14. The six-structuring of space

Five lines of a plane, no three of which go through the same point, always structure the plane as point field in the same way however the lines may be chosen; this was shown in Chapter 12. They determine a cyclic order which becomes evident in the boundary of the ever-present five-sided core. A ring of five three-sided cores surrounds this core.

The dividing up of point space by five planes was outlined in Exercise 5 on page 87. We now take a further step and study the structuring of space by six planes, no three of which belong to the same line and no four of which belong to the same point. In other words, the complete spatial 6-plane. This form is extremely interesting and full of impressive features. To understand this form by realizing it in clear mental images is the aim we set ourselves. Such a 6-plane always produces the same structuring of point space, no matter how the planes are chosen. The six planes evince a particular cyclic ordering, shown by a closed ring of six tetrahedral cores. Two neighbouring cores of the ring "peak" each other: that is, they have in common a vertex and the edge lines radiating from it. The structuring always contains two six-faced principal cores with purely four-sided boundary domains, and it also contains two principal points with special properties. This is just a first indication of the six-structuring's particularities.

The six planes, which we shall call 1, 2, 3, 4, 5, 6 for short, have the 15 lines of intersection:

$$12,\ 13,\ 14,\ 15,\ 16,\ 23,\ 24,\ 25,\ 26,\ 34,\ 35,\ 36,\ 45,\ 46,\ 56.$$

These are the edges of the complete 6-plane. Its twenty vertices are the points of intersection of the six planes taken three at a time, as follows:

$$123,\ 124,\ 125,\ 126,\ 134,\ \ldots\ldots\ldots,\ 456.$$

The 6-plane structures point space into 26 cores, as we shall see.

To construct the form, start by drawing a tetrahedron. Then intersect the tetrahedron with a plane; this gives us a Desargues Configuration (Figure 60). Furthermore intersect the tetrahedron with a second plane, producing a second Desargues Configuration. Lastly the line in which these two planes (which intersect the original tetrahedron) intersect each other must be determined. The drawing is easier than one would at first imagine.

