the general third-order surface contains at most a finite number of straight lines.[1]

An enumeration similar to the above shows that the general surfaces of order higher than the third do not in general contain any straight lines.

§ 25. Schläfli's Double-Six

We begin with some simple considerations concerning the possible positions of straight lines in space. Three skew straight lines a, b, and c define a hyperboloid H. In general, an arbitrary fourth straight line d intersects H at two points, although it may also be tangent to H or lie on H. In the general case, each of the points at which d and H intersect is incident with a straight line lying on H that does not belong to the same family as a, b, c and therefore intersects a, b, and c. Conversely, every straight line that intersects a, b, c, and d, is on H and is incident with one of the points at which d intersects H. Hence there are in general two, and not more than two, straight lines that intersect four given straight lines. In the case where d is tangent to H there is only one (double) straight line that intersects a, b, c, and d. If, on the other hand, there are more than two straight lines that intersect a, b, c, and d, then d must lie on H, and then there are infinitely many straight lines intersecting a, b, c, and d. In this case we say that the four straight lines are in a hyperboloidal position.

In the construction of Schläfli's double-six we start with any straight line 1 and draw three mutually skew straight lines intersecting 1, which we shall call 2′, 3′, and 4′, for reasons that will become apparent later (see Fig. 179). Then we draw another straight line 5′ through 1, which is to have the most general possible position relative to 2′, 3′, and 4′: 5′ will not intersect any of the straight lines 2′, 3′, and 4′, and there will be besides 1 just one straight line—we shall call it 6—that intersects 2′, 3′, 4′, and 5′. Finally we draw a straight line 6′ through 1 which must not intersect 6, 2′, 3′, 4′, or 5′, and which must furthermore be such as to make the positions of the quadruples 2′3′4′6′, 2′3′5′6′, 2′4′5′6′, and 3′4′5′6′ as general as possible. Then there is exactly one straight line 5 in addition to 1 which intersects 2′, 3′, 4′, and 6′, and the straight lines 4, 3, and 2 are defined analogously (e.g. 4 is distinct from 1 and intersects 2′, 3′, 5′, and 6′, etc.). In this way we obtain the system of intersections represented schematically in Fig. 179. It is easily seen that our choice of the straight lines 2′, 3′, 4′, 5′, 6′ precludes the possibility of additional intersections. Turning now to the four straight lines 2, 3, 4, and 5, we shall show that they cannot be in a hyperboloidal position. For if they were, every straight line that

