

# Nonlinear dynamics & chaos

Fractals and Strange Attractors

Lecture X

# Recap

## One-directional maps in discrete time.

Logistic map  $x_{n+1} = rx_n(1 - x_n)$   $x_n \geq 0, r \geq 0$ .

**Routes to chaos:**  
intermittency, periodic  
doubling.

## Universality

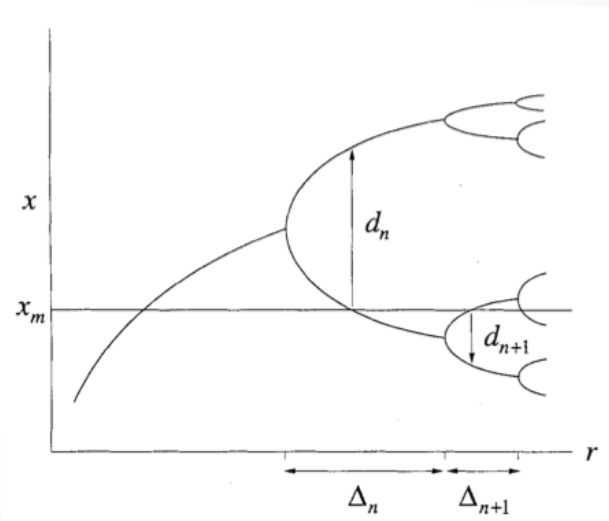
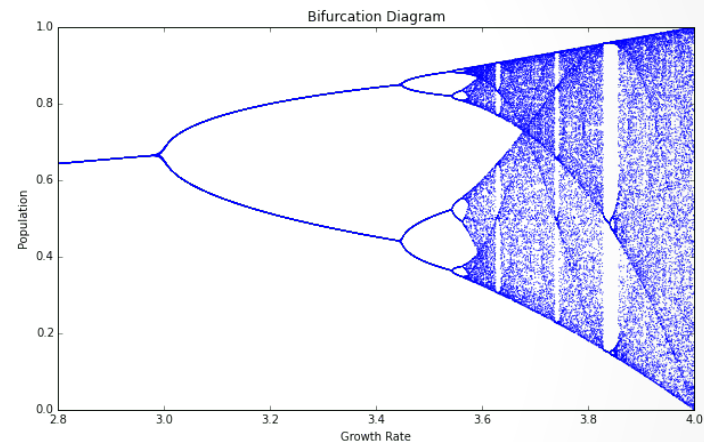
For example,

$$\frac{d_n}{d_{n+1}} \rightarrow \alpha \approx -2.5029$$

as  $n \rightarrow \infty$ ,

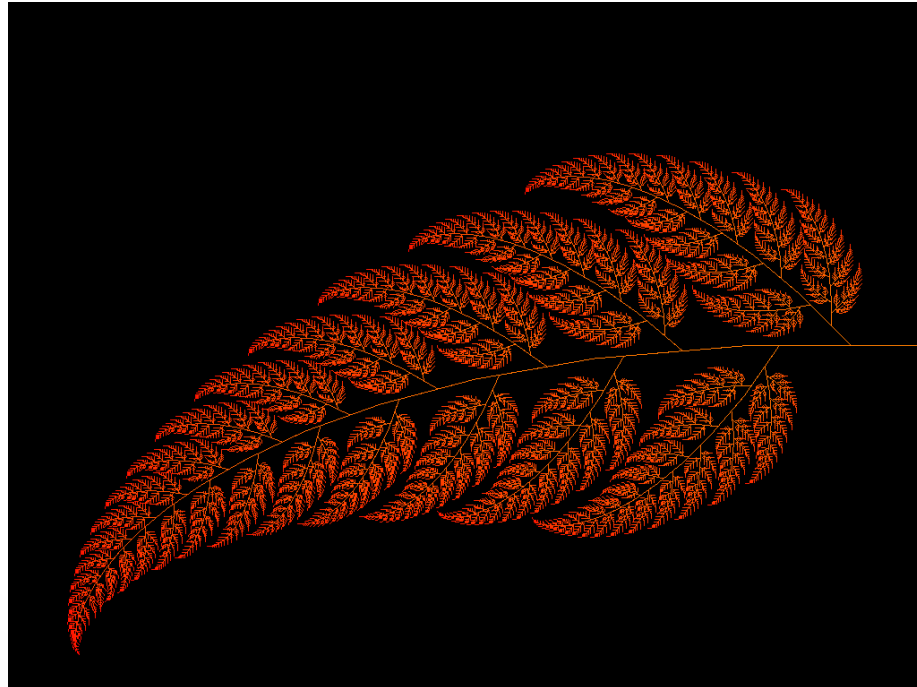
independent of the form of  $f$  in

$$x_{n+1} = f(x_n)$$



# Fractals and Strange Attractors

To hopefully understand the geometry of strange attractors a little better we take this crash course on fractals...