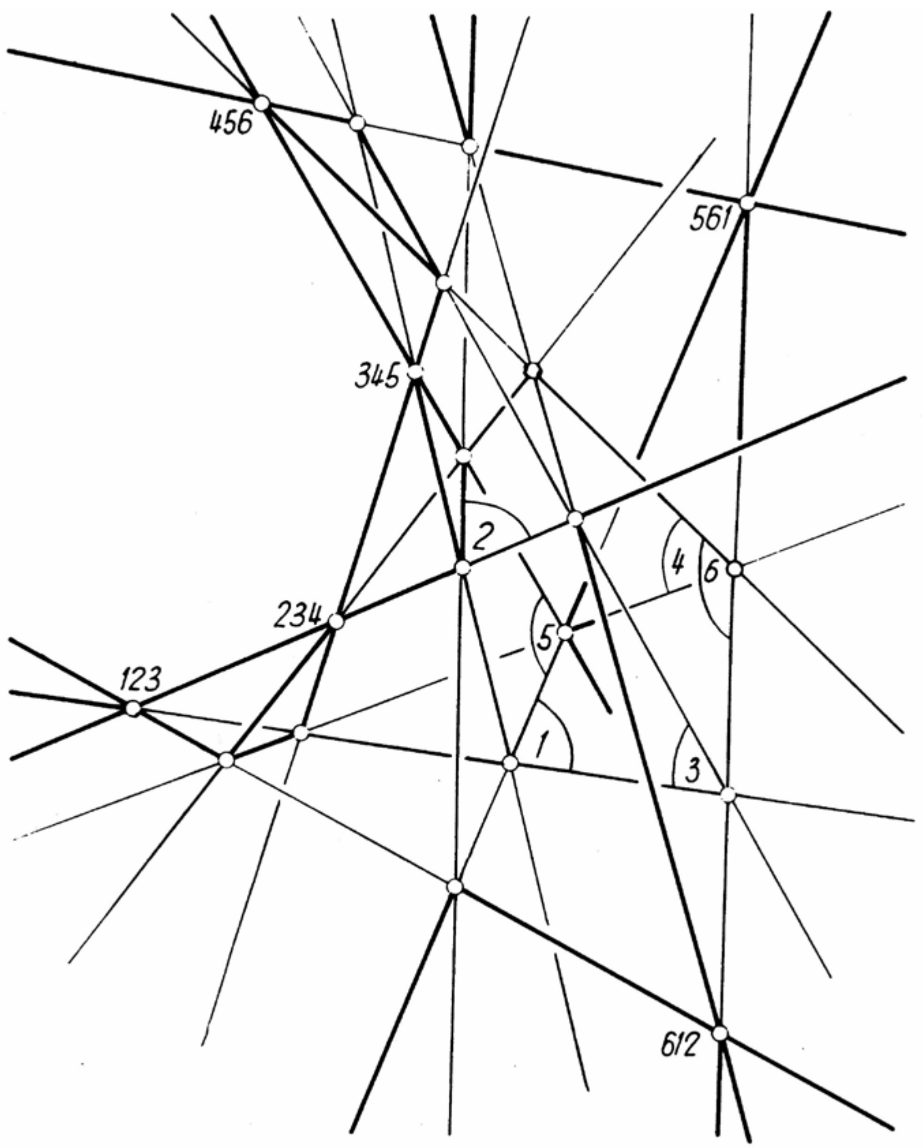
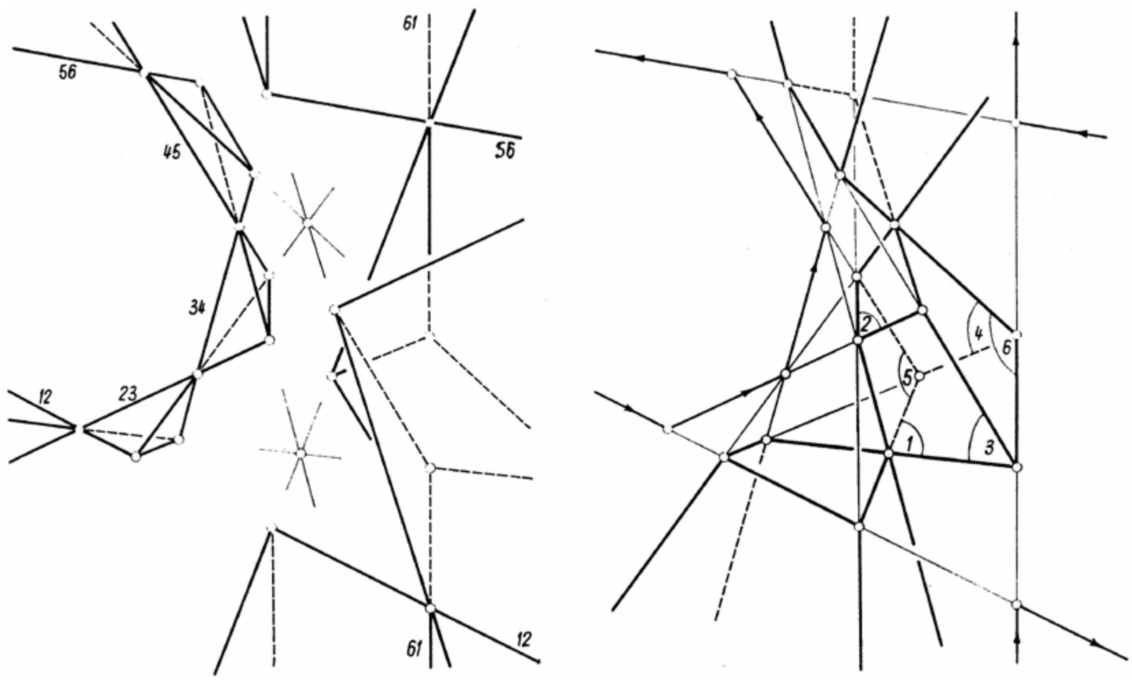


Figure 94

In Figure 94, consider the tetrahedron with apex 345 and base — thought of as horizontal — to the left of and adjoining the number 1. This tetrahedron has been intersected with plane 2 and with plane 6.

In Figure 97, choose the tetrahedron with an edge running from left to right horizontally at the top, and its opposite edge below going back and to the right from vertex 612 at the front. This has been intersected with the two planes 3 and 4.