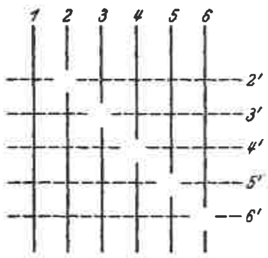

FIG. 179

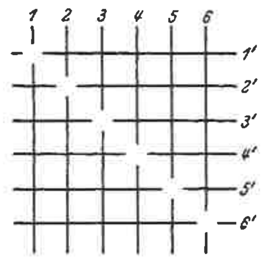

FIG. 180

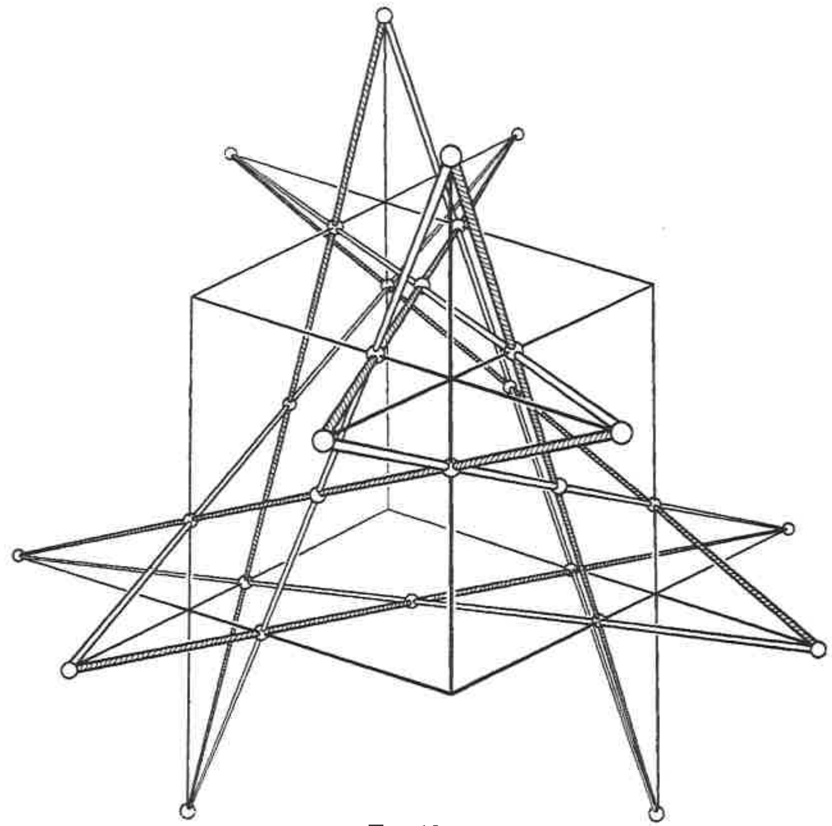
FIG. 181

[1] E.g., there is no straight line on the surface $xyz = 1$ which passes through a finite point of the surface.

intersects three of them would also intersect the fourth, and in particular, this would apply to each of the straight lines 2′, 3′, 4′, and 5′, according to our scheme. Then these four straight lines would also be in a hyperboloidal position, contradicting the conditions of our construction. Thus there are at most two straight lines that meet 2, 3, 4, and 5. But according to our construction, 2, 3, 4, and 5 all intersect 6′. Let us denote the second straight line that intersects 2, 3, 4, and 5, by 1′; we assert that 1′ does not coincide with 6′ and that it cuts 6. Pending the proof of this assertion (to be given below), we may supplement the arrangement represented by Fig. 179, changing it into that of Fig. 180. The latter scheme represents the double-six. It is immediately seen that we are dealing with a regular configuration of points and straight lines whose symbol is $(30_2 12_5)$. A particularly clear and symmetrical form of the double-six can be constructed by suitably choosing one of the straight lines of each set of six on each face of a cube. The arrangement should be apparent from Fig. 181 (cf. also Fig. 102, p. 93).

We must now prove the assertion made above that there is a straight line 1′ distinct from 6′ which meets 2, 3, 4, and 5, and that this 1′ must meet 6. Let us tentatively assume that the first part is already proved and prove on the basis of this assumption that 1′ intersects 6. To this end, we select four points on the straight line 1 and three points on each of the straight lines 2′ to 6′, making sure that none of the points of intersection of the lines under consideration are included among the nineteen points thus chosen. According to the argument of the last section, a third-order surface F_3 can be drawn through these nineteen points. Now F_3, having four points in common with the straight line 1, must contain the entire straight line. Furthermore, F_3 has four points in common with each of the straight lines 2′ to 6′—the three points chosen in the beginning and the point (distinct from these) where the line meets 1; thus F_3 contains 2′ to 6′ as well. From this it follows in turn that F_3 also contains the straight lines 2 to 6, as each of them intersects four straight lines lying on the surface. And finally, F_3 contains 1′ for the same reason. Supposing now that 1′ did not intersect 6, let us consider the straight line l which, like 5′, intersects 2, 3, 4, and 6. As in the construction of 1′, we shall rule out for the time being the case where l coincides with 5′. l cannot coincide with 1′, since it was assumed that 1′ does not meet 6. Since l meets four straight lines lying on F_3, namely 2, 3, 4, and 6, l itself lies on F_3. By our construction, each of the four straight lines l, 1′, 5′, 6′ meets 2, 3, and 4. Hence the four straight lines are in a hyperboloidal position. Then the entire associated hyperboloid must be a part of F_3; this follows directly from the fact that every straight line that intersects l, 1′, 5′, and 6′, lies on F_3, while the set of all such straight lines covers the hyperboloid.

