual of the other.

We shall now take up the configurations (10_3). In order to understand the most important one of these, Desargues' configuration, it is necessary to extend the method of introducing ideal elements, and the principle of duality, from the plane to three-dimensional space.

§ 19. Ideal Elements and the Principle of Duality in Space. Desargues' Theorem and the Desargues Configuration (10_3)

We have arrived at the concept of the projective plane by studying projection in space. Now projective geometry also changes the space as a whole, by the addition of ideal elements, into "projective space," an entity that is in many ways simpler. Only, it is not possible in this case to justify the procedure in visual terms; it is purely abstract. To begin with, we introduce the ideal elements in all the planes of ordinary space according to the principle discussed earlier. Then it appears reasonable to interpret the entity formed by all the ideal points and straight lines as a plane, the "infinitely distant" or "ideal" plane of the space. For, this entity shares with the ordinary planes in space the property that any given plane intersects it in a straight line, the ideal straight line

[2] Frequently the more general theorem, which is called here the fourth Pascal theorem, is also referred to as Pappus' theorem. [*Trans.*]

of the given plane. Every ordinary straight line has only one point, its ideal point, in common with the ideal plane, just as it has only one point in common with any other plane that does not contain the line. Moreover two planes are parallel if and only if they have the same ideal line.[1]

A great many phenomena of the geometry of space are simplified by this point of view. Thus parallel projection can be regarded as a special case of central projection in which the center of projection is an infinitely distant point. Furthermore, to give another example, the difference between the hyperboloid of one sheet and the hyperbolic paraboloid may be characterized by the property that the hyperboloid intersects the ideal plane in a non-degenerate conic whereas the paraboloid intersects it in a pair of generating straight lines of the surface; this distinction amounts to the same thing as the fact explained on page 15, that three skew straight lines lie on a paraboloid rather than on a hyperboloid if and only if they are parallel to a fixed plane; for, this is equivalent to the condition that the three straight lines meet one ideal line, which consequently lies on the surface since it has three points in common with it.

It is clear that all planes of projective space must be regarded as projective planes, so that the principle of duality in the plane is true for them. But the space as a whole is also governed by a different principle of duality as well.

To arrive at this, we proceed as in the plane, compiling the list of axioms by which the incidence of points, straight lines, and planes in space must be regulated if finite and infinitely distant elements are treated alike. The axioms may be formulated as follows:

1. Two planes determine one and only one straight line; three planes that do not pass through a common straight line determine one and only one point.

2. Two intersecting straight lines determine one and only one point and one and only one plane.

3. Two points determine one and only one straight line; three points not on one straight line determine one and only one plane.

This system of axioms remains unaltered if the words "point" and "plane" are interchanged. (The first axiom is interchanged

[1] For, the property of two planes being parallel, and also the property of their having the same ideal line, are each equivalent to the property that parallels to every straight line of one plane can be drawn in the other.

with the third, and the second is unchanged.) The set of remaining axioms of the projective geometry of space is also left unaltered by this interchange. Thus the point and the plane correspond to each other dually, and the straight line corresponds to itself. The set of all points of a surface corresponds dually to the set of all tangent planes to another surface. As was the case with the conics in the plane, the second-order surfaces in space are self-dual.

The simplest and at the same time most important theorem of three-dimensional projective geometry is named after Desargues. Desargues' theorem may be stated as follows (see Fig. 133):

Two triangles ABC and $A'B'C'$ in space being given, let them be so placed that the lines connecting corresponding vertices pass

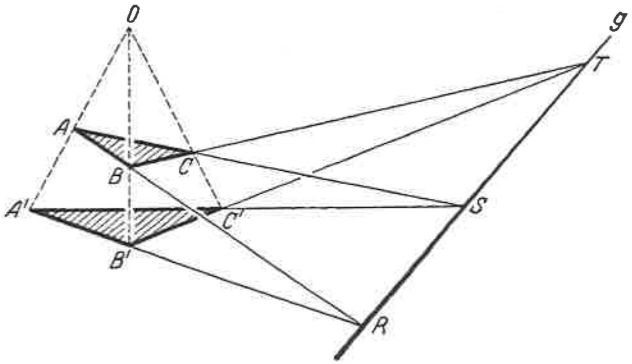

FIG. 133

through a single point O. Then the three pairs of corresponding sides have points of intersection, R, S, and T, and these points of intersection are, moreover, collinear.