# Countable and Uncountable Sets

Are some infinities larger than others? Yes.

George Cantor invented a way to compare different infinite sets in the late 1800's.

Two sets  $X$  and  $Y$  are said to have the same *cardinality*, that is, the same number of elements if there is an invertible mapping that pairs each element  $x \in X$  with precisely one  $y \in Y$ . Such a mapping is called a *one-to-one correspondence*.

The set of natural numbers  $\mathbf{N} = \{1, 2, 3, 4, \dots\}$  is an infinite set.

If another set  $X$  can be put into one-to-one correspondence with  $\mathbf{N}$ , then  $X$  is said to be *countable*. Otherwise  $X$  is *uncountable*.

# Countable Sets

**Example 1.** The set of even numbers  $E = \{2, 4, 6, \dots\}$  is countable because the invertible mapping  $1 \leftrightarrow 2, 2 \leftrightarrow 4, 3 \leftrightarrow 6, \dots$  is one-to-one. So, there are exactly as many even numbers as natural numbers.

**An equivalent characterisation of countable sets:** A set  $X$  is countable if it can be written as a list  $\{x_1, x_2, x_3, \dots\}$ , with every  $x \in X$  appearing somewhere in the list. In other words, given any  $x$ , there is some finite  $n$  such that  $x_n = x$ . Recipe: find an algorithm that systematically counts the elements of  $X$ .

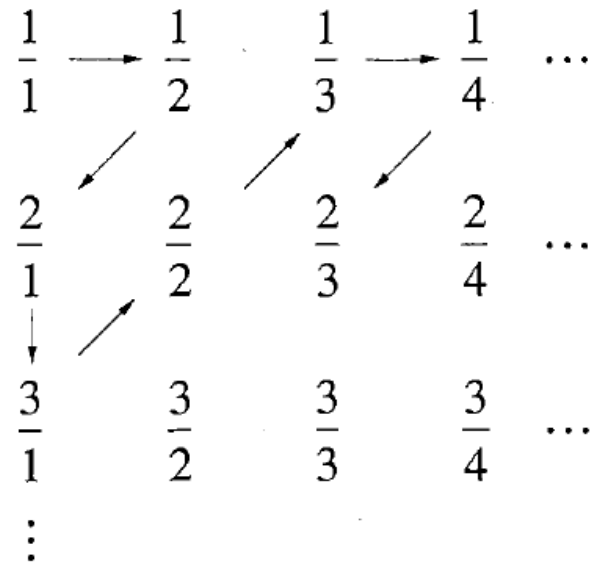
**Example 2.** Show that integers are countable.

*Solution:* An algorithm for listing all integers; the list is  $\{0, 1, -1, 2, -2, 3, -3, \dots\}$ . Any particular integer appears eventually, so the integers are countable.

# Countable Sets

**Example 3.** The positive rational numbers are countable: Make a table where the  $pq$ -th entry is  $p/q$ . Any given  $p/q$  is reached after a finite number of steps.

(Wrong way: By listing  $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots$  in order, we never get to, for example,  $\frac{2}{3}$ .)



# Uncountable Sets

**Example 1.** Let  $X$  denote the set of real numbers between 0 and 1. Show that  $X$  is uncountable.

*Solution:* Proof by contradiction. If  $X$  were countable, we could list all the numbers between 0 and 1 as a set  $\{x_1, x_2, x_3, \dots\}$ .

Rewrite these in decimal form:

$$x_1 = 0.x_{11}x_{12}x_{13} \dots$$

$$x_2 = 0.x_{21}x_{22}x_{23} \dots$$

$$x_3 = 0.x_{31}x_{32}x_{33} \dots$$

$\vdots$  Show that there's a number  $r$  that's not on the list.

**Diagonal argument** by Cantor: The first digit of  $r$  is *anything other than*  $x_{11}$ , the second digit is anything other than  $x_{22}$ . In general, the  $n$ th digit is  $\bar{x}_{nn}$ .