Now, it is easy to prove algebraically that a third-order surface that contains all the points of a second-order surface must consist of the second-order surface and a plane: If $G=0$ and $H=0$ are the equations of the third-order and second-order surface respectively, the polynomial G of the third degree must be divisible by the polynomial H of the second degree, and this can only be the case if G is the product of H and a linear expression. From the conclusion that the surface F_3 defined by our nineteen points must be a degenerate case of this sort, we can easily deduce a contradiction. For, no four of the straight lines 2′, 3′, 4′, 5′, 6′ have a hyperboloidal position; hence at most three of them could be on the hyperboloid that forms a part of F_3. Hence at least two would have to be on the plane that constitutes the other component of F_3 and these two would therefore have a point of intersection, in contradiction to our construction.

If we admit the possibility, previously excluded, that 1′(2345) may coincide with 6′ or l(2346) with 5′, the proof is not essentially changed. In this case, too, we can conclude that the hyperboloid defined by 2, 3, and 4 would have to be a part of F_3. But the limiting process by which this case is derived from the general case can not be justified without the use of algebraic methods.

In the proof of the last incidence relation (1′6) of the double-six we used the fact, interesting in itself, that there is always a third-order surface F_3 that contains this configuration. It is easy to supplement the configuration with several additional straight lines which also lie on F_3. Consider, for instance, the plane spanned by the intersecting straight lines 1 and 2′ and the plane spanned by 1′ and 2 and let (12) denote the line in which the two planes intersect. Then (12) meets the four straight lines 1, 1′, 2, and 2′, all of which lie on F_3; hence (12) also lies on F_3. In all there are fifteen straight lines that bear the same relation to the double-six as (12) and therefore lie on F_3 as well. For, fifteen different pairs can be chosen from the numbers from 1 to 6. We have thus found $2 \times 6 + 15 = 27$ straight lines all lying on F_3.

Among the straight lines of the enlarged configuration that we have obtained in this way there are further incidence relations. In fact, it may be shown that all those pairs of the straight lines denoted by two numbers whose symbols have no number in common, and those only, will have a point of intersection. The proof can be based on the same idea as our proof that 1′ and 6 intersect, and we shall only give an indication of it. For reasons of symmetry it suffices to show that (12) meets (34). To this end, we consider the three straight lines 1, 2, (34), and note that 3′ and 4′ intersect them. If (12) did not intersect (34), there would be a straight line a that would meet the four lines 1, 2, 1′, and (34), and a straight line b that would meet 1, 2, 2′, and (34). b would necessarily be distinct from a, for if they were one and the same straight line, this would meet the four lines 1, 2, 1′, 2′, and would therefore be identical with (12) and yet meet (34), whereas we are assuming for the time being that (34) does not meet (12). Similarly a and b would have to be distinct from 3′ and 4′; for if, say, a coincided with 3′, then 3′ would intersect 1′, in contradiction to our construction. Now a and b, like 3′ and 4′, would have to lie on F_3, and because all of them meet the triple 1, 2, (34), the four straight lines would be hyperboloidal. But we have already seen that it is impossible for F_3 to contain a set of four straight lines in the hyperboloidal position. It follows that (12) does meet (34). For the same reasons it must meet (35), (36), (45), (46), and (56). Since (12) also meets 1, 2, 1′, and 2′, it follows that (12) intersects ten straight lines of the enlarged configuration, and does every one of the straight lines we denoted by two numbers. The same is true for the straight lines of the double-six itself; 1, for example, intersects the five lines 2′ to 6′ and the five lines (12), (13), (14), (15), (16). Accordingly, the configuration consisting of the 27 straight lines on F_3 together with their points of intersection has the symbol $(135_2 27_{10})$. The fact that there are exactly 135 points follows from the equation $135 \times 2 = 27 \times 10$. It can be shown, moreover, that the configuration is regular, and that many different double-sixes can therefore be found in it. Considering in addition the planes spanned by intersecting pairs of lines of the configuration, we can verify by referring to the incidence table that every such

