CS-C2160 Theory of Computation

Lecture 8: Turing Machines

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Topics:

- Turing machines
- Extensions of Turing machines
  - ▶ Multitrack machines
  - ▶ Multitape machines
  - ▶ Nondeterministic machines
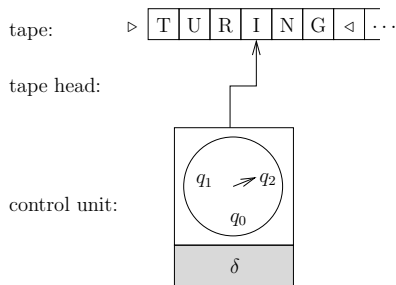- Excursion: The halting problem, first encounter

Material:

- In Finnish: Sections 4.1–4.2 and 6.1 in the Finnish lecture notes
- In English: Sections 3.1–3.3 in the Sipser book, multitrack machines on these slides

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Turing Machines

8.1 Turing machines

Alan Turing 1935–36

tape: ▷ | T | U | R | I | N | G | ◁ | ...

tape head:

control unit:

$q_1 \rightrightarrows q_2$

q_0

δ

- A Turing machine is like a finite automaton and has a tape ...
- but it can move both left and right on the tape
- and it can also write on the tape, not just read it.
- In addition, the tape is unbounded to the right.

The Church-Turing Thesis (∼1936)

Any mechanically (= physically) solvable computational problem can also be solved with a Turing machine.

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Other models of computation that are equivalent to Turing machines:

- Recursive function systems by Gödel and Kleene (1936)
- Church's λ-calculus (1936)
- String rewriting systems by Post (1936) and Markov (1951)
- All current programming languages (when the amount of and access to the memory are not limited)

From the modern point of view:

- *Turing machines ≡ (assembly-language) computer programs*

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Definition 8.1

A *Turing machine*[a] is a tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}),$$

where

- Q is the finite set of *states*,
- Σ is the finite *input alphabet*,
- $\Gamma \supseteq \Sigma$ is the finite *tape alphabet* (we assume $\triangleright, \triangleleft \notin \Gamma$),
- $\delta : (Q - \{q_{\text{acc}}, q_{\text{rej}}\}) \times (\Gamma \cup \{\triangleright, \triangleleft\}) \to Q \times (\Gamma \cup \{\triangleright, \triangleleft\}) \times \{L, R\}$ is the *transition function*,
- $q_0 \in Q$ is the *start state* ($q_0 \neq q_{\text{acc}}$ and $q_0 \neq q_{\text{rej}}$),
- $q_{\text{acc}} \in Q$ is the *accept state*, and
- $q_{\text{rej}} \in Q$ is the *reject state* ($q_{\text{rej}} \neq q_{\text{acc}}$).

---

[a]Sipser's book uses a slightly different (but effectively equivalent) formalisation of Turing machines that (i) does not contain the start- and end-of-tape symbols $\triangleright$ and $\triangleleft$ but (ii) has a special "blank" symbol for the yet-unused positions on the tape.

The interpretation for a value

$$\delta(q,a) = (q',b,\Delta)$$

of the transition function is that when in state q and reading symbol a on the tape, the machine:

- moves to state q',
- writes symbol b at the same position on the tape, and
- moves the tape head one position in direction Δ ($L \sim$ "left", $R \sim$ "right").

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The set of symbols the machine is allowed to write and the move directions are restricted in some cases:

- The transition function is not defined for the states q_{acc} and q_{rej}. When in either of these states, the machine *halts* immediately.
- For all transitions $\delta(q, a) = (q', b, \Delta)$ it is required that:
  - if $a = \triangleright$, then $b = \triangleright$ and $\Delta = R$
    That is, the start-of-tape symbol is never overwritten and the machine cannot move left beyond that symbol (i.e., off the tape).

  - $b = \triangleright$ is allowed only if $a = \triangleright$
    In other words, new start-of-tape symbols cannot be written.

  - $b = \triangleleft$ is allowed only if $a = \triangleleft$ and $\Delta = L$
    The machine does not explicitly write new end-of-tape symbols; they are introduced automatically when the machine moves past (and overwrites) the current end-of-tape symbol.

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- A *configuration* of the machine is a tuple

$$(q, u, a, v) \in Q \times \Gamma^* \times (\Gamma \cup \{\varepsilon\}) \times \Gamma^*,$$

  where possibly $a = \varepsilon$ if also either $u = \varepsilon$ or $v = \varepsilon$.
- Interpretation: the machine is in state q and the contents of the tape are (i) from the beginning to the left of the tape head u, (ii) at the tape head position a and (iii) from the right of the tape head to the end of the tape v.
- When at the very start/end of the tape, $a = \varepsilon$ and $u = \varepsilon/v = \varepsilon$. In the "start" case $u = \varepsilon$, the machine is thought to read the symbol $\triangleright$ and in the "end" case $v = \varepsilon$ the symbol $\triangleleft$.
- The *start configuration on input* $x = a_1 a_2 \ldots a_n$ is the tuple

$$(q_0, \varepsilon, a_1, a_2 \ldots a_n).$$

- A configuration (q, u, a, v) is more compactly denoted as $(q, u\underline{a}v)$ and the start configuration on input x as $(q_0, \underline{x})$