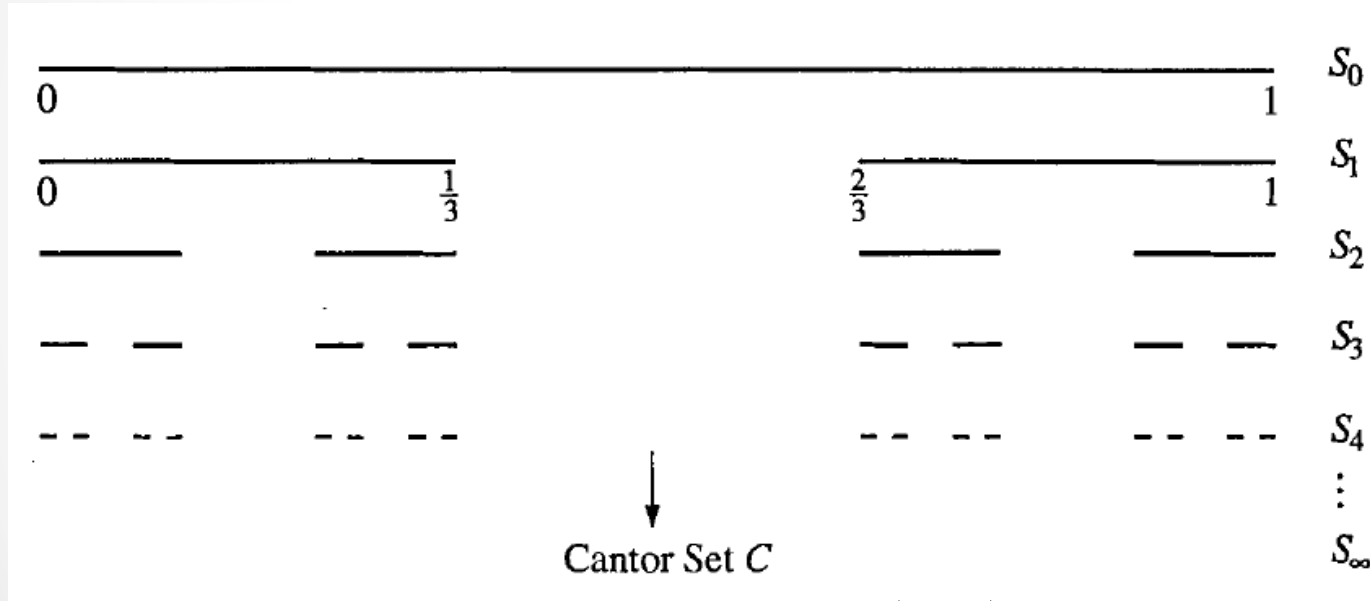
$\Rightarrow r = 0.\bar{x}_{11}\bar{x}_{22}\bar{x}_{33} \dots$  is not on the list  $\rightarrow r \neq x_1, r \neq x_2, \dots$

$\rightarrow X$  is uncountable.

# Cantor Set

The Cantor set is a simple fractal set, related to the geometry of strange attractors.

Construction of the Cantor set:



Start with  $S_0 = [0, 1]$  and remove  $(\frac{1}{3}, \frac{2}{3})$  leaving the endpoints.  $\rightarrow$  Closed intervals of  $S_1$ . Keep repeating  $\rightarrow$  **The limiting set  $C = S_\infty$**  is the Cantor set that consists of an infinite number of infinitesimal pieces separated by various-sized gaps.



# Fractal Properties of the Cantor Set

1.  $C$  has structure at arbitrarily small scales.
2.  $C$  is **self-similar**. In other words,  $C$  contains smaller copies of itself *at all scales*. (Take the left or right of  $C$  and enlarge it by three: the original  $C$  results. **Note:** More general fractals are only approximately self-similar.)
3. The dimension of  $C$  is not an integer. The dimension of  $C$  is  $\ln 2 / \ln 3 \approx 0.63$ .

# Other Properties of the Cantor Set

**$C$  has a measure zero.**

$C$  can be covered by intervals whose total length is arbitrarily small.

Each set  $S_n$  covers completely all the sets coming after it.

→  $C = S_\infty$  is covered by each of the sets  $S_n$ .

→ The total length of  $C$  is less than the total length of  $S_n$ .

The lengths of the sets:  $L_0 = 1, L_1 = \frac{2}{3}, L_2 = \left(\frac{2}{3}\right)^2, \dots, L_n = \left(\frac{2}{3}\right)^n$ .

The length of  $C$ :  $L_\infty = \lim_{n \rightarrow \infty} L_n = 0$ .

**$C$  is uncountable.**

This can be proven by Cantor's diagonal argument.