plane contains a third line of the configuration. This can also be seen by the following simple algebraic argument. Every plane necessarily intersects F_3 in a third-order curve. If the plane contains two straight lines of the configuration, this curve is bound to contain them, and it can be deduced algebraically that the curve must then consist of these two straight lines and a third straight line. It is easy to check by counting that five of our planes pass through each of the twenty-seven straight lines and that the planes number forty-five in all. Thus we see that the configuration is not self-dual, although the double-six, being built up on the self-dual relation of the incidence of two straight lines, is self-dual. The double-six can easily be extended to a configuration that is the dual of the configuration we have just constructed. To this end, we need to add a different set of straight lines $[ik]$ instead of the straight lines (ik), where, for example, [12] passes through the points at which 1 intersects 2′ and 1′ intersects 2. The configuration obtained in this way has the symbol $(45_3 27_5)$.

Let us return to the original configuration of twenty-seven straight lines. We shall show by enumerative methods that there is such a configuration K on every third-order surface F_3. Here, as in all enumerative considerations, the cases where K is partly imaginary or degenerate must also be taken into account. The proof begins with the enumeration of the family of all double-sixes. According to our construction, the choice of the straight line 1 is completely free, and thus involves four parameters; the points where 1 intersects the straight lines 2′ to 6′ depend on another five parameters, and each of the lines 2′ to 6′ can assume ∞^2 positions once its point of intersection with 1 is fixed (thus accounting for ten more parameters). Since the straight lines 1, 2′, 3′, 4′, 5′, and 6′, uniquely define the double-six, we see that there are ∞^{19} double-sixes $(19 = 4 + 5 + 10)$. The family of configurations K has the same number of parameters; for, each configuration of this type is defined by one of the double-sixes in it, and obviously there is only a finite number of double-sixes in any one configuration K. Now we have given a construction for passing an F_3 through any given K; it follows either that the family of the surfaces F_3 constructed in this way comprises ∞^{19} surfaces or, should there be fewer surfaces, that at least ∞^1 configurations K lie on the same F_3, i.e. that F_3 would have to be a ruled surface of the third order. It can be shown, however, that there are less than ∞^{18} ruled surfaces of the third order; hence the F_3 we constructed would have to contain at least ∞^2 double-sixes. But since it was already demonstrated that the F_3 do not contain a hyperboloid and since any ruled surface of order higher than the second contains only one family of straight lines, such an F_3 cannot possibly carry ∞^2 double-sixes. Therefore our surfaces cannot in general be ruled surfaces, and it follows that our construction accounts for not less than ∞^{19} surfaces. On the other hand, as we have mentioned in the last section, there are only ∞^{19} third-order surfaces. From this, the algebraic nature of the figures under consideration being borne in mind, the truth of our assertion that every third-order surface contains a configuration of the type K can be rigorously deduced.

CHAPTER IV

DIFFERENTIAL GEOMETRY

So far we have examined geometrical figures with regard to their overall structure. Differential geometry represents a fundamentally different method of approach. Specifically we will, to start with, investigate curves and surfaces only in the immediate vicinity of any one of their points. For that purpose we compare the vicinity, or "neighborhood," of such a point with a figure which is as simple as possible, such as a straight line, a plane, a circle, or a sphere, and which approximates the curve as closely as possible in the neighborhood under consideration; in this way one obtains, for example, the familiar concept of the tangent to a curve at one of its points.

This approach, known as local differential geometry, or differential geometry in the small, is supplemented by another important point of view, differential geometry in the large: if a continuous geometrical figure is known to have a certain property in the neighborhood of *every one* of its points, then it is possible, as a rule, to deduce certain essential facts relating to the total structure of the figure. If, for example, we are given a plane curve of which it is known that at no point of the curve does a neighboring portion lie entirely on one side of the tangent at the point, then it may be proved that the curve must of necessity be a straight line.

Besides dealing with continuous sets of points, differential geometry also deals with manifolds composed of other elements, e.g. manifolds of straight lines. Problems of this kind arise, for example, in the field of geometrical optics, which is concerned with the study of continuous systems of light rays.

And finally, differential geometry leads to the problem, first posed by Gauss and Riemann, of building up a complete geometrical system on the basis of concepts and axioms that only affect the immediate neighborhood of each point. This gave rise to an abund-