$p = 8$, five different tables can be set up, and all of them can be realized geometrically. One of these configurations (8_4), the so-called Moebius configuration, is geometrically important because it satisfies the last incidence condition automatically and thus expresses a geometric theorem. This configuration consists of two mutually inscribed and circumscribed tetrahedra.

Going on to higher configurations, the number of possibilities keeps growing, and it soon becomes impossible to get an over-all view of them. Thus there are no less than 26 configurations of the type (9_4) that can be realized geometrically. Accordingly, we shall examine in greater detail only two three-dimensional configurations that are particularly important and play a role in other parts of mathematics as well. These are Reye's configuration and Schaefli's double six.

§ 22. Reye's Configuration

Reye's configuration consists of twelve points and twelve planes. It embodies a theorem of projective geometry, so that the last incidence always follows automatically, regardless of the positions of the points and planes. For the time being, however, we shall arrange the points in a special symmetrical order, so as to facilitate the visualization of the configuration.

We shall use as points of the configuration the eight vertices of a cube together with the center of the cube and each of the three ideal points where four parallel edges of the cube meet (Fig. 145). As planes of the configuration we shall use the planes of the six faces and each of the six diagonal planes passing through a pair of opposite edges. In the figure defined in this way, there are six points lying on each plane: four vertices and two ideal points on each of the planes containing a face of the cube, and four vertices, the center of the cube, and an ideal point on each of the diagonal planes. There are six planes through each point: the six diagonal planes pass through the center of the cube, three face planes and three diagonal planes through each vertex, and four face planes and two diagonal planes through each of the ideal points. Thus we have indeed constructed a configuration of points and planes, and its symbol is (12_6).

But the construction may also be interpreted as being a configuration of points and straight lines. To this end, we select some of the

straight lines of intersection of the planes, namely the twelve edges and the four diagonals of the cube. There are three points of the configuration on each of these straight lines: two vertices and one ideal point on each edge, two vertices and the center on each diagonal. Furthermore, there are four straight lines through each point: three edges and one diagonal through each vertex, four diagonals through the center of the cube, and four edges through

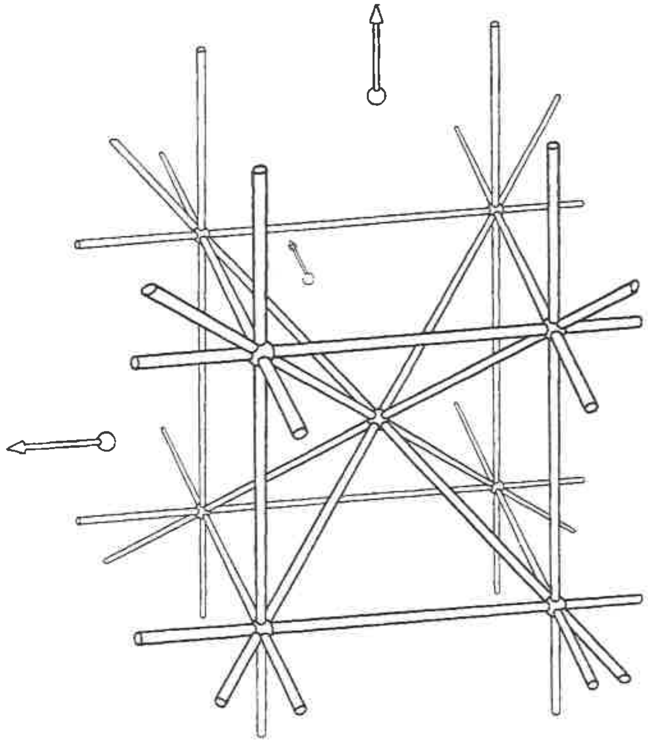

FIG. 145

each ideal point. Hence the points and straight lines of Reye's configuration form a configuration of the type $(12_4 16_3)$.

