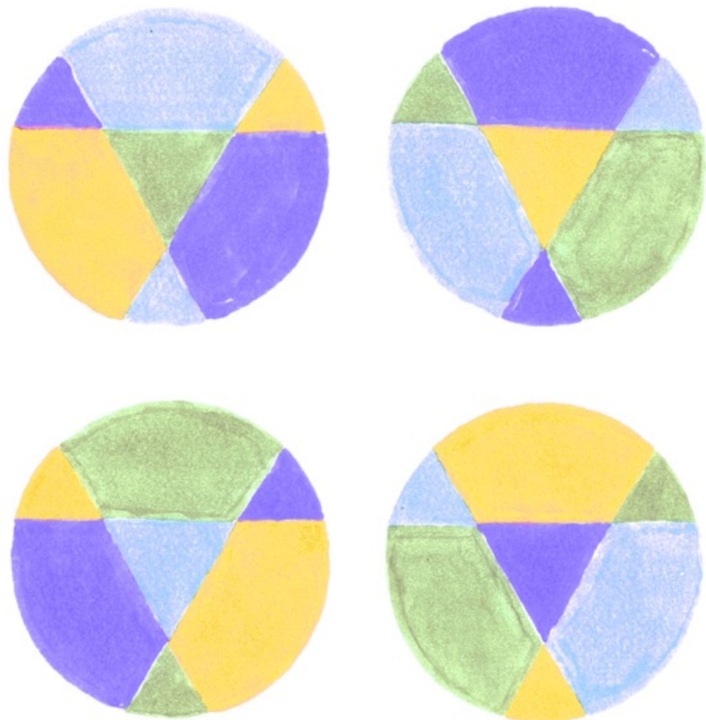
PROJECTIVE GEOMETRY
PART 4



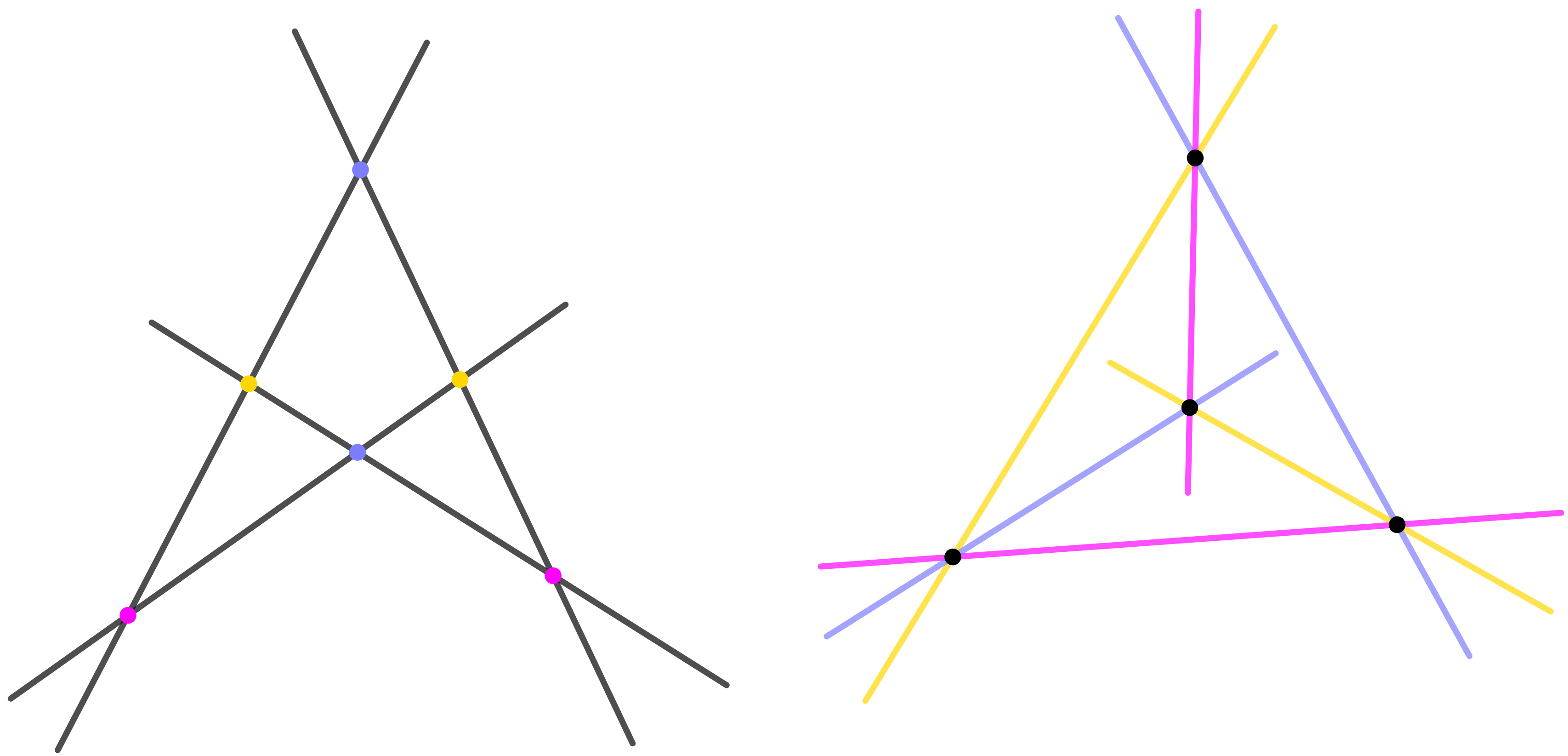
Taneli Luotoniemi

CRYSTAL FLOWER IN HALLS OF MIRRORS 2021