Figures 95 and 96

The cycle of six planes is found as follows. First look for the tetrahedral cores: there are six of them and they successively "peak" each other. Then consider the six tetrahedron edges which connect pairs of the peak vertices in which two tetrahedral cores meet. These edges form a closed path which we call the *principal path*. Label the lines containing the edges of the principal path 12, 23, 34, 45, 56, 61, starting arbitrarily, and the vertices of the path 123 (namely the intersection of 12 and 23), 234 (the intersection of 23 with 34), 345, 456, 561, 612. The planes

$$\begin{aligned} &1=(61,12), \quad 2=(12,23), \quad 3=(23,34),\\ &4=(34,45), \quad 5=(45,56), \quad 6=(56,61), \end{aligned}$$

taken in the sequence 123456 gives the required cycle (123456).

In Figure 94, three of the six tetrahedral cores lie entirely in the finite. One core extends from vertex 123 (bottom left) leftwards over the limit plane and from the right back to a three-sided domain to be found in plane 6. And one core extends from point 456 as apex (top left) to a three-sided domain in plane 1. Finally a tetrahedral core extends, from the edge at the top right, upwards over the limit plane and from below back to the edge on 12. Figure 95 shows, on a smaller scale, the ring of tetrahedra in Figure 94.

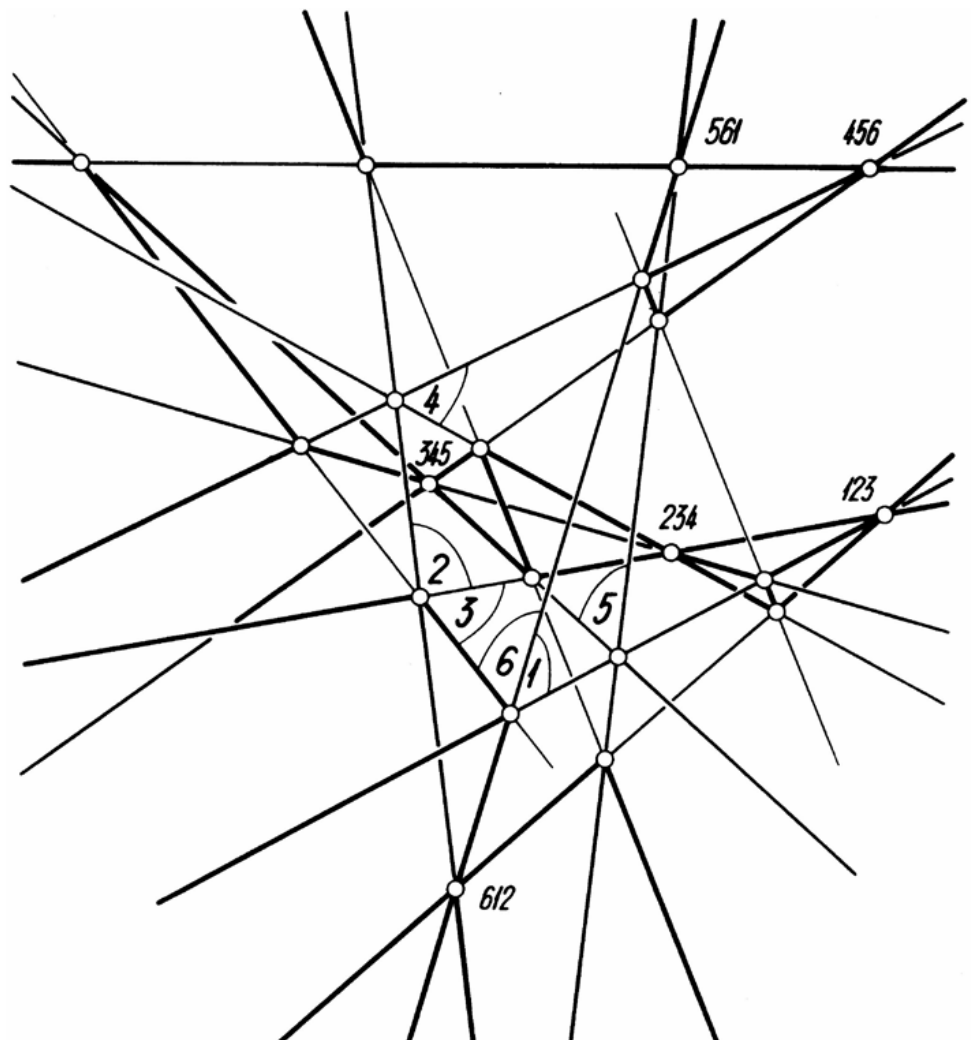


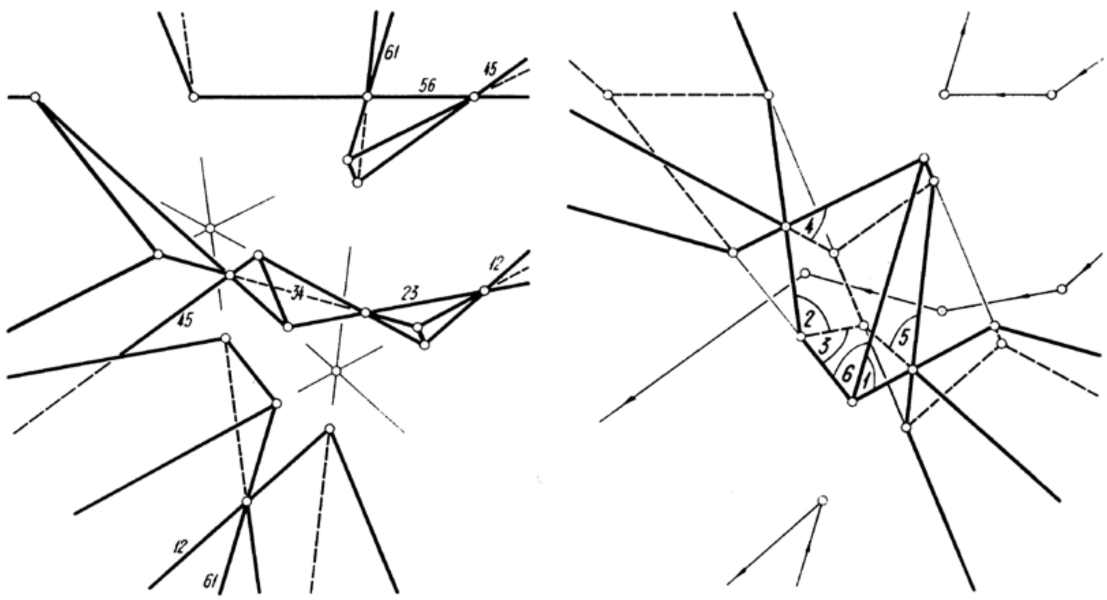
Figure 97

In Figure 97, too, three of the six tetrahedral cores are completely in the finite. Figure 98 shows the ring reduced in size.

Of the twenty vertices of the complete 6-plane, eighteen appear as vertices of the six tetrahedral cores. The other two, the *principal points* of the 6-plane, are the points of intersection 135 and 246. The complete 6-plane thus has two vertices which are qualitatively different from the other eighteen.