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- A configuration (q, w) *leads in one step* to (or *yields*) configuration (q', w'), denoted as

$$(q, w) \underset{M}{\vdash} (q', w'),$$

as determined by the following rules:
  - if $\delta(q, a) = (q', b, R)$, then $(q, u\underline{a}cv) \underset{M}{\vdash} (q', ub\underline{c}v)$;
  - if $\delta(q, a) = (q', b, L)$, then $(q, uc\underline{a}v) \underset{M}{\vdash} (q', u\underline{c}bv)$;
  - if $\delta(q, \triangleright) = (q', \triangleright, R)$, then $(q, \underline{\varepsilon}cv) \underset{M}{\vdash} (q', \underline{c}v)$;
  - if $\delta(q, \triangleleft) = (q', b, R)$, then $(q, u\underline{\varepsilon}) \underset{M}{\vdash} (q', ub\underline{\varepsilon})$;
  - if $\delta(q, \triangleleft) = (q', b, L)$, then $(q, uc\underline{\varepsilon}) \underset{M}{\vdash} (q', u\underline{c}b)$;
  - if $\delta(q, \triangleleft) = (q', \triangleleft, L)$, then $(q, uc\underline{\varepsilon}) \underset{M}{\vdash} (q', u\underline{c})$.

  where $q, q' \in Q$, $u, v \in \Gamma^*$, $a, b \in \Gamma$ and $c \in \Gamma \cup \{\varepsilon\}$.
- Configurations of form (q_{acc}, w) and (q_{rej}, w) do not yield any other configuration. In these configurations the machine *halts*.

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- A configuration (q, w) *leads* to a configuration (q', w'), denoted as

$$(q, w) \underset{M}{\vdash^*} (q', w'),$$

if there is a finite sequence of configurations (q_0, w_0), (q_1, w_1), $\ldots, (q_n, w_n)$, $n \geq 0$, such that

$$(q, w) = (q_0, w_0) \underset{M}{\vdash} (q_1, w_1) \underset{M}{\vdash} \cdots \underset{M}{\vdash} (q_n, w_n) = (q', w').$$

- A Turing machine M *accepts* a string $x \in \Sigma^*$ if

$$(q_0, \underline{x}) \underset{M}{\vdash^*} (q_{\text{acc}}, w) \qquad \text{for some } w \in \Gamma^*;$$

otherwise M *rejects* x.

- The language *recognised* by the machine M is

$$\mathcal{L}(M) = \{x \in \Sigma^\star \mid (q_0, \underline{x}) \underset{M}{\vdash^*} (q_{\text{acc}}, w) \text{ for some } w \in \Gamma^*\}.$$

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Example:

The (regular) language $\{a^{2k} \mid k \geq 0\}$ can be recognised with the Turing machine

$$M = (\{q_0, q_1, q_{\mathsf{acc}}, q_{\mathsf{rej}}\}, \{a\}, \{a\}, \delta, q_0, q_{\mathsf{acc}}, q_{\mathsf{rej}}),$$
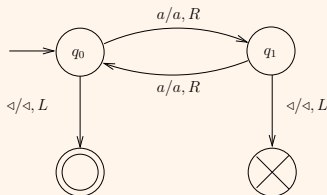
where

$$
\begin{aligned}
\delta(q_0, a) &= (q_1, a, R), \\
\delta(q_1, a) &= (q_0, a, R), \\
\delta(q_0, \triangleleft) &= (q_{\mathsf{acc}}, \triangleleft, L), \\
\delta(q_1, \triangleleft) &= (q_{\mathsf{rej}}, \triangleleft, L).
\end{aligned}
$$

State diagram representation:

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The notations used in the state diagram representation:

q State q

$\rightarrow q_0$ Start state

◎   Accept state (q_{acc})

⊗   Reject state (q_{rej})

$q \xrightarrow{a/b,\,\Delta} q'$ Transition $\delta(q, a) = (q', b, \Delta)$

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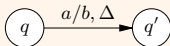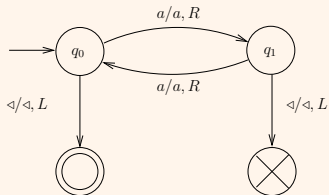
The computation of machine M on input aaa:

$$(q_0, \underline{a}aa) \quad \vdash_M \quad (q_1, a\underline{a}a) \quad \vdash_M (q_0, aa\underline{a})$$
$$\vdash_M \quad (q_1, aaa\underline{\varepsilon}) \quad \vdash_M (q_{\text{rej}}, aa\underline{a}).$$

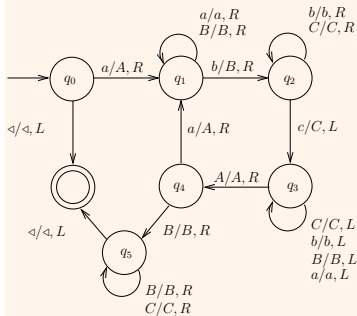The machine halts in state q_{rej} and thus $aaa \notin \mathcal{L}(M)$.

(*Note:* As this machine moves its tape head only to the right and accepts/rejects at the end of the string, it is effectively a finite automaton.)