PROJECTIVE CONFIGURATIONS

COMPLETE QUADRILATERAL

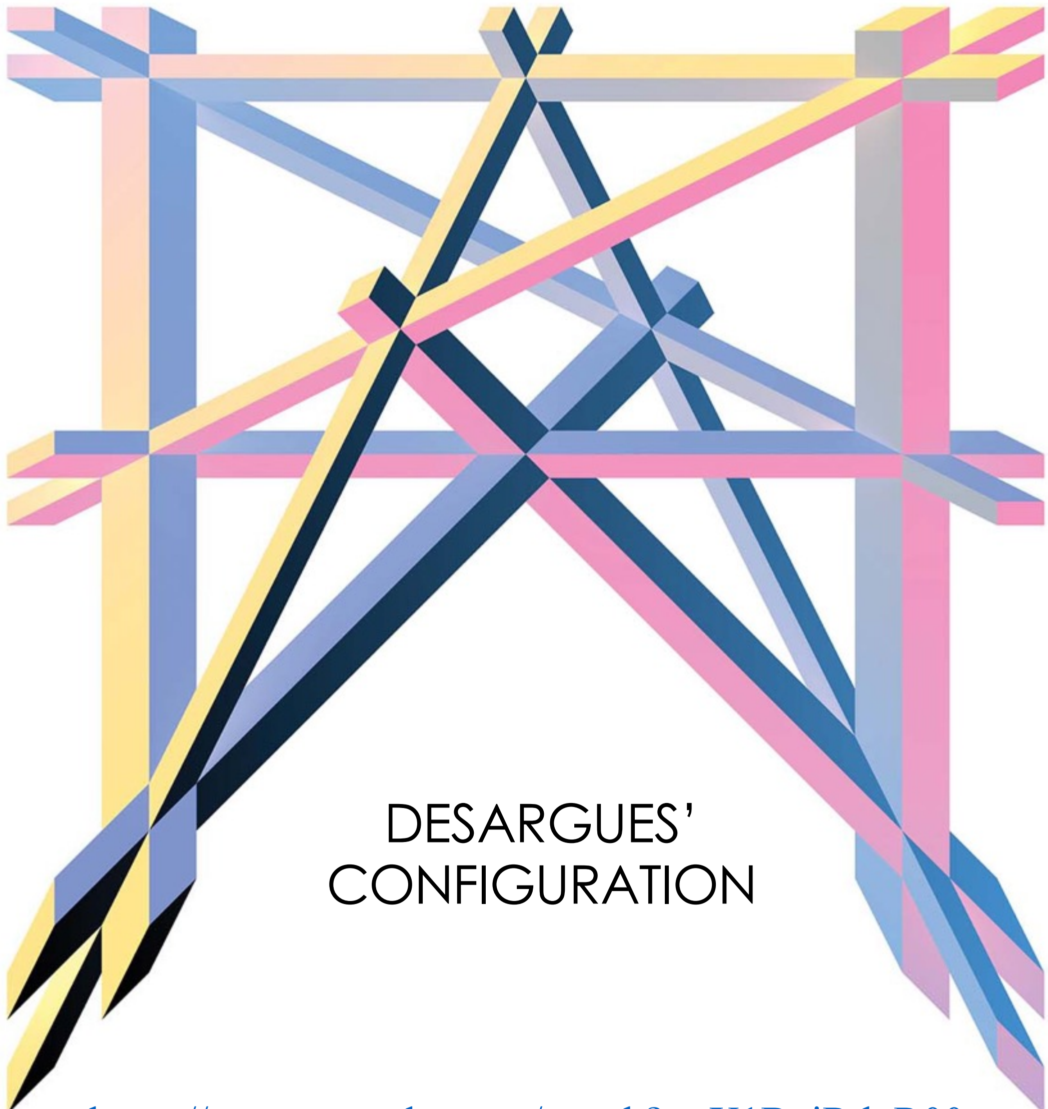
(FOUR LINES IN GENERAL POSITION INTERSECTING EACH OTHER IN PROJECTIVE PLANE)