The principal points are opposite vertices of *two* six-faced cores with four-sided boundary domains; these two cores we call the *principal cores*. Planes 1 and 2 contain opposite faces of one of the principal cores, as do planes 3 and 4, and 5 and 6. The second principal core, linked with this one, has opposite faces in planes 2 and 3, 4 and 5, 6 and 1. The remarkable positions in relation to each other of the two principal cores repay scrutiny.

In Figure 96, the principal cores of Figure 94 are shown reduced in size; the same is done in Figure 99 for the principal cores of Figure 97.



Figures 98 and 99

The opposite faces of one of the principal core intersect in lines 12, 34, 56; the opposite faces of the other intersect in lines 23, 45, 61.

Of the twenty vertices of the complete 6-plane, fourteen (twice eight minus two) are claimed as vertices of the two principal cores. The other six are precisely the vertices of the principal path.

Once the cycle of six planes has been found, everything else appears in the most beautiful order. For example, successive tetrahedral cores are formed from the planes

$$1234,\ 2345,\ 3456,\ 4561,\ 5612,\ 6123.$$

On each line of the configuration there are four points. The cycle (123456) gives them in their natural order. For example, the four points on 12 arise in their natural order as the intersections of 12 with the planes 3, 4, 5, 6 respectively; similarly, the four points on 24 are the intersections of 24 with the planes 1, 3, 5, 6.

In the configuration there are six planar structurings each composed of five lines. That is, each of the six planes is intersected by the other five in five lines, which determine, in the plane in question, a five-sided domain with a ring of five three-sided domains.