# Dimension of Self-Similar Fractals

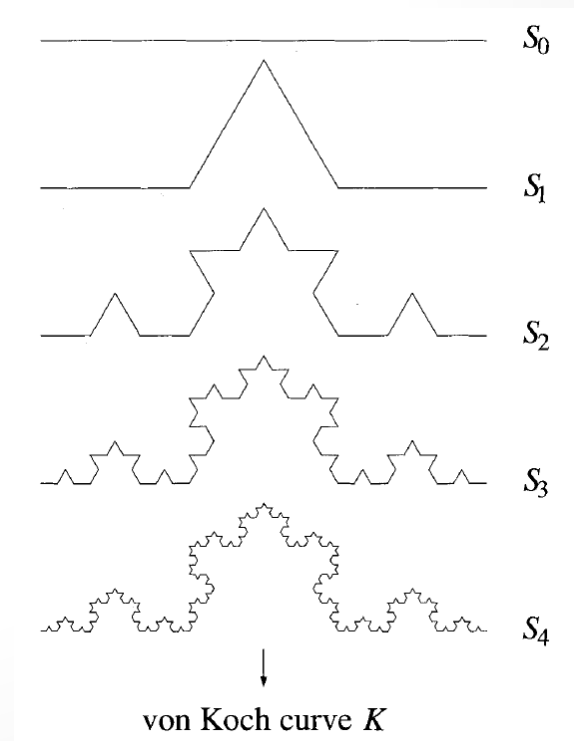
A **conventional intuitive idea of the dimension** is that it is the minimum number of coordinates needed to describe every point in the set. For example, every point on a smooth curve is determined by one number.

This idea of the dimension breaks down for fractal objects.

Consider the *von Koch curve*.

Start with a line segment  $S_0$ .

Generate  $S_1$  by deleting the middle third of  $S_0$  and replacing it with the other two sides of an equilateral triangle. Repeat recursively: the limiting set  $K = S_\infty$  is the Koch curve.



# Dimension of Self-Similar Fractals

The Koch curve  $K$  has infinite arc length.

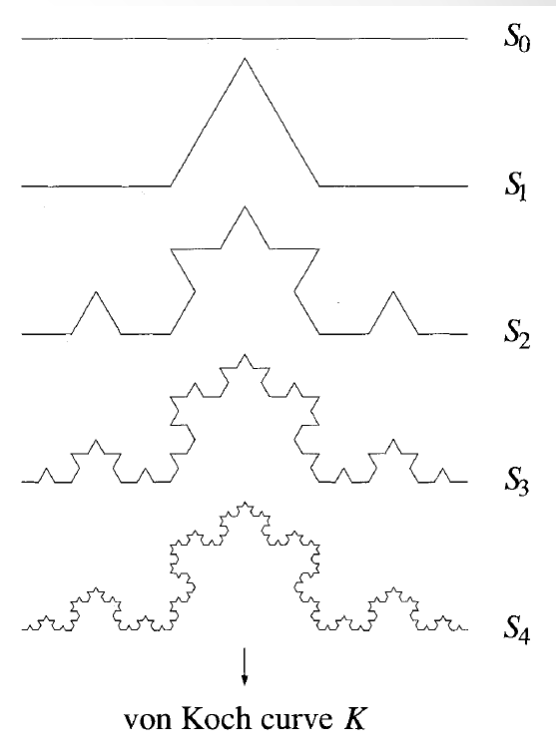
1. The length of  $S_0$  is  $L_0$ .

2.  $S_1$ :  $L_1 = \frac{4}{3}L_0$ .

3. Recursion:  $L_n = \left(\frac{4}{3}\right)^n L_0 \rightarrow \infty$  as  $n \rightarrow \infty$ .

By similar reasoning, the arc length between any two points on  $K$  is infinite. So, points on  $K$  are not determined by their arc length from a particular point; every point is infinitely far from each other!  $\rightarrow K$  is more than one-dimensional.

On the other hand,  $K$  does not have an area, so it is not two-dimensional either.  $\rightarrow 1 < d < 2$ .

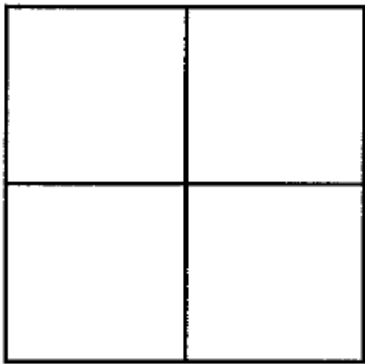


# Dimension of Self-Similar Fractals

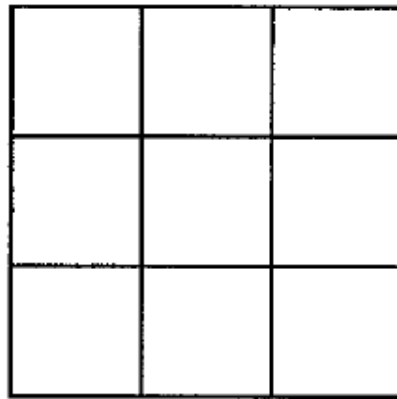
## Similarity Dimension

The simplest fractals are self-similar.

→ Define dimension by extending an elementary observation about classical self-similar sets like lines, squares, or cubes.



$$m = 4$$
$$r = 2$$



$$m = 9$$
$$r = 3$$

$m$  is the number of copies  
 $r$  is the scale factor

Shrink the square by a factor of 3 in each direction: it takes nine of the small squares to equal the whole.

