The space duals of the complete plane n-point and the complete plane n-line are the *complete n-plane on a point* and the *complete n-line on a point* respectively. They are the projections from a point, of the plane n-line and the plane n-point respectively.

15. Configurations. The figures defined in the preceding section are examples of a more general class of figures of which we will now give a general definition.

DEFINITION. A figure is called a *configuration*, if it consists of a finite number of points, lines, and planes, with the property that each point is on the same number a_{12} of lines and also on the same number a_{13} of planes; each line is on the same number a_{21} of points and the same number a_{23} of planes; and each plane is on the same number a_{31} of points and the same number a_{32} of lines.

A configuration may conveniently be described by a square matrix:

	1 point	2 line	3 plane
1 point	a_{11}	a_{12}	a_{13}
2 line	a_{21}	a_{22}	a_{23}
3 plane	a_{31}	a_{32}	a_{33}

In this notation, if we call a point an element of the first kind, a line an element of the second kind, and a plane one of the third kind, the number a_{ij} $(i \neq j)$ gives the number of elements of the jth kind on every element of the ith kind. The numbers a_{11}, a_{22}, a_{33} give the total number of points, lines, and planes respectively. Such a square matrix is called the *symbol* of the configuration.

A tetrahedron, for example, is a figure consisting of four points, six lines, and four planes; on every line of the figure are two points of the figure, on every plane are three points, through every point pass three lines and also three planes, every plane contains three lines, and through every line pass two planes. A tetrahedron is therefore a configuration of the symbol

4	3	3
2	6	2
3	3	4

The symmetry shown in this symbol is due to the fact that the figure in question is self-dual. A triangle evidently has the symbol

3	2
2	3

Since all the numbers referring to planes are of no importance in case of a plane figure, they are omitted from the symbol for a plane configuration.

In general, a complete plane n-point is of the symbol

n	$n-1$
2	$\frac{1}{2}n(n-1)$

and a complete space n-point of the symbol

n	$n-1$	$\frac{1}{2}(n-1)(n-2)$
2	$\frac{1}{2}n(n-1)$	$n-2$
3	3	$\frac{1}{6}n(n-1)(n-2)$

Further examples of configurations are figs. 14 and 15, regarded as plane figures.

EXERCISE

Prove that the numbers in a configuration symbol must satisfy the condition

$$a_{ij}a_{ii} = a_{ji}a_{jj} \qquad (i, j = 1, 2, 3)$$

16. The Desargues configuration. A very important configuration is obtained by taking the plane section of a complete space five-point. The five-point is clearly a configuration with the symbol

5	4	6
2	10	3
3	3	10

and it is clear that the section by a plane not on any of the vertices is a configuration whose symbol may be obtained from the one just given by removing the first column and the first row. This is due to the fact that every line of the space figure gives rise to a point in the plane, and every plane gives rise to a line. The configuration in the plane has then the symbol

10	3
3	10

We proceed to study in detail the properties of the configuration just obtained. It is known as the *configuration of Desargues.*

We may consider the vertices of the complete space five-point as consisting of the vertices of a triangle A, B, C and of two points O_1, O_2

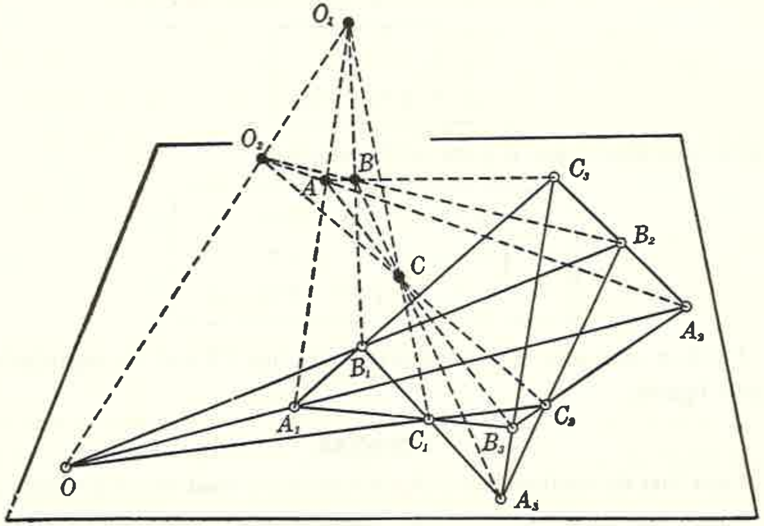

Fig. 14

not coplanar with any two vertices of the triangle (fig. 14). The section by a plane α not passing through any of the vertices will then consist of the following:

A triangle $A_1B_1C_1$, the projection of the triangle ABC from O_1 on α.
A triangle $A_2B_2C_2$, the projection of the triangle ABC from O_2 on α.
The trace O of the line O_1O_2.
The traces A_3, B_3, C_3 of the lines BC, CA, AB respectively.
The trace of the plane ABC, which contains the points A_3, B_3, C_3.
The traces of the three planes AO_1O_2, BO_1O_2, CO_1O_2, which contain respectively the triples of points OA_1A_2, OB_1B_2, OC_1C_2.