The first part of the theorem is easy to prove. By the second axiom for space, the two intersecting straight lines AA' and BB' define a common plane. The straight lines AB and $A'B'$ also lie in this plane, whence it follows, by the second axiom for incidence in the plane, that these two straight lines have a point of intersection R. (R may be a finite or an ideal point.) The existence of the two other points of intersection, S and T, is proved analogously.

The truth of the second part of the theorem is easy to see in the case where the triangles are in different planes. In this case the planes of the triangles determine a common—ordinary or ideal—straight line of intersection (by Axiom 1 for space). Of every pair of corresponding sides of the triangles one lies in one of these planes and the other lies in the other plane. Since we have seen that the sides of such a pair intersect, their point of intersection must be on the straight line that the two planes have in common. This proves Desargues' theorem for the general case.

But it is precisely the special case where the triangles are coplanar that is of particular importance. Here we may apply a method of proof similar to the proof for Brianchon's theorem, in which we project a spatial figure onto the plane. We only need show that every plane Desargues figure is a projection of a three-dimensional Desargues figure. To this end, we connect all the points and straight lines of the plane Desargues figure with a point S out-

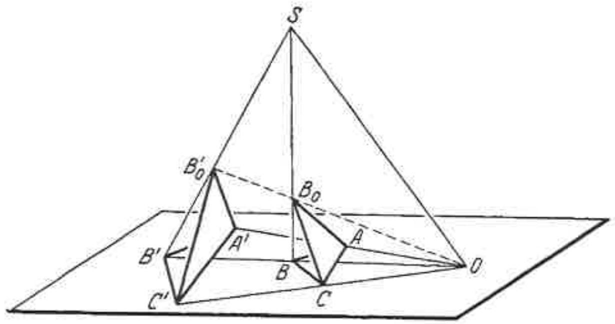

FIG. 134

side the plane of the figure (see Fig. 134). We then pass a plane through the straight line AC intersecting BS at a point B_0 distinct from S, and draw OB_0. The straight lines OB_0 and $B'S$ are coplanar and therefore have a point of intersection B_0'. But now the triangles AB_0C and $A'B_0'C'$ form a three-dimensional Desargues figure, since all the straight lines connecting corresponding vertices pass through O. Projecting the line in which the planes of these triangles intersect from S onto the original plane, we get a straight line on which the pairs of corresponding sides of the original triangles ABC and $A'B'C'$ must intersect. This completes the proof of Desargues' theorem.

The principle of duality for the plane and the one for space both lead to interesting consequences of Desargues' theorem. To begin with, it is readily seen that the converse of the theorem is also true; i.e. the existence of a Desargues line containing the points of inter-

section of pairs of corresponding sides of the two triangles implies the existence of the Desargues point through which the lines connecting corresponding vertices pass. In the case where the triangles are coplanar, the converse of Desargues' theorem proves to be the same as the theorem we obtain from Desargues' theorem by applying the principle of duality in the plane. We can elucidate this by writing the two theorems side by side, as follows:

Let three pairs of points AA', BB', CC' be given, such that the three lines determined by the pairs pass through a common point. Then the three points of intersection of the pairs of straight lines AB and $A'B'$, BC and $B'C'$, CA and $C'A'$, lie on one straight line.

Let three pairs of straight lines aa', bb', cc' be given, such that the points of intersection of the pairs lie on one straight line. Then the lines joining the pairs of points (ab) and $(a'b')$, (bc) and $(b'c')$, (ca) and $(c'a')$, pass through a common point.

Let us examine the figure (Fig. 135) consisting of the vertices and sides of two coplanar Desargues triangles together with the lines joining pairs of corresponding vertices, the points where pairs of corresponding sides meet, the Desargues point O, and the

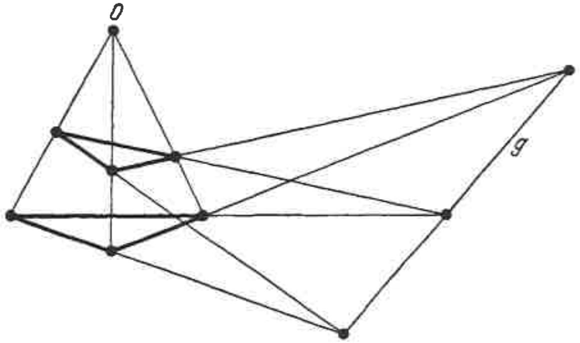

FIG. 135

Desargues line g. It is a simple matter of counting to see that the figure is a configuration of type (10_3). It is called the Desargues configuration. This configuration shares with Pascal's configuration the property that the last incidence condition is automatically satisfied when the figure is constructed step by step from its table. Furthermore, the Desargues configuration, like Pascal's, is self-dual. This is seen to be true because the configuration represents both Desargues' theorem and its converse, and the latter is the dual of the former.