We can also see, if we count them, that three planes pass through each of the lines and that four lines lie on each plane. The straight lines on any one of the planes together with the six points of the configuration lying in the plane constitute a complete quadrilateral.

Reye's configuration appears in various geometrical contexts. An example is the system of centers of similitude of four spheres, which we shall now study.

The term *centers of similitude* of two circles or spheres denotes the two points that divide the line joining the centers of the circles or spheres in the ratio of their radii. The point on the segment that lies between the centers is called the *internal* center, the one on the extension of the segment the *external* center of similitude. If we are dealing with circles, and each of them lies outside the other, the internal center of similitude is the point of intersection of the two straight lines tangent to the circles on opposite sides, and the external center of similitude is the point of intersection of the straight lines tangent to the circles on the same side

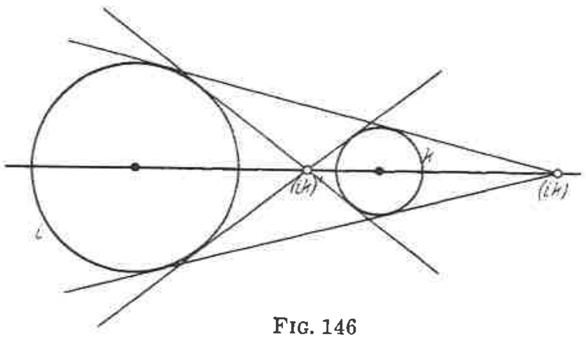

FIG. 146

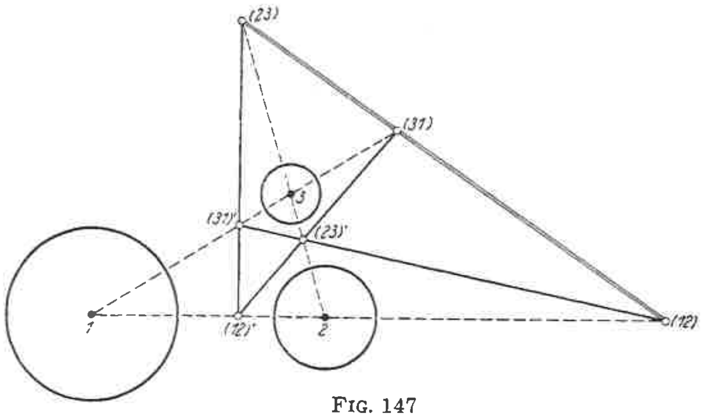

FIG. 147

(see Fig. 146). By rotating this figure about the straight line containing the centers we get an analogous property relating the centers of similitude of two spheres with common tangents to the spheres. (But in addition the spheres have many common tangents that do not pass through a center of similitude.) We shall use the symbols (ik) and (ik') respectively for the external and internal centers of similitude of two circles or spheres i and k.

Let us now consider three circles or spheres, 1, 2, and 3. They have three internal centers of similitude and three external centers of similitude, making six in all. We shall assume that the centers

of the circles or spheres are not collinear but form a triangle; no two of the centers of similitude can then coincide, and the six can not all be collinear. By a theorem of Monge, the three external centers of similitude, (12), (23), and (31), are collinear, and each external center of similitude is collinear with the two internal centers of similitude that belong to different pairs of circles or spheres, e.g. (31) with (12)′ and (23)′ (see Fig. 147).[1] Accordingly, all the centers of similitude lie on four straight lines, which are called the axes of similitude of 1, 2, and 3. Monge's theorem may be summarized by saying that the centers of similitude and axes of similitude constitute the six points and four lines of a complete quadrilateral in which the centers of 1, 2, and 3 form the diagonal triangle. We shall denote the axes of similitude by the following symbols: (123) for the straight line containing the external centers of similitude, (1′23) for the straight line on which (23), (12)′, and (13)′ lie, etc.