The configuration may then be considered (in ten ways) as consisting of two triangles $A_1B_1C_1$ and $A_2B_2C_2$, perspective from a point O and

having homologous sides meeting in three collinear points A_3, B_3, C_3. These considerations lead to the following fundamental theorem:

Theorem 1. The Theorem of Desargues.* *If two triangles in the same plane are perspective from a point, the three pairs of homologous sides meet in collinear points; i.e. the triangles are perspective from a line.* (A, E)

Proof. Let the two triangles be $A_1B_1C_1$ and $A_2B_2C_2$ (fig. 14), the lines A_1A_2, B_1B_2, C_1C_2 meeting in the point O. Let B_1A_1, B_2A_2 intersect in the point C_3; A_1C_1, A_2C_2 in B_3; B_1C_1, B_2C_2 in A_3. It is required to prove that A_3, B_3, C_3 are collinear. Consider any line through O which is not in the plane of the triangles, and denote by O_1, O_2 any two distinct points on this line other than O. Since the lines A_2O_2 and A_1O_1 lie in the plane (A_1A_2, O_1O_2), they intersect in a point A. Similarly, B_1O_1 and B_2O_2 intersect in a point B, and likewise C_1O_1 and C_2O_2 in a point C. Thus $ABCO_1O_2$, together with the lines and planes determined by them, form a complete five-point in space of which the perspective triangles form a part of a plane section. The theorem is proved by completing the plane section. Since AB lies in a plane with A_1B_1, and also in a plane with A_2B_2, the lines A_1B_1, A_2B_2, and AB meet in C_3. So also A_1C_1, A_2C_2, and AC meet in B_3; and B_1C_1, B_2C_2, and BC meet in A_3. Since A_3, B_3, C_3 lie in the plane ABC and also in the plane of the triangles $A_1B_1C_1$ and $A_2B_2C_2$, they are collinear.

Theorem 1′. *If two triangles in the same plane are perspective from a line, the lines joining pairs of homologous vertices are concurrent; i.e. the triangles are perspective from a point.* (A, E)

This, the converse of Theorem 1, is also its plane dual, and hence requires no further proof.

Corollary. *If two triangles not in the same plane are perspective from a point, the pairs of homologous sides intersect in collinear points; and conversely.* (A, E)

A more symmetrical and for many purposes more convenient notation for the Desargues configuration may be obtained as follows: Let the vertices of the space five-point be denoted by P_1, P_2, P_3, P_4, P_5 (fig. 15). The trace of the line P_1P_2 in the plane section is then naturally denoted by P_{12}, — in general, the trace of the line P_iP_j by P_{ij} ($i, j = 1, 2, 3, 4, 5, i \neq j$). Likewise the trace of the plane $P_iP_jP_k$ may

* Girard Desargues, 1593–1662.

be denoted by l_{ijk} $(i, j, k = 1, 2, 3, 4, 5)$. This notation makes it possible to tell at a glance which lines and points are united. Clearly a point is on a line of the configuration if and only if the suffixes of the point are both among the suffixes of the line. Also the third point on the line joining P_{ij} and P_{jk} is the point P_{ki}; two points are on the same line if and only if they have a suffix in common, etc.