We next consider the result obtained from the three-dimensional case of Desargues' theorem on applying the principle of duality *in space*. We get the following juxtaposition:

Let three pairs of points AA', BB', CC', be given such that the three lines determined by the pairs pass through a common point. Then the three points of intersection of the pairs of straight lines AB and $A'B'$, BC and $B'C'$, CA and $C'A'$, lie on one straight line.

Let three pairs of planes $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, be given such that the three lines of intersection determined by the pairs lie in one plane. Then the three planes containing the pairs of straight lines $(\alpha\beta)$ and $(\alpha'\beta')$, $(\beta\gamma)$ and $(\beta'\gamma')$, $(\gamma\alpha)$ and $(\gamma'\alpha')$, pass through one straight line.

Fig. 136 illustrates the theorem that appears in the right-hand column. In this theorem the two triangles are replaced by two trihedral angles formed by the planes α, β, γ and α', β', γ', respectively. Paralleling what we have done in the case of the plane Desargues figure, we shall now examine the three-dimensional figure consisting of the two Desargues trihedra together with the planes determined by pairs of corresponding edges, the lines of intersection of corresponding pairs of faces, the "Desargues plane" ($\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ in Fig. 136), and the "Desargues line" (VW in the figure). The intersection of this three-dimensional figure with any plane that does not contain any of the points V, W, X, Y, Z is a plane Desargues configuration, since the Desargues trihedra intersect the plane in Desargues triangles. To the planes and straight lines of the space figure there correspond the straight lines and points of the plane configuration. However, the three-dimensional figure has an intrinsic symmetry that is not reflected in the plane figure. The space figure consists of all the connecting straight lines and plane of the

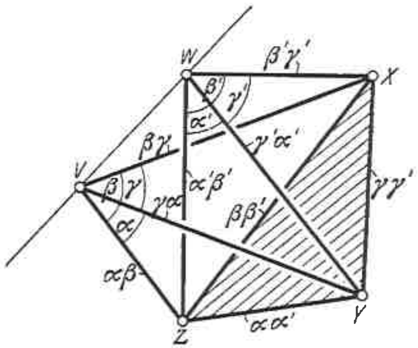

FIG. 136

five points V, W, X, Y, Z, and the roles of the five points are completely equivalent. Conversely, every complete five-point in space becomes a three-dimensional Desargues figure if two of the vertices are arbitrarily chosen as vertices of the Desargues trihedra.[2] From the fact that all the straight lines and all the planes of the spatial figure play the same role, it follows that the same is true for the points and the straight lines of the plane Desargues configuration. This proves that the Desargues configuration is regular, so that the choice of the Desargues point or the Desargues line in the configuration can be made quite arbitrarily.[3]

We shall now represent the Desargues configuration as a pair of mutually inscribed and circumscribed pentagons. To this end, we first look for any pentagons at all in the configuration, where it is required that all the vertices and sides of the polygon be elements of the configuration and no three consecutive vertices be collinear. The problem is considerably simplified by going back to the five-point in space. The vertices of the plane polygon are associated with the corresponding edges of the five-point in space. Since it is required that any two consecutive vertices of the plane polygon lie on a straight line of the configuration, the corresponding edges must be in one plane and must therefore intersect. To ensure that no three consecutive vertices are collinear, we need only see to it that the corresponding edges are not coplanar; this would happen if and only if three consecutive edges formed a triangle. By passing through the vertices V, W, X, Y, Z of the three-dimensional five-point in any order, say in the order in which they are written, we obtain a closed polygonal path of the kind we need; in the plane

[2] The only condition the five points must satisfy is that they be in general position, i.e. that no four of them be coplanar and hence no three of them collinear.

[3] By a complete n-point in space we mean the set of all the straight lines and planes connecting n points in general position in space. As in the case $n=5$, the section of the complete n-point, for any value of n, by a plane that does not pass through any of the vertices is a configuration. These configurations are regular and of type $p=\frac{n(n-1)}{2}$, $\gamma=n-2$, $g=\frac{n(n-1)(n-2)}{6}$, $\pi=3$.