Shrink the square by a factor of 2 in each direction: it takes four of the small squares to equal the whole.

Reduce the linear dimension by  $r$ : it takes  $r^2$  smaller squares to equal the original.

# Dimension of Self-Similar Fractals

In dimension 3: Shrink a cube by a factor of 2 in each direction: it takes eight of the small cubes to equal the whole.

Reduce the linear dimension by  $r$ : it takes  $r^3$  smaller cubes to equal the original.

**Definition:** Suppose that a self-similar set is composed of  $m$  copies of itself scaled down by a factor of  $r$ . Then the *similarity dimension*  $d$  is the exponent defined by  $m = r^d$ , or equivalently,

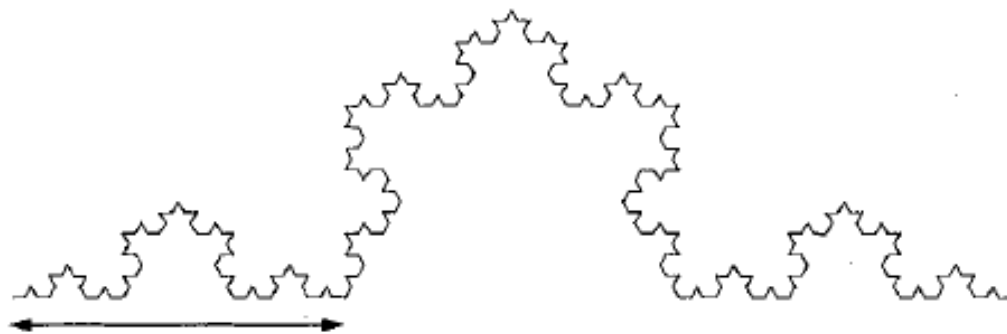
$$d = \frac{\ln m}{\ln r}.$$

# Dimension of Self-Similar Fractals

**Example 1.** Find the similarity dimension of the Cantor set  $C$ .

*Solution:*  $C$  is composed of two copies of itself, each scaled down by a factor of 3. So  $m = 2$  when  $r = 3$ . Therefore,  $d = \ln 2 / \ln 3 \approx 0.63$ .

**Example 2.** von Koch curve has a similarity dimension of  $d = \ln 4 / \ln 3 \approx 1.26$ , because the curve is made up of four equal pieces, each of which is similar to the original but scaled down by a factor of 3 in both directions. So,  $m = 4$ , when  $r = 3$ .



**Figure 11.3.4**

# Dimension of Self-Similar Fractals

## More General Cantor Sets

To obtain a new kind of Cantor set, divide an interval into some other odd number of equal pieces, delete every other piece and repeat.

A closed set is called a *topologically Cantor set* if it satisfies:

1.  $S$  is totally disconnected, i.e.,  $S$  contains no connected subsets other than single points. So, all points are separated from each other.
2.  $S$  contains no isolated points, i.e., every point in  $S$  has a neighbour arbitrarily close by: given any point  $p \in S$  within a distance  $\varepsilon > 0$ , there is some other point  $q \in S$  within a distance  $\varepsilon$  of  $p$ .

The definition says nothing about self-similarity, which is a geometric rather than a topological property. Cross sections of strange attractors are often topological Cantor sets, which are not necessarily self-similar.

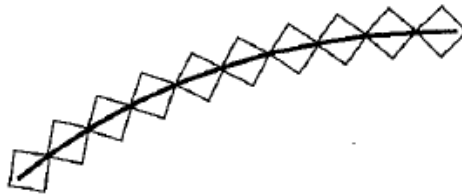


# Box Dimension

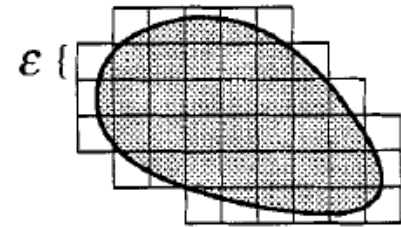
We generalise the notion of dimension to deal with fractals that are not self-similar. There are various definitions, all sharing the idea of “measurement at scale  $\varepsilon$ ”.

## “Definition”.

“Measurement”:  
cover the set  
with boxes of  
size  $\varepsilon$ .



$$N(\varepsilon) \propto \frac{L}{\varepsilon}$$



$$N(\varepsilon) \propto \frac{A}{\varepsilon^2}$$

Let  $S$  be a subset of  $D$ -dimensional Euclidian space, and let  $N(\varepsilon)$  be the minimum number of  $D$ -dimensional cubes of side  $\varepsilon$  needed to cover  $S$ . How does  $N(\varepsilon)$  depend on  $\varepsilon$ ?