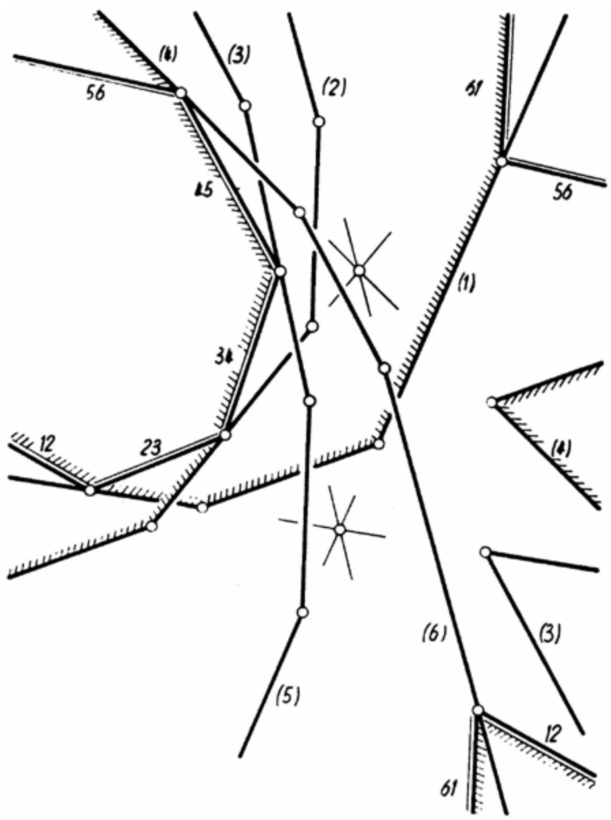


Figure 100

The five-sided domain in plane 1 is obtained as follows: we take the lines 12, 13, 14, 15, 16 in that order, giving the cycle $(12, 13, 14, 15, 16)$ of the five lines; from it the corresponding domain is easily determined. Similarly we have, for example, in plane 3 the cycle $(31, 32, 34, 35, 36)$, in plane 4 the cycle $(41, 42, 43, 45, 46)$, and so on. To simplify identification, Figure 100 shows, reduced in size, the six five-sided domains $(1), (2), \ldots$, in planes $1, 2, \ldots$ respectively, of the configuration of Figure 94 (domains (1) and (4) are slightly emphasized). These are joined together in a characteristic way, in that pairs of them have an edge of the principal path in common: the five-sided domains (1) and (2) have the path edge on line 12 in common, (2) and (3) the edge on 23, and so on. Each such five-sided domain is the common boundary of two six-faced cores. If we differentiate between front and back of these domains then the number of five-sided domains is twelve.

Figure 101 represents a case in which both principal points 135 and 246 belong to the limit plane; it is therefore a matter of the interpenetration of two triangular prisms' faces. In the case of Figure 101 five of the tetrahedral cores are entirely in the finite. Figure 102 shows, reduced in size, the six five-sided domains of the configuration of Figure 101.

In order to be able to state in a concise way which planes bound a core, as well as the nature of the boundary domains, we introduce the *characteristic* of a core. This consists of a sequence of six numbers: the first relates to plane 1, the second to plane 2, the third to plane 3, and so on. The number itself is *the number of segments bounding the domain* lying in the plane in question. Thus the characteristic (330033) represents the tetrahedral core which is bounded by each of the planes 1, 2, 5, 6 in a three-sided domain, and in whose formation planes 3 and 4 are not involved. The characteristic (553443) represents a six-faced core which involves all six planes. Planes 1 and 2 each bound the core in a five-sided domain, planes 3 and 6 each in a three-sided, and planes 4 and 5 each in a four-sided domain.

The two principal cores have the same characteristic, namely (444444). One should ascertain that one principal cores can be seen as an interpenetrating system of the three tetrahedra 1234, 3456 and 5612, and the other as an interpenetrating system of 2345, 4561, 6123.

The other 24 cores are uniquely determined by their characteristics. First there are the six tetrahedral cores

$$(333300),\ (033330),\ (003333),\ (300333),\ (330033),\ (333003).$$

Then there are twelve five-faced cores bounded by two three-sided and three four-sided domains, namely

$$\begin{array}{cccccc} (334440), & (444330), & (044433), & (033444), & (403344), & (304443), \\ (330444), & (440334), & (444033), & (433044), & (443304), & (344403). \end{array}$$