It follows that a configuration of the special type where $p=l$ is only obtained in the case $n=5$. Other regular configurations can be obtained by using n-points in general position in higher-dimensional spaces. All these configurations are called "polyhedral."

configuration it furnishes a pentagon of the required type. But the edges of the three-dimensional five-point that were not used in this path constitute a second three-dimensional polygon of the same kind. For, two unused edges pass through every vertex of the five-point in space, since every vertex is incident with four edges in all,

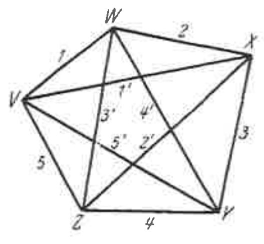

FIG. 137a

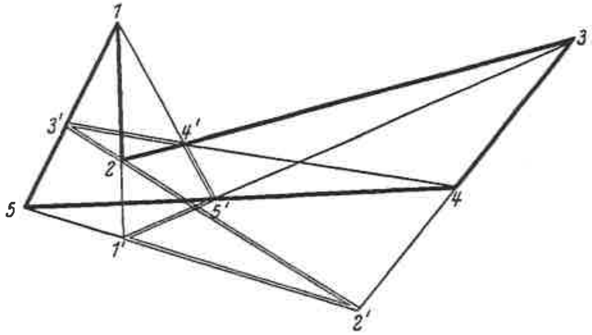

FIG. 137b

two of which were used up for the first path. This second polygonal path corresponds to a second pentagon in the configuration, and a simple enumeration reveals that this must be inscribed in the first pentagon. Because of symmetry, the first pentagon is also inscribed in the second pentagon. Figs. 137a and 137b illustrate

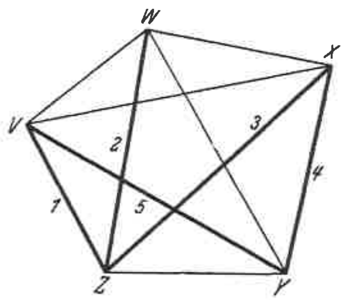

FIG. 138

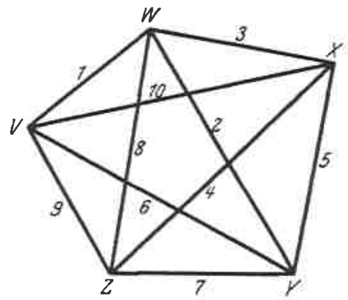

FIG. 139

the way in which the three-dimensional arrangement and the plane pair of pentagons are related.

We can also find other types of systems of five edges of the five-point in space corresponding to pentagons contained in the plane configuration. An example is given in Fig. 138. But it can be verified that it is then impossible to arrange the five remaining edges

cyclically in such a way that any two consecutive edges have a common point and no three consecutive edges form a triangle. Hence the construction given in the beginning exhausts all the possibilities. Since an automorphism of the configuration corresponds to every permutation of the vertices and since the decomposition of the five-point in space into two polygonal paths is completely determined by the order of the vertices in the first path, we see that, leaving aside automorphisms, there is only one possible decomposition of the Desargues configuration into two mutually inscribed pentagons.

The question of whether, and in how many ways, the Desargues configuration can be considered as a self-inscribed and self-circumscribed decagon, can be settled by the same method. It is

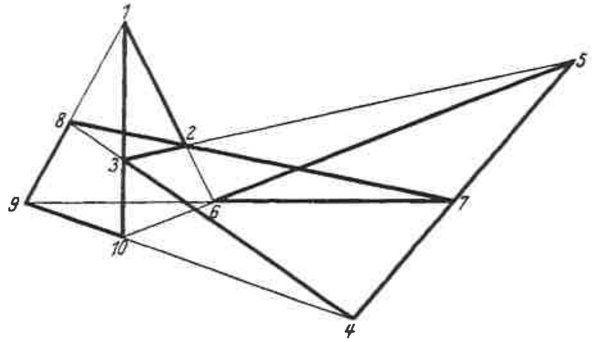

FIG. 140

found that the arrangement of edges in space corresponding to such a decagon can always be chosen as indicated in Fig. 139. Accordingly there is one way, and except for automorphisms only one way, of interpreting the Desargues configuration as a ten-sided polygon inscribed in and circumscribed about itself (Fig. 140). The figure exhibits a certain regularity; if we move along the sides of the decagon from the point 1 to the point 2, from 2 to 3, etc., in order, then one vertex is omitted on each side, and the numbers of the omitted vertices form a sequence in which pairs of successive numbers differ alternately by 1 and 3 (the vertex 5 is omitted on side 23, 8 on 34, 7 on 45, 10 on 56, etc.). Another feature of the decagon revealed by the three-dimensional arrangement is that the sides belong alternately to two mutually inscribed pentagons.