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Example:

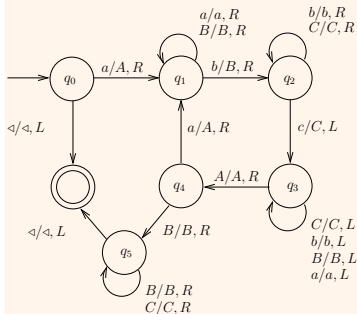A Turing machine recognising the (non-context-free) language $\{a^k b^k c^k \mid k \geq 0\}$:
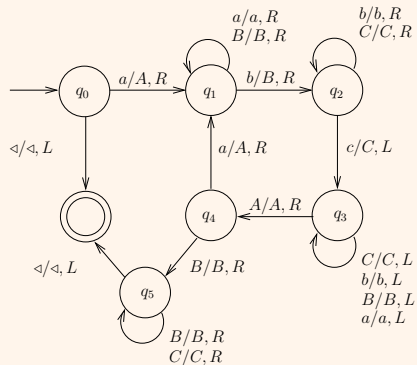
For the sake of clarity, the reject state is not drawn here, but again one interprets that all the "missing transitions" implicitly lead to the reject state.

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The computation of the machine on input $aabbcc$:

$$(q_0, \underline{a}abbcc) \vdash (q_1, A\underline{a}bbcc) \vdash$$
$$(q_1, Aa\underline{b}bcc) \vdash (q_2, AaB\underline{b}cc) \vdash$$
$$(q_2, AaBb\underline{c}c) \vdash (q_3, AaB\underline{b}Cc) \vdash$$
$$(q_3, Aa\underline{B}bCc) \vdash (q_3, A\underline{a}BbCc) \vdash$$
$$(q_3, \underline{A}aBbCc) \vdash (q_4, A\underline{a}BbCc) \vdash$$
$$(q_1, AA\underline{B}bCc) \vdash (q_1, AAB\underline{b}Cc) \vdash$$
$$(q_2, AABB\underline{C}c) \vdash (q_2, AABBC\underline{c}) \vdash$$
$$(q_3, AABB\underline{C}C) \vdash (q_3, AAB\underline{B}CC) \vdash$$
$$(q_3, AA\underline{B}BCC) \vdash (q_3, A\underline{A}BBCC) \vdash$$
$$(q_4, AA\underline{B}BCC) \vdash (q_5, AAB\underline{B}CC) \vdash$$
$$(q_5, AABB\underline{C}C) \vdash (q_5, AABBC\underline{C}) \vdash$$
$$(q_5, AABBCC\underline{\varepsilon}) \vdash (q_{\text{acc}}, AABBC\underline{C}).$$

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The computation of the machine on input $aabcbc$:

$(q_0, \underline{a}abcbc) \vdash (q_1, A\underline{a}bcbc) \vdash$
$(q_1, Aa\underline{b}cbc) \vdash (q_2, AaB\underline{c}bc) \vdash$
$(q_3, Aa\underline{B}Cbc) \vdash (q_3, A\underline{a}BCbc) \vdash$
$(q_3, \underline{A}aBCbc) \vdash (q_4, A\underline{a}BCbc) \vdash$
$(q_1, AA\underline{B}Cbc) \vdash (q_1, AAB\underline{C}bc) \vdash$
$(q_{\text{rej}}, AABC\underline{b}c).$

Note

The definition of "language recognised by a machine" *does not require that the machine halts* on strings that do not belong to the language.

Example:

A Turing machine that enters an infinite loop on some inputs:
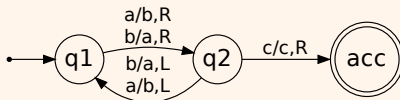
The computation on input abc:

$(q_1, \underline{a}bc) \vdash (q_2, b\underline{b}c) \vdash (q_1, \underline{b}ac) \vdash (q_2, a\underline{a}c) \vdash$
$(q_1, \underline{a}bc) \vdash (q_2, b\underline{b}c) \vdash (q_1, \underline{b}ac) \vdash (q_2, a\underline{a}c) \vdash \dots$

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Example:

A Turing machine that has an infinite computation using an unbounded amount of tape on some inputs:
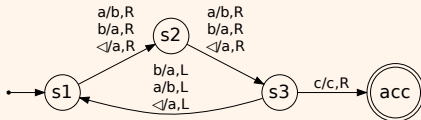
The computation on input a:

$(q_1, \underline{a}) \vdash (q_2, b\underline{\varepsilon}) \vdash (q_3, ba\underline{\varepsilon}) \vdash (q_1, b\underline{a}a) \vdash (q_2, bb\underline{a}) \vdash (q_3, bbb\underline{\varepsilon}) \vdash (q_1, bbb\underline{a}) \vdash (q_2, bba\underline{a}) \vdash (q_3, bbab\underline{\varepsilon}) \vdash (q_1, bba\underline{b}a) \vdash \dots$

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Extensions of Turing Machines

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- Could we obtain even stronger models of computation if we extended the definition of Turing machines with new features?

- We could, for instance, allow multiple read/write tapes or non-determinism (like we did earlier with finite automata).

- In the following we study some such extensions and ...

- show that all the languages that one can recognise with such extended machines, can also be recognised with standard Turing machines.

- Witness the Church-Turing Thesis: *Any mechanically solvable computational problem can be solved with a (standard) Turing machine.*

- The extensions are also useful in designing machines for more complex purposes. (Cf. the uses of NFA as a design tool for DFA.)

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8.2 Multitrack machines

- The tape of the Turing machine now consists of k parallel "tracks" that are all read and written in one computation step.