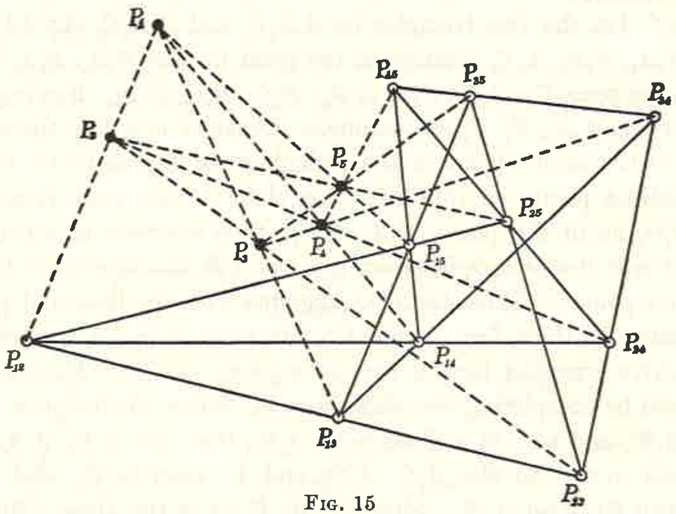

FIG. 15

EXERCISES

1. Prove Theorem 1′ without making use of the principle of duality.

2. If two complete n-points in different planes are perspective from a point, the pairs of homologous sides intersect in collinear points. What is the dual theorem? What is the corresponding theorem concerning any two plane figures in different planes?

3. State and prove the converse of the theorems in Ex. 2.

4. If two complete n-points in the same plane correspond in such a way that homologous sides intersect in points of a straight line, the lines joining homologous vertices are concurrent; i.e. the two n-points are perspective from a point. Dualize.

5. What is the figure formed by two complete n-points in the same plane when they are perspective from a point? Consider particularly the cases $n = 4$ and $n = 5$. Show that the figure corresponding to the general case is a plane section of a complete space $(n + 2)$-point. Give the configuration symbol and dualize.

6. If three triangles are perspective from the same point, the three axes of perspectivity of the three pairs of triangles are concurrent; and conversely. Dualize, and compare the configuration of the dual theorem with the case $n = 4$ of Ex. 5 (cf. fig. 15, regarded as a plane figure).

17. Perspective tetrahedra. As an application of the corollary of the last theorem we may now derive a theorem in space analogous to the theorem of Desargues in the plane.

THEOREM 2. *If two tetrahedra are perspective from a point, the six pairs of homologous edges intersect in coplanar points, and the four pairs of homologous faces intersect in coplanar lines; i.e. the tetrahedra are perspective from a plane.* (A, E)

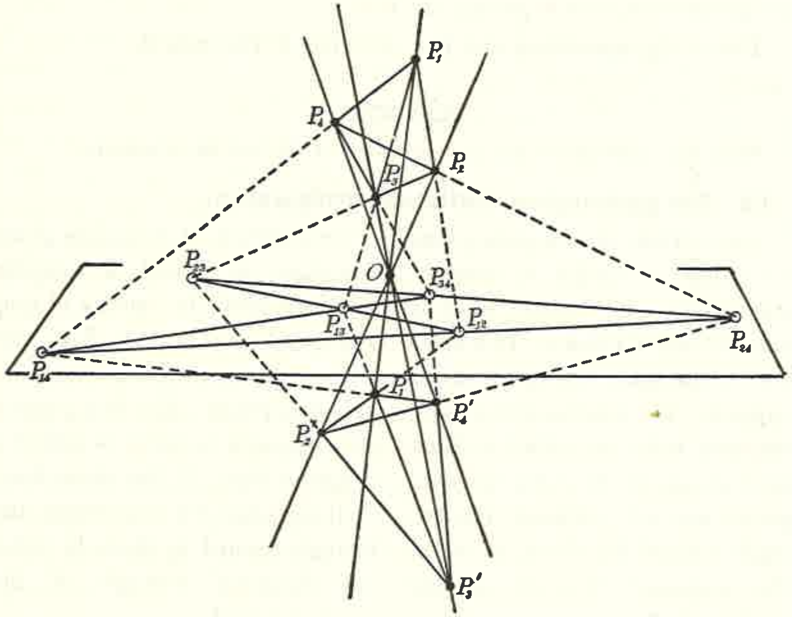

FIG. 16

Proof. Let the two tetrahedra be $P_1P_2P_3P_4$ and $P_1'P_2'P_3'P_4'$, and let the lines P_1P_1', P_2P_2', P_3P_3', P_4P_4' meet in the center of perspectivity O. Two homologous edges P_iP_j and $P_i'P_j'$ then clearly intersect; call the point of intersection P_{ij}. The points P_{12}, P_{13}, P_{23} lie on the same line, since the triangles $P_1P_2P_3$ and $P_1'P_2'P_3'$ are perspective from O (Theorem 1, Cor.). By similar reasoning applied to the other pairs of perspective triangles we find that the following triples of points are collinear:

$$P_{12},\ P_{13},\ P_{23};\ \ P_{12},\ P_{14},\ P_{24};\ \ P_{13},\ P_{14},\ P_{34};\ \ P_{23},\ P_{24},\ P_{34}.$$

The first two triples have the point P_{12} in common, and hence determine a plane; each of the other two triples has a point in common with each of the first two. Hence all the points P_{ij} lie in the same plane. The lines of the four triples just given are the lines of intersection of the pairs of homologous faces of the tetrahedra. The theorem is therefore proved.