With this preparation we turn to the consideration of four spheres 1, 2, 3, 4 whose centers are not all in one plane, so that, moreover, no three of the centers can be on one straight line (cf. Fig. 148, p. 140). We shall see that all the centers of similitude and axes of similitude of these spheres collectively constitute the points and straight lines of a Reye configuration. Since six different pairs can be selected from the spheres 1, 2, 3, 4, and since each pair gives rise to an external and an internal center of similitude, there are twelve centers of similitude in all. Also we have the right number, 16, of axes of similitude, for there are four different ways we can select three out of the four spheres, and each set of three spheres gives rise to four different axes of similitude, e.g. (123), (1′23), (12′3), and (123′). Each axis is incident with three points, e.g. (123) is incident with (12), (23), and (13). Similarly, every point is incident with four different axes, e.g. (12)

[1] *Proof*: Let the radii of 1, 2, and 3 be equal to r_1, r_2, and r_3, respectively. Then the external centers of similitude divide the sides of the triangle formed by the centers in the ratios $-\frac{r_1}{r_2}, -\frac{r_2}{r_3}, -\frac{r_3}{r_1}$. The product of these ratios is -1, and it follows by a theorem of Menelaus that the external centers of similitude are collinear. If two of the external centers of similitude are replaced by the corresponding internal centers of similitude, two of the ratios change their sign. The product is therefore still -1, so that we once more have three collinear points.

is incident with (123), (123'), (124), and (124'), and (12)' with (1'23), (12'3), (1'24), and (12'4).

We thus see that the centers and axes of similitude do indeed form a configuration and that its type is $(12_4, 16_3)$. To see that it is identical with Reye's configuration, we need to find twelve suitable planes. First we take the four planes containing the centers of three spheres each. The points and axes lying on any one of these planes form a complete quadrilateral, as in Reye's configuration. To get eight more planes with this property, we simply take all the remaining planes spanned by any two axes that intersect at a point of the configuration. Two axes of this kind must certainly belong to different number triples, for, any two axes associated with the same set of three numbers, e.g. (123) and (1'23), define the plane containing the centers of three spheres (1, 2, and 3 in our case), so that nothing new is obtained. Let us begin with two axes containing only external centers of similitude, e.g. (123) and (124). They span a plane that contains (12). In addition, this plane contains the other four points of those axes, i.e. (13), (23), (14), and (24). But (23) and (24) also lie on the axis (234) which contains as well the remaining external center of similitude (34). Hence all six external centers of similitude lie on the single plane we have been considering. This plane also contains the remaining "external" axes (134) and (234); thus it is incident with six points and four straight lines, as it should be. We proceed to the case of two intersecting axes one of which is "external" and one "internal" and which are associated with two different number triples. Since their point of intersection must be an external center and since all the numbers play the same role, we may pick the axes (123) and (124') as a representative pair. Apart from their point intersection, (12), these axes contain the points (13), (23), (14)', and (24)'. By the same reasoning as before, we see that the axes (134') and (234') and the point (34)' are also in the plane of (123) and (124'). Thus the three internal centers of similitude defined by the sphere 4 together with the three other spheres are in a single plane with the three external centers of similitude of the spheres 1, 2, and 3. There must be altogether four planes of this kind. Only the case based on two intersecting internal axes of similitude remains to be considered. Of course the last plane considered above contains three internal axes which intersect in

pairs; but the points of intersection are always internal centers of similitude, so that the case of two axes intersecting at an external center of similitude is still open. Let us begin, then, with two internal axes, say (123') and (124'), which intersect at an external center of similitude—(12) in this case. Apart from the point of intersection, the plane of these axes contains the points (13)', (23)', (14)', and (24)'. Hence this plane also contains the axes (1'34) and (2'34) and the point (34). Thus there are four internal axes of similitude in this plane, and it meets the opposite edges 1, 2 and 3, 4 of the tetrahedron 1, 2, 3, 4 at the external centers of similitude and the remaining edges at the internal centers of similitude. There are three planes of this type, since a tetrahedron has three pairs of opposite edges. Thus we have obtained altogether $1 + 4 + 3 = 8$ planes.