Finally there are six six-faced cores bounded by two three-sided, two four-sided and two five-sided domains, namely

$$(553443),\ (355344),\ (435534),\ (443553),\ (344355),\ (534435).$$

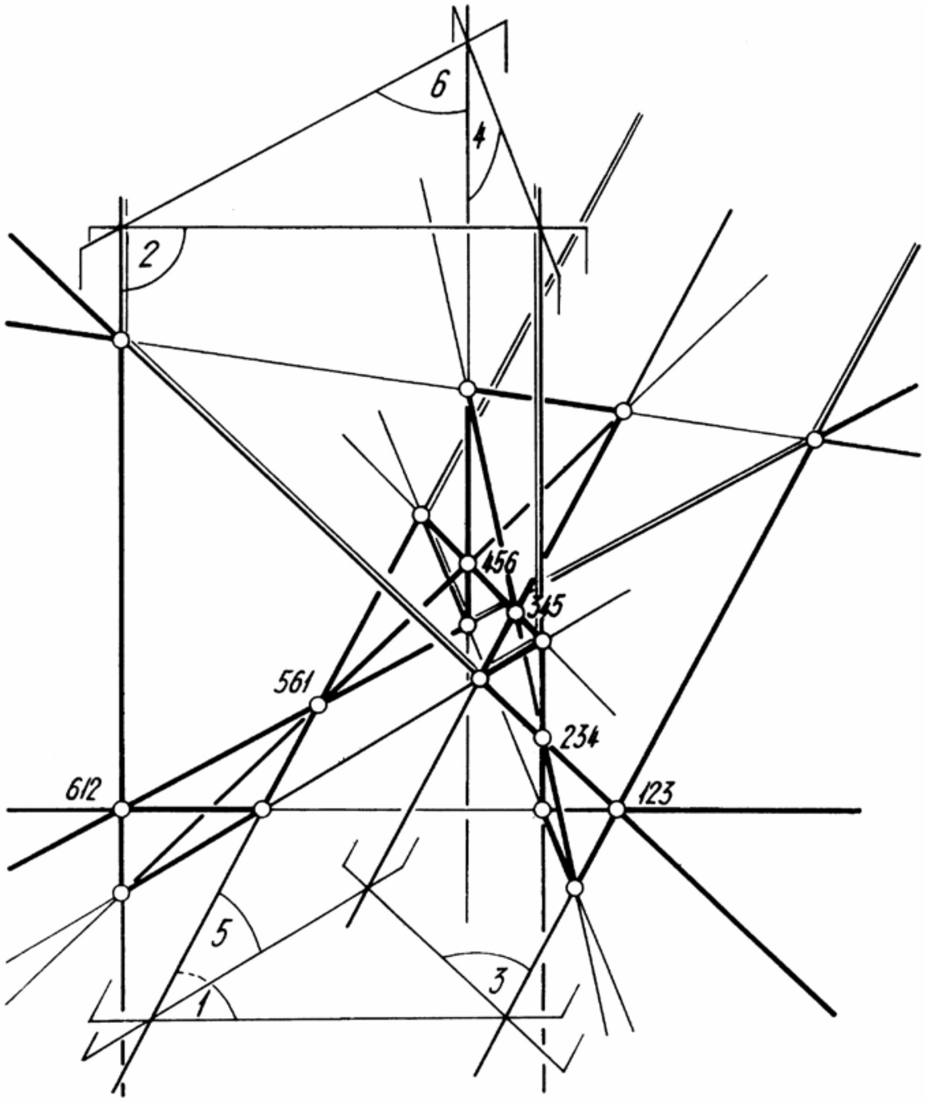


Figure 101

For example, in Figure 101 the cores (435534) and (443553) are not difficult to recognize.

Of special interest is the question of how the hexahedron, whose structure we studied earlier (pages 45, 46 and 84), comes about as a degenerate form of the

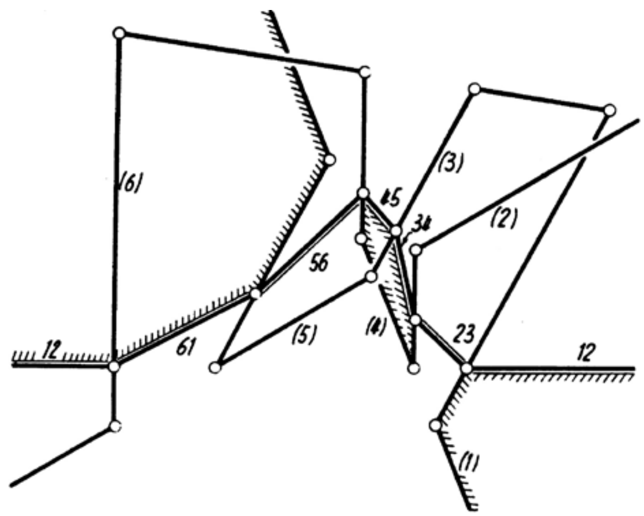


Figure 102

general complete 6-plane. We reach a deeper understanding of the cube, and in general of the Fundamental Structure, if we can see how the cube structure is connected with the general 6-plane. To this end we show, with the help of the figures, how to effect the transition to the special 6-plane, to the hexahedron.