A	L	A	N	#	#	#	#		
M	A	T	H	I	S	O	N		⋯
T	U	R	I	N	G	#	#		

tape head:

- The transition function of such a machine is of form

$$\delta(q, (a_1, \ldots, a_k)) = (q', (b_1, \ldots, b_k), \Delta),$$

where $a_1, \ldots, a_k$ are the symbols read on tracks $1, \ldots, k$, $b_1, \ldots, b_k$ the symbols written over them, and $\Delta \in \{L, R\}$ is the move direction as before.

- In the beginning of a computation, the input string is placed on the first track and the other tracks contain special "blank symbols" # in the same positions.

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- Formally, a *k-track Turing machine* is a tuple

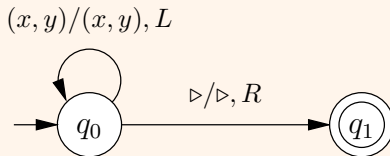$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}),$$

where the other components are as in the standard model but the transition function is:

$$\delta : (Q - \{q_{\text{acc}}, q_{\text{rej}}\}) \times (\Gamma^k \cup \{\triangleright, \triangleleft\}) \to Q \times (\Gamma^k \cup \{\triangleright, \triangleleft\}) \times \{L, R\}.$$

- The "leads to" relation $\vdash_M$, start configuration etc. are defined similarly as in the standard model.

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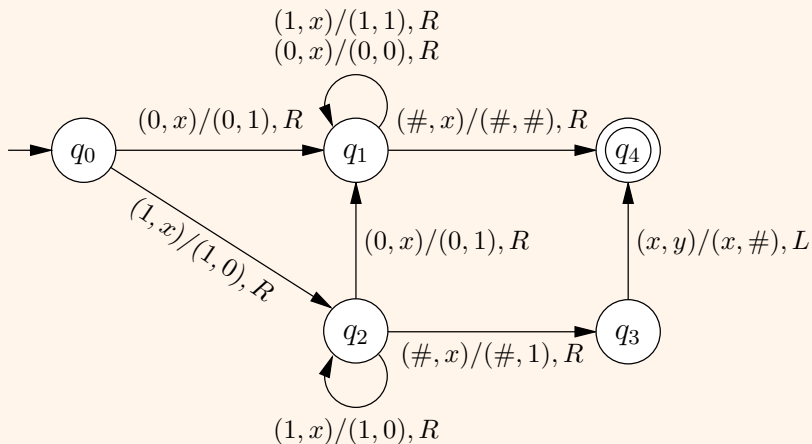Aalto University / Dept. Computer Science
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Example:

A 2-track Turing machine `rewind` that rewinds the tape head to the beginning of the tape:

(The notation $(x, y)/(x, y), L$ is here a shorthand meant to cover all the transitions that can be obtained by replacing the variables x and y with some tape alphabet symbols.)

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Example:

A 2-track Turing machine `succ` that computes the successor of a number given on track 1 to track 2 (the numbers are written in binary, least significant bit first):
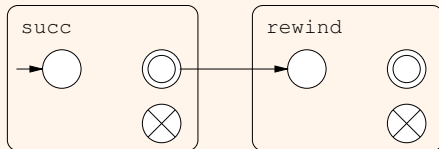
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The computation of the machine on input number $3_{10} = 011_2$ (provided in lsb first representation on track 1):
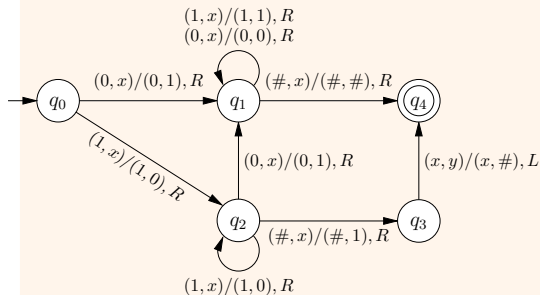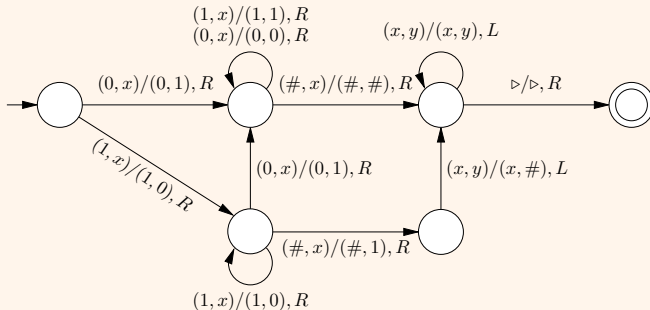

$$
\left(q_0, \begin{array}{cccc} 1 & 1 & 0 & \# \\ \underline{\#} & \# & \# & \# \end{array} \right) \vdash
$$

$$
\left(q_2, \begin{array}{cccc} 1 & 1 & 0 & \# \\ 0 & \underline{\#} & \# & \# \end{array} \right) \vdash
$$

$$
\left(q_2, \begin{array}{cccc} 1 & 1 & 0 & \# \\ 0 & 0 & \underline{\#} & \# \end{array} \right) \vdash
$$

$$
\left(q_1, \begin{array}{cccc} 1 & 1 & 0 & \# \\ 0 & 0 & 1 & \underline{\#} \end{array} \right) \vdash
$$

$$
\left(q_4, \begin{array}{ccccc} 1 & 1 & 0 & \# & \\ 0 & 0 & 1 & \# & \underline{\varepsilon} \end{array} \right)
$$

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The sequential composition of the 2-track Turing machines `succ` and `rewind`:

which means the following:

Lemma 8.1

If a language L can be recognised with a k-track Turing machine, then it can be recognised with a standard Turing machine as well.