For the sake of clarity, we shall set up the two tables that give the incidence relations between the points and the planes and between the points and the lines, respectively. The faces of the tetrahedron are labelled I, II, III, and IV, where I is the face opposite the point 1. The plane of the external centers of similitude is called e_a, the four planes containing three external and three internal centers are called e_1, e_2, e_3, e_4 respectively, according to the number of the exceptional sphere, and the three remaining planes are denoted by (12, 34), (13, 24), and (14, 23) respectively, according to the exceptional pair of opposite edges of the tetrahedron. For the sake of brevity, parentheses are omitted in the notation for points and straight lines.

	Planes											
	I	*II*	*III*	*IV*	e_a	e_1	e_2	e_3	e_4	(12, 34)	(13, 24)	(14, 23)
Points	23	13	12	12	12	23	13	12	12	12	13	14
	24	14	14	13	13	24	14	14	13	34	24	23
	34	34	24	23	14	34	34	24	23	13'	12'	12'
	23'	13'	12'	12'	23	12'	12'	13'	14'	14'	14'	13'
	24'	14'	14'	13'	24	13'	23'	23'	24'	23'	23'	24'
	34'	34'	24'	23'	34	14'	24'	34'	34'	24'	34'	34'

	Planes											
	I	*II*	*III*	*IV*	e_a	e_1	e_2	e_3	e_4	(12, 34)	(13, 24)	(14, 23)
Lines	234	134	124	123	123	234	134	124	123	123'	12'3	1'23
	2'34	1'34	1'24	1'23	124	1'23	12'3	123'	124'	124'	1'24	12'4
	23'4	13'4	12'4	12'3	134	1'24	12'4	13'4	134'	1'34	134'	13'4
	234'	134'	124'	123'	234	1'34	2'34	23'4	234'	2'34	23'4	234'

The configuration is depicted in Fig. 148.[2] That this configuration is identical with that of Fig. 145 becomes manifest on moving the three points (12), (12)′, and (34) to infinity in mutually perpendicular directions; the three points then assume the positions of the

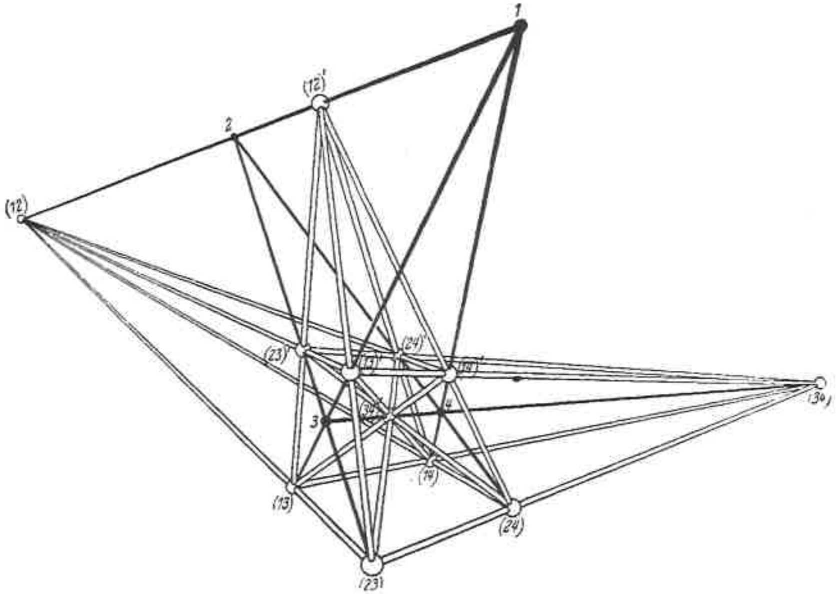

FIG. 148

ideal points of the configuration given in Fig. 145. The eight points (13), (14), (23), (24), (13)′, (14)′, (23)′, and (24)′ become the vertices of the cube, and (34)′ becomes the center of the cube. But the points 1 and 2 also move to infinity. In order to find the four spheres belonging to Fig. 145 it is consequently necessary to extend the definition of center of similitude by the addition of limiting cases. First, the external center of similitude of two equal circles or spheres must be defined as the ideal point on the line connecting the centers (see Fig. 149). Furthermore, the centers of similitude of a sphere k