QUADRILATERAL / QUADRANGLE

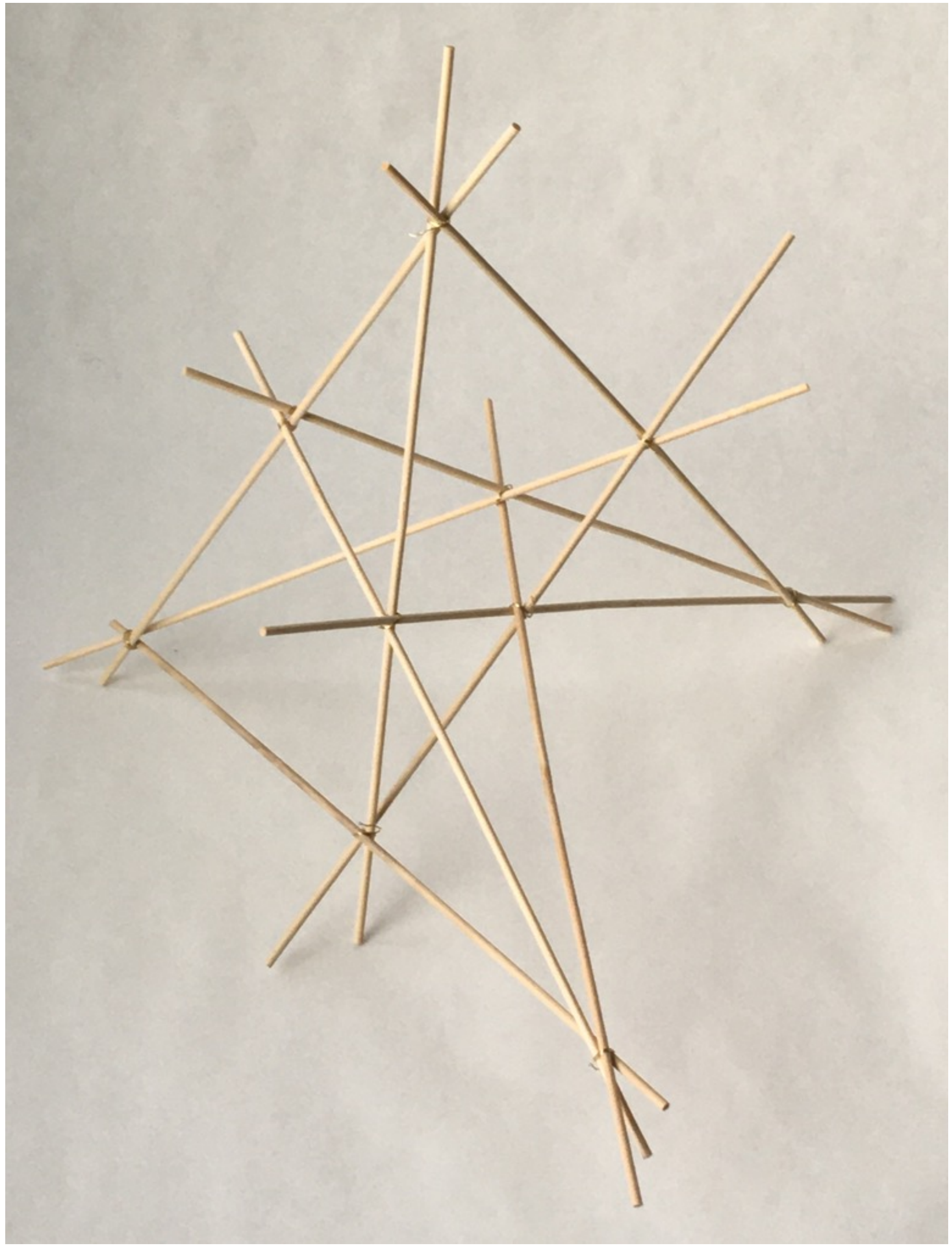
COMPLETE PENTAHEDRON

(FIVE PLANES IN GENERAL POSITION INTERSECTING EACH OTHER IN PROJECTIVE SPACE)