Proof

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$ be a k-track Turing machine recognising the language L. An equivalent standard Turing machine $\widehat{M}$ can be constructed as follows:

$$\widehat{M} = (\widehat{Q}, \Sigma, \hat{\Gamma}, \hat{\delta}, \hat{q}_0, q_{\text{acc}}, q_{\text{rej}}),$$

where $\widehat{Q} = Q \cup \{\hat{q}_0, \hat{q}_1, \hat{q}_2\}$, $\hat{\Gamma} = \Sigma \cup \Gamma^k$ and for all $q \in Q$ we have
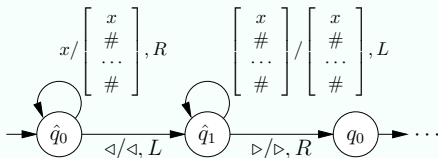
$$\hat{\delta}(q, \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}) = (q', \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}, \Delta),$$

$$\text{when} \quad \delta(q, (a_1, \ldots, a_k)) = (q', (b_1, \ldots, b_k), \Delta).$$

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In the beginning of its computation, the machine $\widehat{M}$ must "lift" the input string to the (simulating) first track, meaning that it replaces the input $a_1 a_2 \ldots a_n$ with

$$\begin{bmatrix} a_1 \\ \# \\ . \\ . \\ \# \end{bmatrix} \begin{bmatrix} a_2 \\ \# \\ . \\ . \\ \# \end{bmatrix} \cdots \begin{bmatrix} a_n \\ \# \\ . \\ . \\ \# \end{bmatrix}.$$

For this, the transition function of $\widehat{M}$ includes, in addition to the transitions copied from M, a small "preprocessor" sub-machine

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8.3 Multitape machines
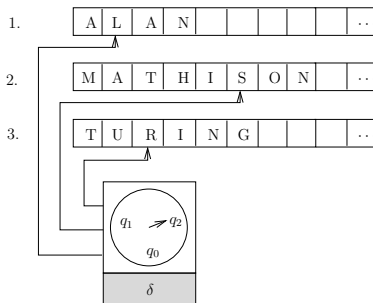
- We now allow a Turing machine to have k independent tapes, each with its own tape head.
- The machine reads and writes all the tapes in each step.
- In the beginning of the computation, the input is on the first tape, the other tapes are empty, and all the tape heads point to the beginning of their tapes.

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- The transitions of such a machine are of form

$$\delta(q, a_1, \ldots, a_k) = (q', (b_1, \Delta_1), \ldots, (b_k, \Delta_k)),$$

where

  - $a_1, \ldots, a_k$ are the symbols read from the tapes $1, \ldots, k$,
  - $b_1, \ldots, b_k$ are the symbols written on the tapes $1, \ldots, k$, and
  - $\Delta_1, \ldots, \Delta_k \in \{L, R, S\}$ are the move directions of the tape heads (S means "stay", i.e., the tape head is not moved).

- Formally, a *k-tape Turing machine* is a tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}),$$

  where all the other components are as in the standard model, but the transition function is of form:

$$\delta : (Q - \{q_{\text{acc}}, q_{\text{rej}}\}) \times (\Gamma \cup \{\triangleright, \triangleleft\})^k \to Q \times ((\Gamma \cup \{\triangleright, \triangleleft\}) \times \{L, R, S\})^k.$$

- The "leads to" relations and other concepts are defined similarly as in the standard model.

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Example:

A 3-tape Turing machine `Rew2` that moves the tape head of the second tape to the beginning (other tape heads stay in their original places):
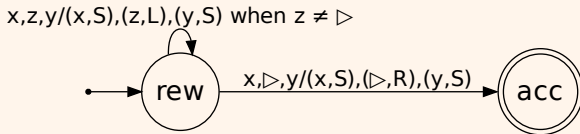
Here x, y, and z are again parameters for abbreviations and, for instance, the transition "$x, z, y/(x, \text{S}), (z, \text{L}), (y, \text{S})$ when $z \neq \rhd$" represents all the possible transitions that can be obtained by replacing x, y and z with any tape alphabet symbols (excluding the case $z = \rhd$).

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A 3-tape machine `4Succ2` that computes the successor of the number given on tape 2. Numbers are in base 4, in "least significant digit first" order and terminated by a semicolon. The machine shall be started in a situation where the tape head of the second tape is in the beginning.
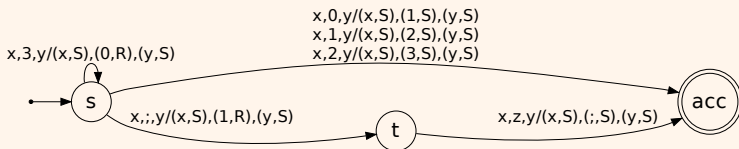
E.g., computing the successor 100_4 of 33_4 (i.e., 15_{10}):

$$
\begin{pmatrix} & ab\underline{b}a \\ \text{s,} & 3\underline{3}; \\ & \underline{1}11; \end{pmatrix} \vdash
\begin{pmatrix} & ab\underline{b}a \\ \text{s,} & 0\underline{3}; \\ & \underline{1}11; \end{pmatrix} \vdash
\begin{pmatrix} & ab\underline{b}a \\ \text{s,} & 00\underline{;} \\ & \underline{1}11; \end{pmatrix} \vdash
\begin{pmatrix} & ab\underline{b}a \\ \text{t,} & 001\underline{\varepsilon} \\ & \underline{1}11; \end{pmatrix} \vdash
$$

$$
\begin{pmatrix} & ab\underline{b}a \\ \text{acc,} & 001\underline{;} \\ & \underline{1}11; \end{pmatrix}
$$

Example:

The sequential composition of machines `4Succ2` and `Rew2`. It computes the successor of the number given on tape 2 (in base 4, "least significant digit first" order and ;-terminated) and rewinds the second tape head to the beginning after the computation.