In the latter the lines of intersection of opposite faces form a 3-side in a plane. If 1 and 2, 3 and 4, 5 and 6 are pairs of opposite faces of a hexahedron (for example A^- and A_1^-, B^- and B_1^-, C^- and C_1^- in Figure 16) then the lines of intersection 12, 34, 56 all belong to one plane, whereas in a general 6-plane they are skew. Each set of four planes 1, 2, 3, 4 and 3, 4, 5, 6 and 5, 6, 1, 2 goes through a point (namely C^+, A^+ and B^+ respectively). Hence the tetrahedral cores produced by 1234, 3456 and 5612 must all have shrunk to a point.

The core determined by 2345 of interest here is the tetrahedral domain adjoining the cube's edge on 25, with opposite edge on $34 = C^+A^+$. Another of the cores in question, produced by 4561, shares an edge with the cube on 41, while its opposite edge belongs to line $56 = A^+B^+$. The cube's edge on 63 and $12 = B^+C^+$ have a corresponding significance for 6123.

The second principal core turns into the six-faced core bounded by six three-sided planar domains (page 84) which is attached to vertices 135 and 246 of the hexahedron. This transformed principle core is joined by three more six-faced cores which are of the same type and on an equal footing with it; they are connected to the other three pairs of opposite vertices. In order for the ordinary hexahedron, and with it the Fundamental Structure, to materialize from the general 6-plane, the duality of the two principal cores must be abolished.

First Remark. We have outlined only the simplest properties of the six-structuring here. Many remarkable things could still be said. Incidentally, the complete 6-plane configuration is connected with the possible forms of the so-called cubic space curves and the cubic developables.

Second Remark. The structuring of point space by the general 6-plane was described in this Chapter in a direct, pictorial way. The actual proof (that is, from **A** and **O**) that six generally positioned planes always produce it, is rather laborious. We can bring something surprising to full consciousness here. In themselves, **A** and **O** are undoubtedly simple and to begin with uninteresting. Yet these simple axioms give rise to the 6-plane having the remarkable properties we have described. It is extraordinarily important actually to experience this contradiction — call it a tension if you will. A first consequence is to realize that **A** and **O** obviously contain much more than one had at first suspected.

With seven planes, different types of structuring are possible. Finding the number of different types is a difficult problem. As for the types of cores in the general case of any number of dividing planes, apparently only the following is known so far: in all divisions of space created by n planes (n greater than 3) there exist at least n tetrahedral cores.

The number V_n of vertices, the number C_n of cores, the number F_n of face portions bounding the cores and the number S_n of segments bounding the cores are easy to state. In fact

$$\begin{aligned} V_n &= \tfrac{1}{6}n(n-1)(n-2), & S_n &= \tfrac{1}{2}n(n-1)(n-2), \\ F_n &= \tfrac{1}{2}n(n-1)(n-2)+n, & C_n &= \tfrac{1}{6}n(n-1)(n-2)+n. \end{aligned}$$

On the other hand, to mathematical thinking, access to the different qualities of the various structurings is largely closed even today.

Third Remark. We have only described one aspect of six-structuring. The polar aspect is the structuring of plane space by six points. To understand the structuring of space into 26 surrounds of planes, by "polarizing" what has been described in this chapter, is an interesting though not particularly easy exercise.

EXERCISES

Take any tetrahedron, cut it with two planes neither of which goes through the tetrahedron's vertices and determine the cycle of the six planes. Drawing the ring of tetrahedral cores, the two principal cores, the characteristic positions of the six six-faced cores, each containing two five-sided planar cores in its boundary — these all provide rich material for various sorts of exercises. A true picture of the 6-plane is only obtained when these drawings are carried out for various different relative positionings of the six dividing planes.