https://www.youtube.com/watch?v=X1BuiRdpR00

Stick model of the complete pentahedron



COMPLETE PENTALATERAL

(FIVE LINES IN GENERAL POSITION INTERSECTING EACH OTHER IN PROJECTIVE PLANE)

COMPLETE HEXAHEDRON

(SIX PLANES IN GENERAL POSITION INTERSECTING EACH OTHER IN PROJECTIVE SPACE)

Selecting the layout for a stick model of a configuration amounts to adjusting how *the plane at infinity* is situated with respect to the configuration portrayed.

We can illustrate this by adding a sixth plane into our Desargues' configuration.

How many points and lines there are in *the complete hexahedron*, and how are they incident with each other?

The reason why the Desargues’ stick configuration always has the same layout, is that six planes in general position will always intersect each other in projective space in the same configuration of **fifteen lines** (with four points along each) and **twenty points** (with three lines through each).

What kinds of cells there are in the configuration?

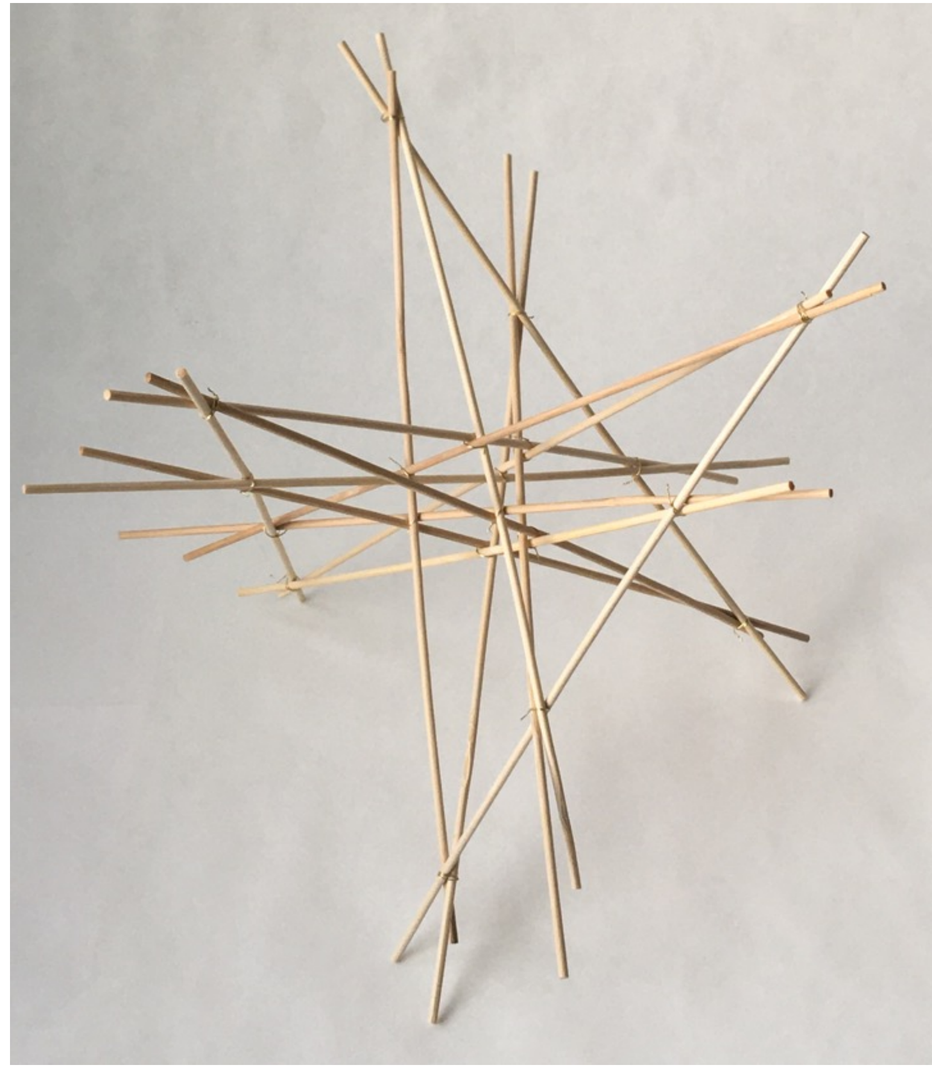
The complete hexahedron partitions the space into six tetrahedral cells, twelve triangular prisms, and two cuboids. There are also six cells bounded by two triangles, two quadrilaterals, and two pentagons each.