THEOREM 2'. *If two tetrahedra are perspective from a plane, the lines joining pairs of homologous vertices are concurrent, as likewise the planes determined by pairs of homologous edges ; i.e. the tetrahedra are perspective from a point.* (A, E)

This is the space dual and the converse of Theorem 2.

EXERCISE

Write the symbols for the configurations of the last two theorems.

18. The quadrangle-quadrilateral configuration.

DEFINITION. A complete plane four-point is called a *complete quadrangle.* It consists of four vertices and six sides. Two sides not on the same vertex are called *opposite.* The intersection of two opposite sides is called a *diagonal point.* If the three diagonal points are not collinear, the triangle formed by them is called the *diagonal triangle* of the quadrangle.*

DEFINITION. A complete plane four-line is called a *complete quadrilateral.* It consists of four sides and six vertices. Two vertices not on the same side are called *opposite.* The line joining two opposite vertices is called a *diagonal line.* If the three diagonal lines are not concurrent, the triangle formed by them is called the *diagonal triangle* of the quadrilateral.*

The assumptions A and E on which all our reasoning is based do not suffice to prove that there are more than three points on any line. In fact, they are all satisfied by the triple system (1), p. 3 (cf. fig. 17). In a case like this the diagonal points of a complete quadrangle are collinear and the diagonal lines of a complete quadrilateral concurrent, as may readily be verified. Two perspective triangles cannot exist in such a plane, and hence the Desargues theorem becomes

* In general, the intersection of two sides of a complete plane n-point which do not have a vertex in common is called a *diagonal point* of the n-point, and the line joining two vertices of a complete plane n-line which do not lie on the same side is called a *diagonal line* of the n-line. A complete plane n-point (n-line) then has $n(n-1)(n-2)(n-3)/8$ diagonal points (lines). Diagonal points and lines are sometimes called *false vertices* and *false sides* respectively.

trivial. Later on we shall add an assumption* which excludes all such cases as this, and, in fact, provides for the existence of an infinite number of points on a line. A part of what is contained in this assumption is the following:

ASSUMPTION H_0. *The diagonal points of a complete quadrangle are noncollinear.*

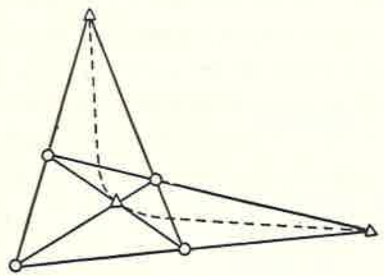

FIG. 17

Many of the important theorems of geometry, however, require the existence of no more than a finite number of points. We shall therefore proceed without the use of further assumptions than A and E, understanding that in order to give our theorems meaning *there must be postulated the existence of the points specified in their hypotheses.* In most cases the existence of a sufficient number of points is insured by Assumption H_0, and the reader who is taking up the subject for the first time may well take it as having been added to A and E. It is to be used in the solution of problems.

We return now to a further study of the Desargues configuration. A complete space five-point may evidently be regarded (in five ways) as a tetrahedron and a complete four-line at a point. A plane section of a four-line is a quadrangle and the plane section of a tetrahedron is a quadrilateral. It follows that (in five ways) the Desargues configuration may be regarded as a quadrangle and a quadrilateral. Moreover, it is clear that the six sides of the quadrangle pass through the six vertices of the quadrilateral. In the notation described on page 41 one such quadrangle is $P_{12}, P_{13}, P_{14}, P_{15}$ and the corresponding quadrilateral is $l_{234}, l_{235}, l_{245}, l_{345}$.

The question now naturally arises as to placing the figures thus obtained in special relations. As an application of the theorem of Desargues we will show how to construct † a quadrilateral which has the same diagonal triangle as a given quadrangle. We will assume in our discussion that the diagonal points of any quadrangle form a triangle.

* Merely saying that there are more than three points on a line does not insure that the diagonal points of a quadrangle are noncollinear. Cases where the diagonal points are collinear occur whenever the number of points on a line is $2^n + 1$.

† To *construct* a figure is to determine its elements in terms of certain given elements.