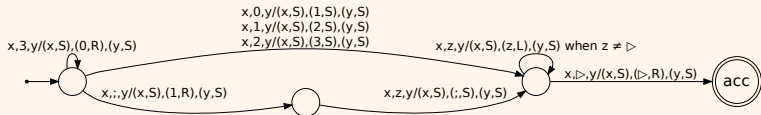The machine shall be started in a situation where the tape head of the second tape is in the beginning.

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Lemma 8.2

If a language L can be recognised with a k-tape Turing machine, then it can be recognised with a standard Turing machine as well.

Proof

Let

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$$

be a k-tape Turing machine recognising the language L. We can simulate it with a $2k$-track Turing machine $\widehat{M}$ as follows. The odd tracks $1, 3, 5, \ldots, 2k-1$ of $\widehat{M}$ correspond to the tapes $1, 2, \ldots, k$ of M and for each odd track, the following even track contains exactly one $\uparrow$ symbol that indicates the tape head position on the tape of the odd track.

1.	A	L	A	N	#	#	#	#	
2.		$\uparrow$							
3.	M	A	T	H	I	S	O	N	
4.					$\uparrow$				
5.	T	U	R	I	N	G	#	#	
6.			$\uparrow$						

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- In the beginning, the input string is placed on the first track as usual, and in its first move $\widehat{M}$ writes the $\uparrow$ symbols to the first positions of the even tracks.

- After this, $\widehat{M}$ operates by "sweeping" across the tape forwards and backwards.

- On each forward sweep from the beginning to the end, $\widehat{M}$ collects information about which symbols are at the positions indicated by the $\uparrow$ symbols, i.e., at the tape head positions of the simulated machine M.

- Based on this information, $\widehat{M}$ then performs a backward sweep to the beginning and makes the changes on its multitrack tape (writes tape symbols, moves tape head markers $\uparrow$) that correspond to the changes made by a single transition of the simulated machine M.

The multitrack machine $\widehat{M}$ can then be simulated with a standard Turing machine, as presented in Lemma 8.1.

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8.4 Nondeterministic machines

- Formally, a *nondeterministic Turing machine* is a tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\mathsf{acc}}, q_{\mathsf{rej}}),$$

where the other components are as in the standard model but the transition function is of form:

$$\delta : (Q - \{q_{\mathsf{acc}}, q_{\mathsf{rej}}\}) \times (\Gamma \cup \{\triangleright, \triangleleft\}) \to \mathcal{P}(Q \times (\Gamma \cup \{\triangleright, \triangleleft\}) \times \{L, R\}).$$

- The interpretation of a value

$$\delta(q, a) = \{(q_1, b_1, \Delta_1), \dots, (q_k, b_k, \Delta_k)\}$$

of the transition function is that, when in state q and reading tape symbol a, the machine can act according to *some* triple (q_i, b_i, Δ_i) in the list.

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- For nondeterministic machines, the configurations, "leads to" relations etc. are defined as for the standard deterministic machines, except that the condition $\delta(q, a) = (q', b, \Delta)$ is replaced with the nondeterminic version $(q', b, \Delta) \in \delta(q, a)$.

- Because of this, the "leads to" relation $\underset{M}{\vdash}$ is no longer single-valued, meaning that a configuration (q, w) can now have many possible successor configurations (q', w') (i.e., those for which $(q, w) \underset{M}{\vdash} (q', w')$ holds).

- The language recognised by the machine M is now

$$\mathcal{L}(M) = \{x \in \Sigma^\star \mid (q_0, \underline{x}) \underset{M}{\vdash^*} (q_{\text{acc}}, w) \text{ for some } w \in \Gamma^*\}.$$

- That is, a string x belongs to the language recognised by a nondeterministic machine M if *some* finite sequence of configurations leads from the start configuration to an accepting configuration.

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Example: Recognising composite numbers with nondeterministic Turing machines

A non-negative integer n is a *composite* number if it has integer factors $p, q \geq 2$ s.t. $pq = n$. A non-negative integer that is not composite is either *unit* (1) or a *prime* number.

Assume that we already have a deterministic Turing machine `check_mult` that recognises the language

$$\mathcal{L}(\text{check\_mult}) = \{n \# p \# q \mid n, p, q \text{ are binary numbers and } n = pq\}.$$

In addition, let `go_start` be a deterministic Turing machine that moves the tape head to the beginning of the tape.