As the plane at infinity can positioned in several distinct fashions with respect to the six planes of the configuration, there exists different layouts for the stick figure.

What layout do you prefer? E.g. can you make a layout where there is an intact version of each cell type?

Stick model of the complete hexahedron

Nice layout for a planar portrayal?



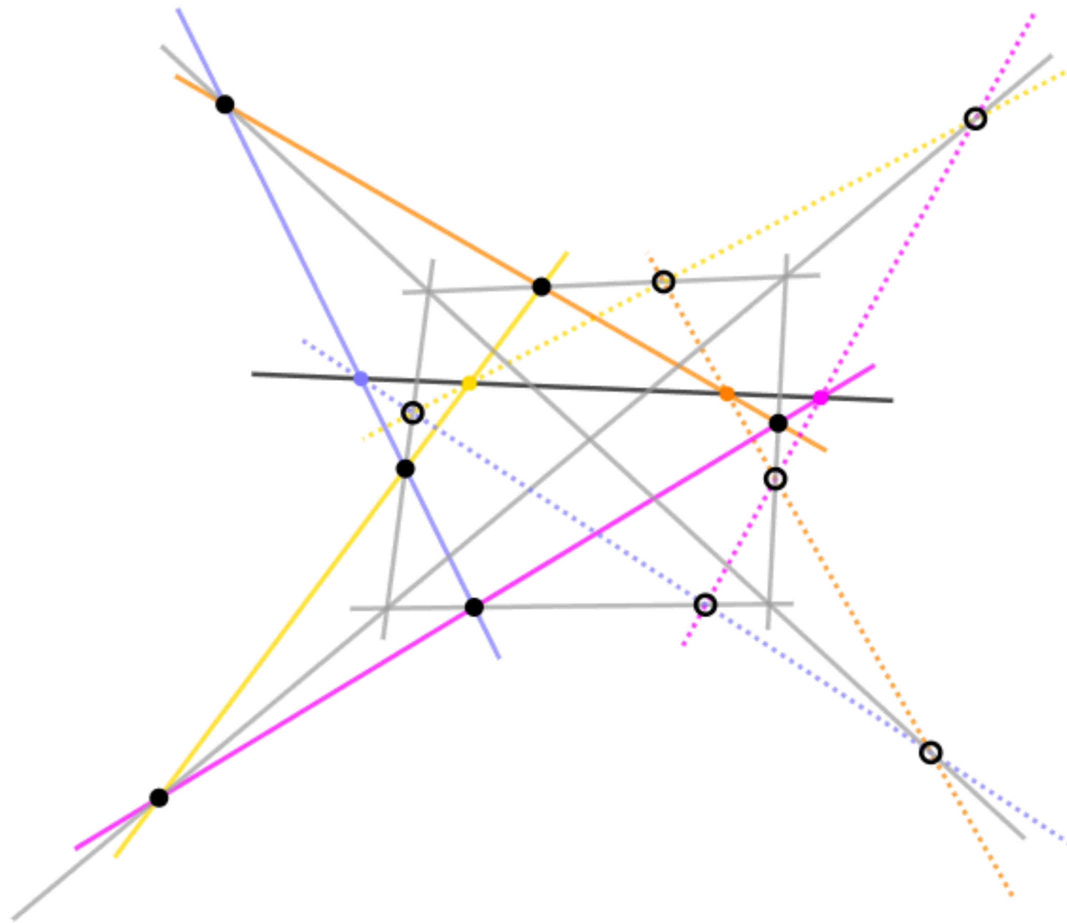
Maximum 3D symmetry?
(three-fold rotation)

The complete hexahedron embodies a theorem about two perspective quadrilaterals – a natural consequence of Desargues' theorem. It states:

> Two quadrilaterals are in perspective from a line
>
> (i.e., the intersection points of the corresponding sides are collinear)
>
> if and only if
>
> the lines determined by their corresponding vertices form a quadrangle.