Desargues' configuration is not the only configuration with the symbol (10_3). In fact, there are nine other possibilities for the schematic table of such a configuration. One of these tables has the same property as the table for (7_3), namely that its configuration cannot be realized either in the real plane or in terms of complex coordinates, because its equations are incompatible. On the other hand, the remaining eight configurations of the form (10_3), like the configurations (9_3), can all be constructed with a ruler alone. But they are differentiated from the Desargues configuration by the fact that the last incidence condition is not automatically satisfied in their construction. Thus they do not express a geometrical theorem and are therefore not as important as the configuration of Desargues. One of these configurations is drawn in Fig. 141. It also represents a self-inscribed and self-circumscribed decagon if the points are taken in the numerical order given in the figure, but here the numbers of the vertices successively omitted on the sides of the polygon always differ by 1. In this arrangement all the vertices play the same role, and the sides are interchangeable with the vertices. It follows that the configuration is regular and self-dual.

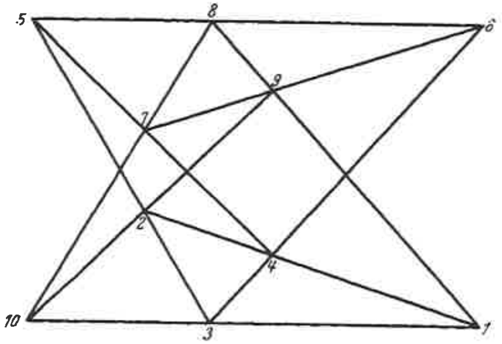

FIG. 141

§ 20. Comparison of Pascal's and Desargues' Theorems

We have found Desargues' theorem and the last of Pascal's theorems to be analogous in many ways. Both theorems were proved by the projection of three-dimensional figures. Both theorems gave rise to configurations, and quite similar configurations at that, both configurations were regular and self-dual, both could be constructed with a ruler alone, the last incidence in both occurred automatically, and both could be regarded as self-inscribed and self-circumscribed polygons.

Nevertheless there is a fundamental difference between the two theorems. The space figure used in the proof of Desargues' theorem can be constructed on the basis of the given axioms for incidence

in space, without the assumption of any additional axioms. The Pascal-Brianchon configuration, on the other hand, was obtained by studying a second-order surface. To be sure, the core of the proof appears to be purely a consideration of the incidence relations between the points, straight lines, and planes of a hexagon in space, but on closer examination it is found that the construction of such hexagons in space is essentially equivalent to the construction of a ruled surface of the second order and that the possibility of such a construction cannot be proved from the axioms of incidence alone.

In the first chapter we introduced the conic sections and quadric surfaces on the basis of metric considerations. It might therefore be thought that Pascal's theorem could not be proved without comparisons of lengths and angles. But the curves and ruled surfaces of the second order can also be generated without the help of metric methods, by using the method of projection. By this method, the points of a given straight line can be mapped into the points of any other straight line in such a way that any three pre-assigned points on the first line go into three pre-assigned points on the second line and all harmonic sets of points on the first line become harmonic sets on the second. The first straight line is then said to be mapped projectively onto the second straight line. The construction of such a mapping (or "projectivity") requires only the axioms of incidence in the plane and in space. But the proof that the mapping is uniquely determined for all the points of the straight lines by the two given conditions—that harmonic sets become harmonic sets and that the mapping of three points is given—requires more than just these axioms. We need for this purpose an axiom of continuity which we shall formulate presently. But once the uniqueness of the projectivity in the given sense is proved, we can define the most general ruled surface of the second order as the surface swept out by a variable straight line that always connects corresponding points in a projectivity of two fixed skew straight lines. It then follows from the uniqueness property of the projectivity that a second family of straight lines also lies on the surface defined in this way. If the straight lines related by the projectivity are not skew but intersecting, then the straight line connecting pairs of corresponding points moves in a plane and envelops a curve of the second order. All the properties