A smooth curve of length  $L$ ,  $N(\varepsilon) \propto L/\varepsilon$ .

A planar region of area bounded by a smooth curve  $L$ ,  $N(\varepsilon) \propto A/\varepsilon^2$ .

# Box Dimension

$\Rightarrow N(\varepsilon) \propto 1/\varepsilon^d$ . This power law holds also for most fractal sets  $S$ , where  $d$  is not an integer.

**Definition of box dimension**  $d = \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}$ , if the limit exists.

**Example 1.** Find the box dimension of the Cantor set.

*Solution:* The Cantor set is covered by each of the sets  $S_n$  used in its construction. Each  $S_n$  consists of  $2^n$  intervals of length  $(1/3)^n$ . If we pick  $\varepsilon = (1/3)^n$ , we need all  $2^n$  intervals to cover  $C$ . Hence,  $N = 2^n$  when  $\varepsilon = (1/3)^n$ .  $\varepsilon \rightarrow 0$  as  $n \rightarrow \infty$ :

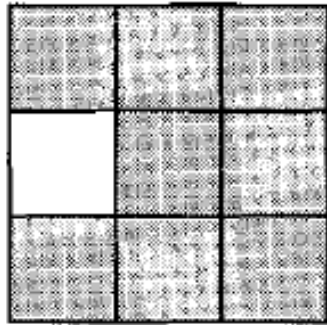
$$d = \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = \frac{n \ln 2}{n \ln 3} = \frac{\ln 2}{\ln 3}$$

This agrees with the similarity dimension.

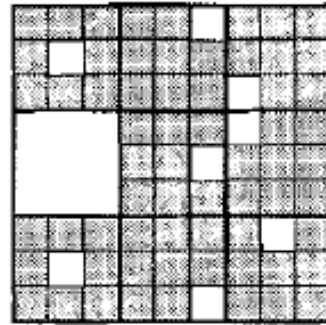
**The trick:** discrete sequence  $\varepsilon = (1/3)^n$  that tends to zero as  $n \rightarrow \infty$  in spite of the definition saying that we should let  $\varepsilon \rightarrow 0$  continuously. The limiting value does not change due to this.

# Box Dimension

**Example 2.** A fractal that is not self-similar constructed by recursion: divide a square region into nine equal squares, after which one of the small squares is selected at random and discarded.



$S_1$



$S_2$

Pick the unit of length equal to the side of the original square:  $S_1$  is covered by  $N = 8$  squares of side  $\varepsilon = (1/3)^2$ . Generally,  $N = 8^n$  when  $\varepsilon = (1/3)^n$ . Hence,

$$d = \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = \frac{n \ln 8}{n \ln 3} = \frac{\ln 8}{\ln 3}$$

# Pointwise and Correlation Dimensions

How to estimate the fractal dimension of a strange attractor?

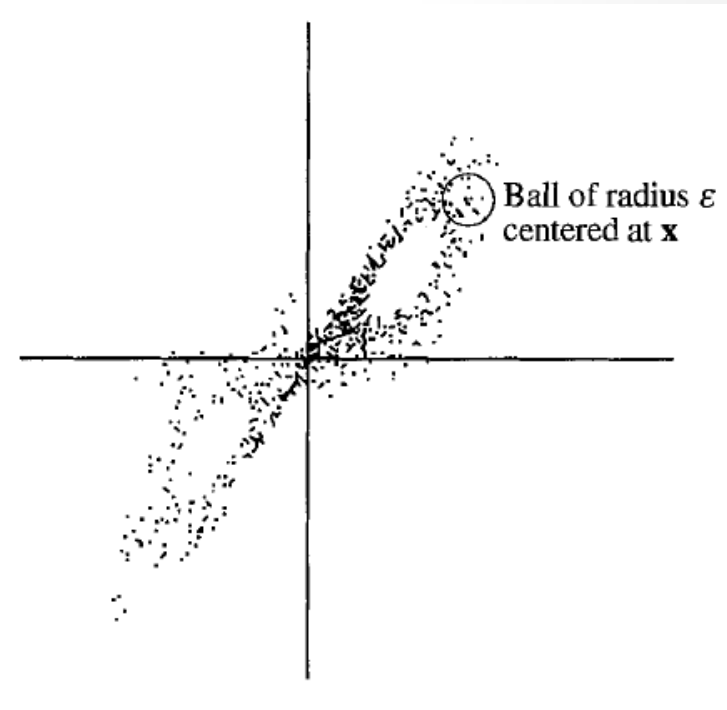
Generate a set of very many points  $\{\mathbf{x}_i, i = 1, \dots, n\}$  on the attractor.

Computing box dimension is impractical. Grassberger & Procaccia introduced a more efficient approach:

Fix a point  $\mathbf{x}$  on the attractor  $A$ .