By observing the situation in three-dimensional stick figure, the quadrilaterals can be thought of as cross sections of a complete tetrahedron, the axis of perspectivity being the intersection line of their planes.

Can you find an instance of the theorem for each of the lines of your configuration?



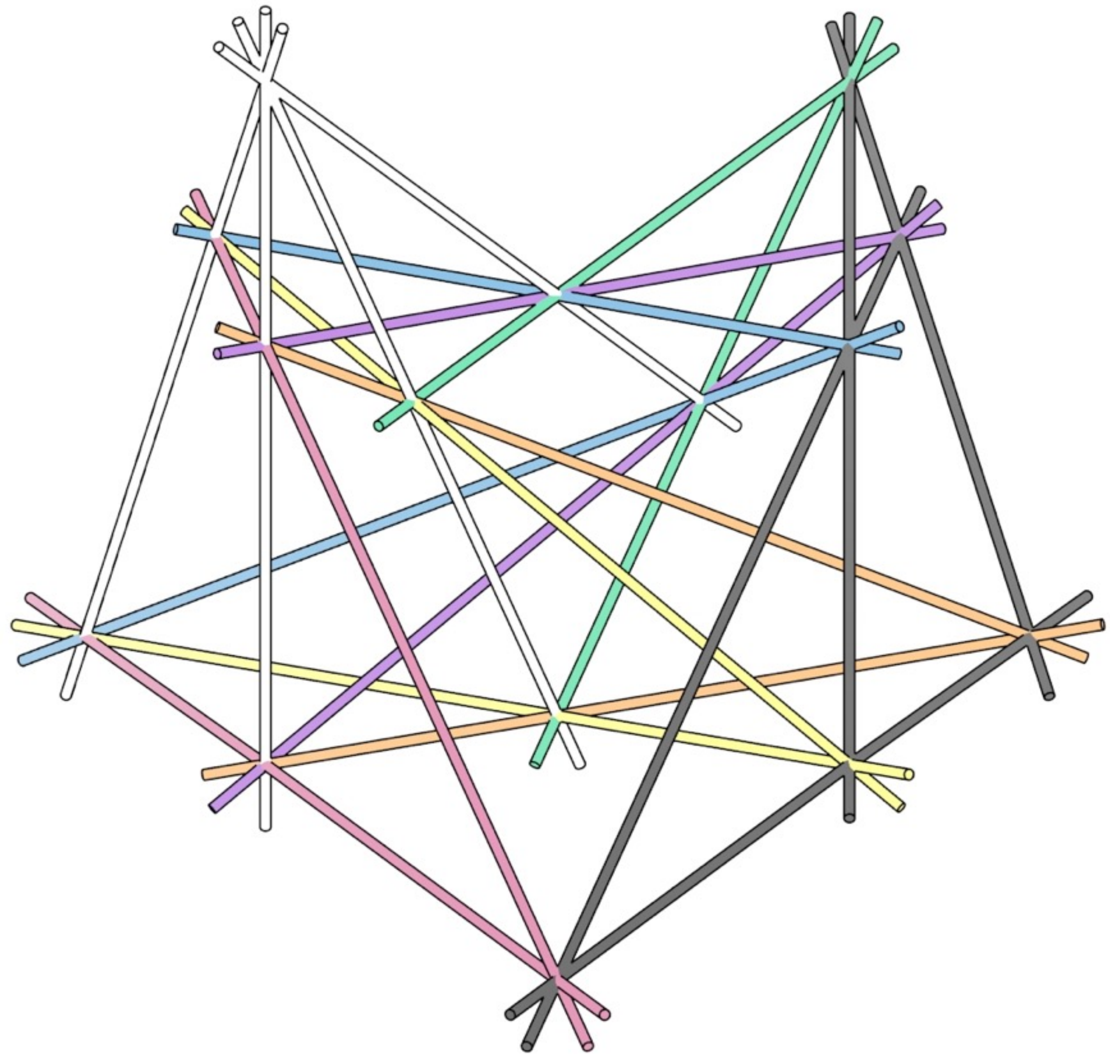
As mentioned previously, there exists a duality – a perfect correspondence between points and lines in the projective plane that extends also to any statements involving them.