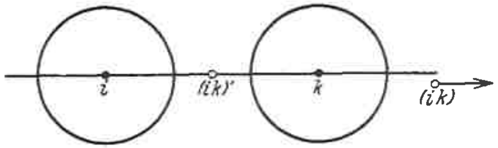

FIG. 149

[2] Viewed as a *plane* figure, Fig. 148 represents a plane configuration of type $(12_4, 16_3)$ consisting of the centers and axes of similitude of four coplanar circles. The centers of the circles are also at 1, 2, 3, and 4, and the radii may be chosen to be the same as in the three-dimensional case.

and a plane e (Fig. 150) must be defined as the extremities (ke) and $(ke)'$ of the diameter of k that is perpendicular to e. For, if e is replaced by a family of spheres K tangent to e at the point P where the extension of the diameter meets e, it is seen that the centers of similitude of k and K approach (ke) and $(ke)'$ as the diameter of K increases to infinity. Finally we consider the case of two planes e and f intersecting in a straight line g (Fig. 151). The centers of similitude must be defined in this case as the ideal points having directions that are perpendicular to g and bisect the two angles formed by e and f. This definition too may be justified by a limiting process, as follows: Replace g by the circle of intersection of two congruent spheres tangent at a fixed point of g to e and f respectively, and then let the radius of the spheres increase to infinity.

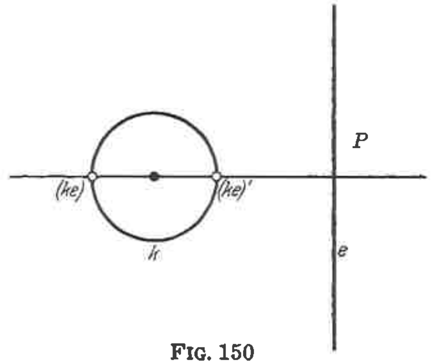

FIG. 150

With these definitions, we are in a position to interpret Reye's configuration in its original version also, as a system of centers of similitude. Let the spheres 3 and 4 have their centers at the midpoints of the front and back faces of the cube in Fig. 145. Let the radii be equal and of such length that each sphere goes through the four corners of the face on which its center lies. Let 1 and 2 be any two planes that are respectively perpendicular to the two diagonals of the faces under consideration. Then the points of the configuration are the centers of similitude of 1, 2, 3, and 4, arranged in the same order as in Fig. 148.

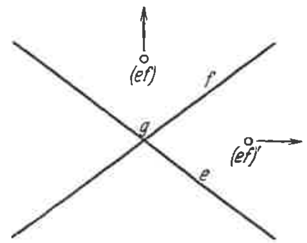

FIG. 151

Instead of this limiting case, we may consider the configuration based on four equal spheres with their centers at the vertices of a regular tetrahedron. Here the external centers of similitude must be at the ideal points of the six edges of the tetrahedron, so that the ideal plane belongs to the configuration and constitutes, in our notation, the plane e_a. The internal centers of similitude are the midpoints of the edges; they form the six vertices of a regular octahedron (see Fig. 152). All the face-planes of the octahedron belong to the configuration, being the face-planes I, II, III, and IV, of the tetrahedron and the planes called e_1, e_2, e_3, and e_4 in our notation. The three remaining planes of the configuration are the three planes of symmetry of the octahedron. The straight lines of the configuration are the four ideal lines of the face-planes of the tetrahedron