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Furthermore, let `gen_int` be the following nondeterministic machine. It writes an *arbitrary* binary number (in the most-significant-bit-first order) that is greater than 1 at the end of the tape:

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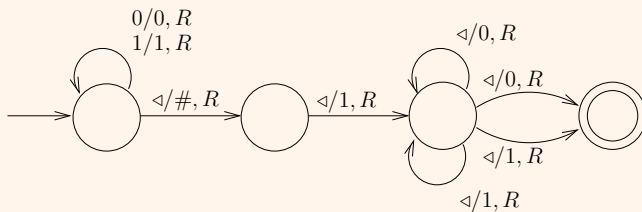
Some computations of machine `gen_int` on input string '0':

$$q_1, \underline{0}$$
$$|$$
$$q_1, 0\underline{\varepsilon}$$
$$|$$
$$q_2, 0\#\underline{\varepsilon}$$
$$|$$
$$q_3, 0\#1\underline{\varepsilon}$$

$q_4, 0\#10\underline{\varepsilon}$ $q_3, 0\#10\underline{\varepsilon}$ $q_4, 0\#11\underline{\varepsilon}$ $q_3, 0\#11\underline{\varepsilon}$

⋮ ⋮ ⋮ ⋮

$q_4, 0\#100\underline{\varepsilon}$ $q_3, 0\#100\underline{\varepsilon}$ $q_4, 0\#101\underline{\varepsilon}$ $q_3, 0\#101\underline{\varepsilon}$

⋮ ⋮ ⋮ ⋮   ⋮ ⋮ ⋮ ⋮

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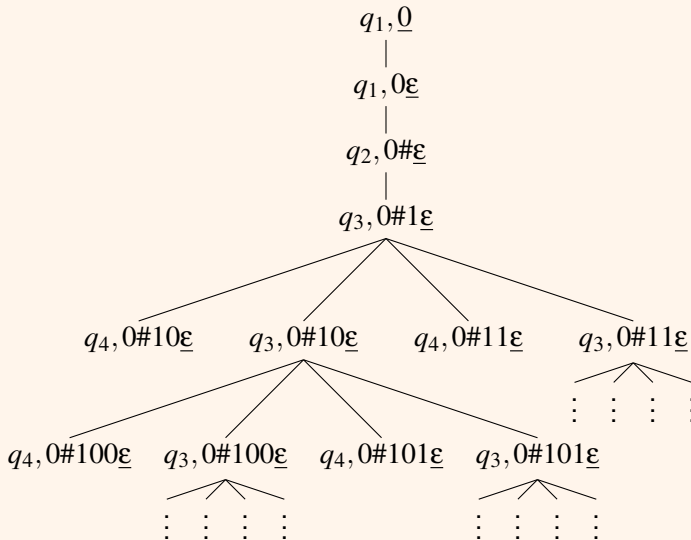
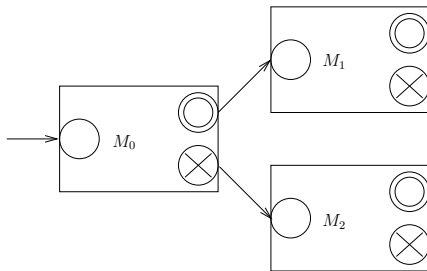A nondetermistic Turing machine that recognises the language

$$\mathcal{L}(\texttt{test\_composite}) = \{n \mid n \text{ is a binary compound number}\}$$

can now be constructed by combining the above mentioned machines:
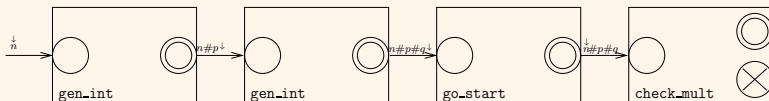
The resulting machine accepts an input binary string n if and only if there exist binary numbers $p, q \geq 2$ for which $n = pq$ holds — that is, if and only if n is a composite number.

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Note. A common diagram notation for an "if-then-else" combination of Turing machines:

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Theorem 8.3

If a language L can be recognised with a nondeterministic Turing machine, then it can be recognised with a standard deterministic Turing machine as well.
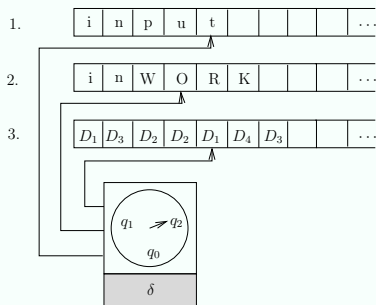
Proof (idea)

- Let

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$$

  be a nondeterministic Turing machine recognising the language L.

- One can simulate M with a 3-tape deterministic machine $\widehat{M}$ that systematically explores all the computations of M until it finds a computation that ends in an accepting configuration — if such a computation exists.

- The 3-tape machine $\widehat{M}$ can then be transformed into a standard deterministic machine as presented in Lemmas 8.1 and 8.2.

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In more (but not full) detail:

- On tape 1, $\widehat{M}$ stores the input string.

- On tape 2, $\widehat{M}$ simulates the work tape of machine M.

- In the beginning of each simulated computation, $\widehat{M}$ copies the input string from tape 1 to tape 2 and erases any spurious symbols that were left on tape 2 from the simulation of the previous computation.

- On tape 3, $\widehat{M}$ stores the "index" of the current computation of M to be simulated.