Let  $N(\varepsilon)$  denote the number of points on  $A$  inside a ball of radius  $\varepsilon$  about  $\mathbf{x}$ .

Most of the points in the ball are unrelated to the immediate portion of the trajectory through  $\mathbf{x}$ , but come from later parts that happen to pass close to  $\mathbf{x}$ .  $\rightarrow N(\varepsilon)$  measures how frequently a typical trajectory visits an  $\varepsilon$  neighbourhood of  $\mathbf{x}$ .



# Pointwise and Correlation Dimensions

Vary  $\varepsilon$ . As  $\varepsilon$  increases the number of points in the ball typically grows as

$$N_x(\varepsilon) \propto \varepsilon^d,$$

where  $d$  is the *pointwise dimension* at  $\mathbf{x}$ .

The pointwise dimension can depend significantly on  $\mathbf{x}$ .  $\rightarrow$   
Average  $N_x(\varepsilon)$  over many  $\mathbf{x}$  to obtain an overall dimension of  $A$ :

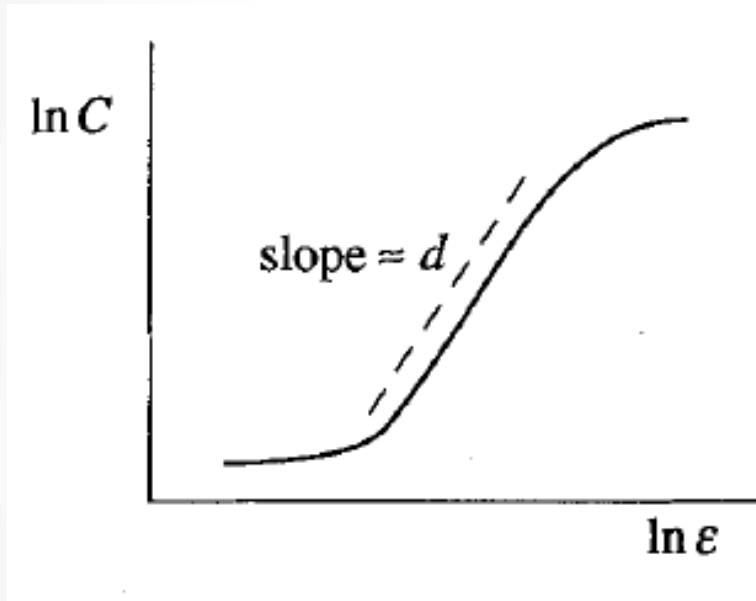
$$C(\varepsilon) \propto \varepsilon^d,$$

where  $d$  is called the *correlation dimension*.

The correlation dimension takes account of the density of points on the attractor, whereas the box dimension weights all occupied boxes equally. In general,  $d_{\text{correlation}} \leq d_{\text{box}}$  (usually very close).

# Pointwise and Correlation Dimensions

To estimate  $d$ , plot  $\log C(\varepsilon)$  vs.  $\log \varepsilon$ .



Saturation: at large  $\varepsilon$ , the  $\varepsilon$ -balls engulf the whole attractor and  $N_x(\varepsilon)$  can grow no further.

Minimum: extremely small  $\varepsilon$ -balls include only  $\mathbf{x}$ .

In the scaling region it holds:

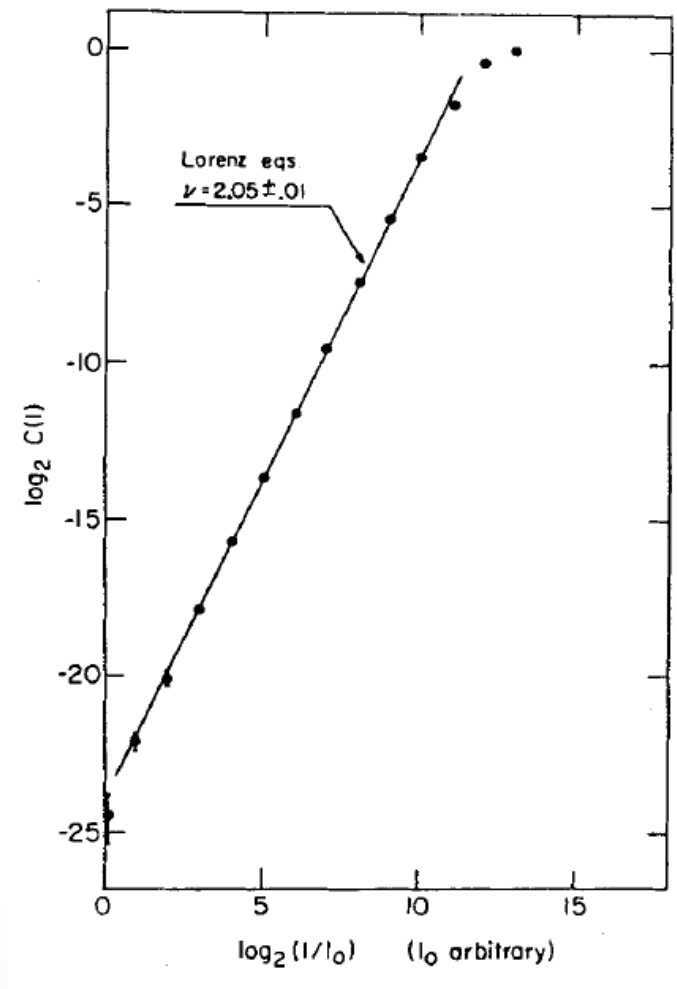
(minimum separation of points on  $A$ )  $\ll \varepsilon \ll$  (diameter of  $A$ )