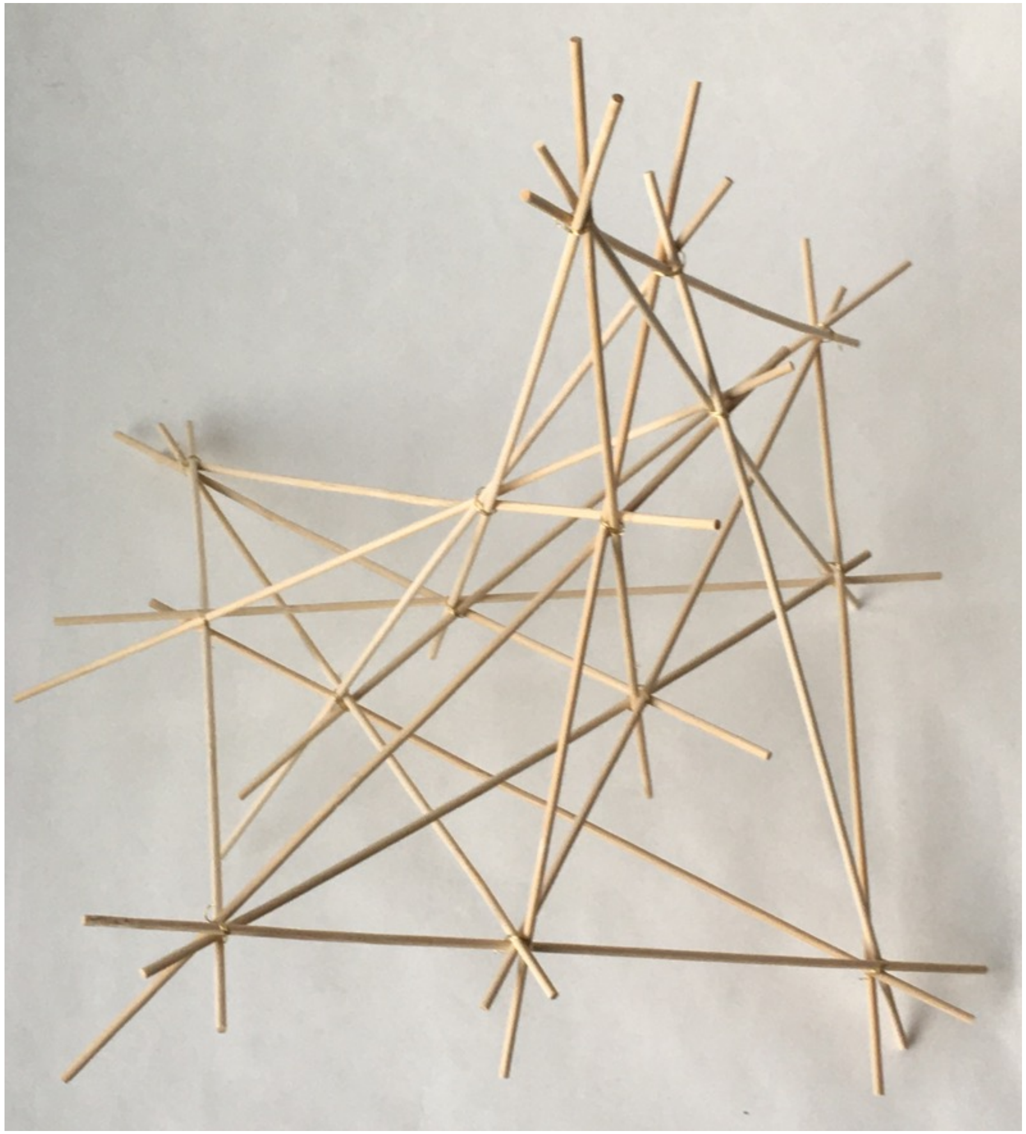
A plane-dual of the quadrilateral theorem above can thus be formulated as:

> two quadrangles are in perspective from a point
> if and only if
> the meets of their corresponding edges are the vertices of a quadrilateral.

PERSPECTIVITY BETWEEN TWO TETRAHEDRA

In three-dimensional projective space this fact becomes a generalization of the Desargues' theorem, stating that two tetrahedra are in perspective from a point (white) if and only if they are also in perspective from a plane (i.e., the intersection points of the corresponding edges are coplanar (black).





Stick model for
the ‘complete hexachoron’ configuration