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1. | i | n | p | u | t | | | | | ... |

2. | i | n | W | O | R | K | | | | ... |

3. | D_1 | D_3 | D_2 | D_2 | D_1 | D_4 | D_3 | | | ... |

$q_1 \nearrow q_2$
q_0

δ
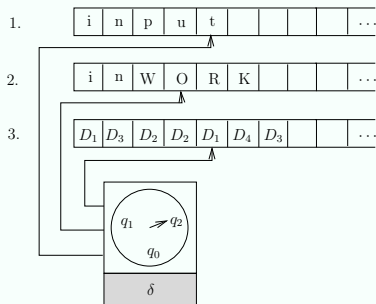
- Let r be the size of the biggest alternative-moves set in the transition function of M.

- Then machine $\widehat{M}$ has special tape symbols $D_1, \ldots, D_r$ and it enumerates all possible sequences of these on tape 3 in lexicographic ("shortlex") order: ε, D_1, D_2, $\ldots$, D_r, $D_1 D_1$, $D_1 D_2$, $\ldots$, $D_1 D_r$, $D_2 D_1$, $\ldots$

- For each such generated sequence, $\widehat{M}$ simulates one (possibly incomplete) computation of machine M, in which the nondeterministic moves are made according to the sequence currently listed on tape 3.

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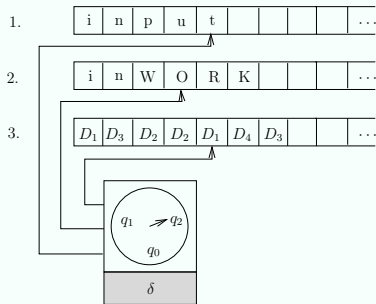
- For instance, if tape 3 contains the sequence $D_1 D_3 D_2$, then the simulated computation takes the first choice in the first move, the third in the second, and the second in the third.

- If this computation did not end up in an accepting configuration, the next sequence $D_1 D_3 D_3$ is generated on tape 3 and a new simulation is performed.

- If the sequence on tape 3 is not valid because it contains a too large choice number at some point, the simulated computation is simply cancelled and the next sequence is generated.

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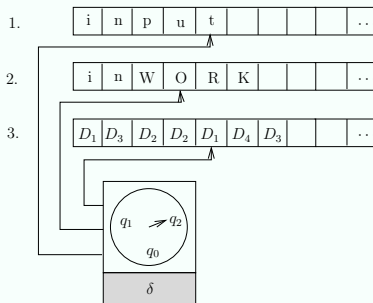
Clearly this systematic simulation of computations of M leads $\widehat{M}$ to accept the input string if and only if M has an accepting computation. If M has no accepting computations on an input string, then machine $\widehat{M}$ does not halt.[a]

---

[a]With a bit more bookkeeping, the machine $\widehat{M}$ could also reject the input string (and halt) if for some n, all the computations of M of length n or less halt in a rejecting configuration. Even in this case, $\widehat{M}$ would obviously not halt if M had some nonhalting computations.

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* Excursion: The Halting Problem, First Encounter

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8.5 The halting problem

- As will be seen in Lecture 9, there are infinitely many more languages than Turing machines (or C/Python/Scala... programs).
- As languages correspond to decision problems, this means that *not all decision problems can be solved*.
- What about concrete examples of such undecidable problems?
- The best-known example is so-called *Turing's halting problem* (Alan Turing, 1936).
- In terms of C programs, we can formulate this result as follows:

Claim

There is no C function $halt(p,x)$ that, given the source code string p of some C function and an input string x for p, outputs 1 if the execution of p on input x eventually terminates, and 0 if the execution of p on x never terminates. Here it is assumed that the programs can access an unlimited amount of memory.

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Proof

Suppose, contrary to the claim, that such a function `halt` existed. By using this hypothetical function, we construct another function `confuse` as shown in the program code box below.

Let `c` denote the presented source code string of `confuse`, and study what happens if we run `confuse` on its own source code:

```c
void confuse(char *q){
    int halt(char *p, char *x){
    ... /* Body of function 'halt' */
    }
    if (halt(q,q) == 1) while (1);
}
```

`confuse(c)` **halts**
$$\Leftrightarrow$$
`halt(c,c) == 1`
$$\Leftrightarrow$$
`confuse(c)` **does not halt!**

As we obtained a contradiction, we must deduce that the hypothetical halting testing function `halt` cannot exist.

In fact, as will be seen in the next lecture, there are *lots of such undecidable problems*.

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The same in Python

The file `haltingTester.py` containing the hypothetical halting testing function `doesHalt`:

```python
# (C) 2013 H. Ackerfrau
def doesHalt(sourceName, inputName):
    """Returns true if the program in file 'sourceName' halts
        when it is run on the input file 'inputName', false otherwise."""
    fs = open(sourceName, "r")
    fi = open(inputName, "r")
    ...
    return result

if __name__ == '__main__':
    source = sys.argv[1]
    input = sys.argv[2]
    halts = doesHalt(source, input)
    print(source+" halts on "+input+": "+halts)
```

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The file `confuse.py`:

```python
# (C) 2013 H. Ackerfrau
def doesHalt(sourceName, inputName):
    """Returns true if the program in file 'sourceName' halts
        when it is run on the input file 'inputName', false otherwise."""
    fs = open(sourceName, "r")
    fi = open(inputName, "r")
    ...
    return result

if __name__ == '__main__':
    sourceAndInput = sys.argv[1]
    halts = doesHalt(sourceAndInput, sourceAndInput)
    if halts:
        while True:
            pass
    print("I'll now halt :-)")
```

Does the execution of the command

$$\texttt{python confuse.py confuse.py}$$

terminate?

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