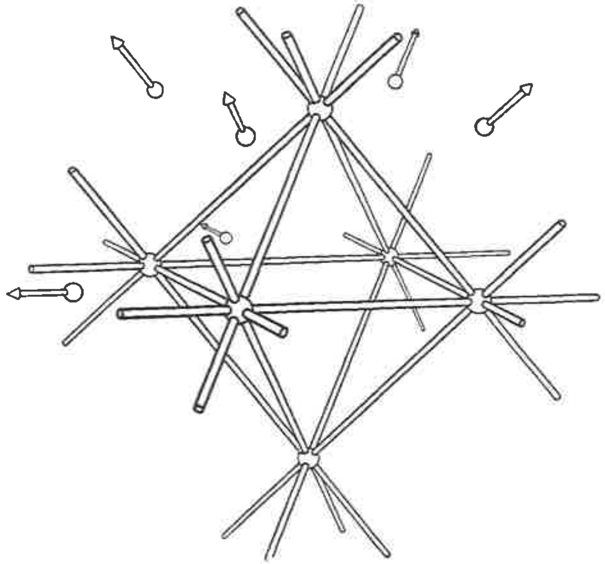

FIG. 152

(external axes of similitude) and the twelve edges of the octahedron (internal axes of similitude).

In the second chapter we have already pointed out how the cube and the octahedron are related. In accordance with § 19, we may say that the cube and the octahedron correspond dually to each other. Similarly, it can be shown more generally that the points and planes of Fig. 152 correspond dually to the planes and points of Fig. 145; the vertices and faces of the cube correspond to the faces and vertices respectively of the octahedron, the center of the cube and the six planes through it correspond to the ideal plane and the six points on it in Fig. 152, and the three ideal points associated with the cube correspond to the three planes of symmetry of the octahedron.[3] It follows that Reye's configuration of points

[3] This correspondence is produced by a polarity with respect to the inscribed sphere of the cube.

and planes is self-dual. Of course the two dual Reye configurations obtained from the cube and octahedron look quite different, but for the purposes of projective geometry, all Reye configurations must be considered as identical.[4]

We shall now show that Reye's configuration also has the other important property of symmetry that we observed in some plane configurations, viz., that it is regular. This is by no means evident from the foregoing discussion. Indeed, the planes belong to four different classes relative to the system of centers of similitude, and in the realization of the configuration either by a cube or an octahedron, both the points and the planes play different sorts of roles. In the following section, we shall obtain Reye's configuration by a method that reveals the equivalence of all the elements. To this end, we need to learn more about the regular polyhdra of three-dimensional and four-dimensional space. For, the figures of four-dimensional space can be projected into three-dimensional space in the same way that the figures of three-dimensional space can be projected into a plane, and a suitable projection of one of the four-dimensional figures gives us Reye's configuration.

§ 23. Regular Polyhedra in Three and Four Dimensions, and their Projections

In Chap. II we listed the five regular polyhedra of three-dimensional space. Among these, the tetrahedron plays an anomalous role in that it is self-dual, whereas the four remaining polyhedra are mutually dual in pairs—the octahedron with the cube, and the dodecahedron with the icosahedron. Possibly this singularity of the tetrahedron is connected with a second phenomenon that distinguishes it from the other polyhedra; the others are symmetrical with respect to a point, which means that the vertices come in pairs that are symmetrical about the center, and the same is true for the edges and the faces (e.g. the straight line connecting any vertex of a cube with the center meets the cube at a second vertex). The tetrahedron, however, is not symmetrical with respect to a point, (does not have "central symmetry"); the straight line connecting

[4] We obtain a projective generalization of the octahedron by starting with any system of projective coordinates in space; in every case the unit points on the six coordinate axes and the six points of intersection of these axes with the unit plane are the points of a Reye configuration.