# Pointwise and Correlation Dimensions

Grassberger & Procaccia estimated the correlation dimension of the **Lorenz attractor** for the standard parameter values  $r = 28$ ,  $\sigma = 10$ ,  $b = 8/3$ .

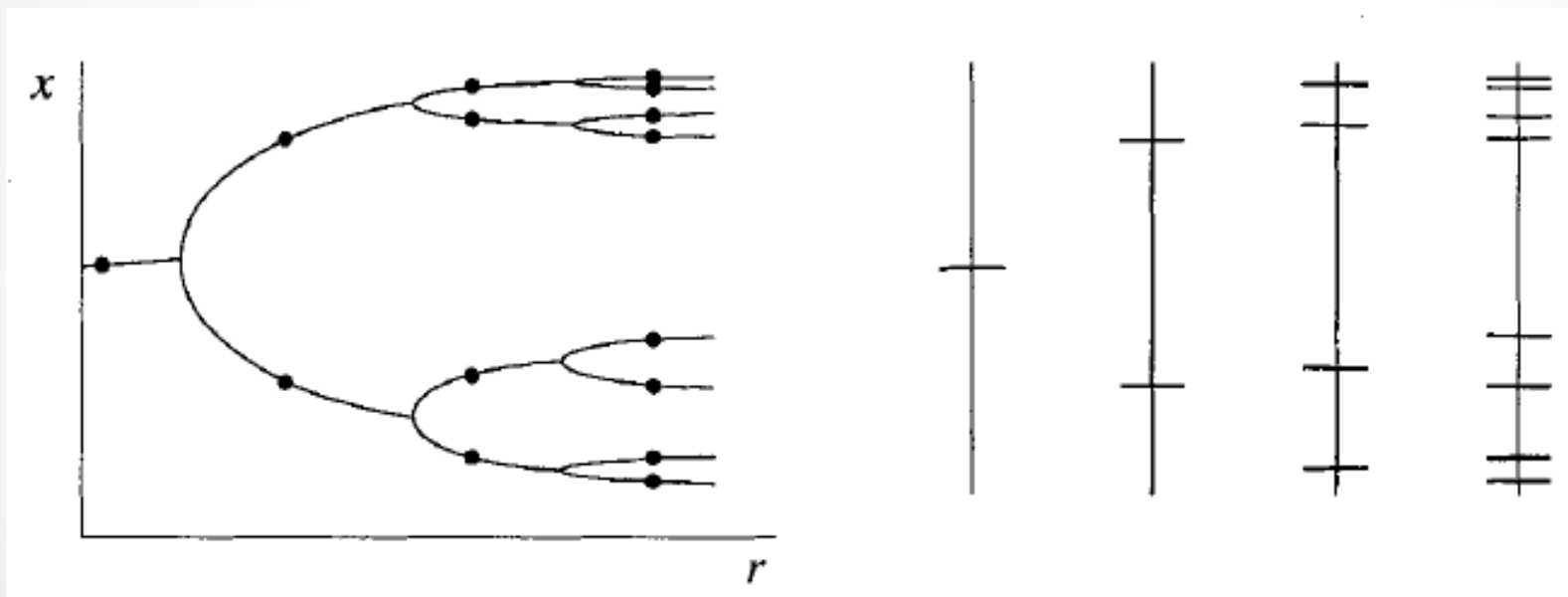
Runge-Kutta, time step 0.25,  
15,000 points:

$$d_{\text{corr}} = 2.05 \pm 0.01$$



# Pointwise and Correlation Dimensions

For the **logistic map**  $x_{n+1} = rx_n(1 - x_n)$  at the parameter value  $r = r_\infty = 3.5699456 \dots$  corresponding to the onset of chaos, the attractor is a Cantor-like set, although it is not strictly self-similar.



The attractor looks like a  $2^n$ -cycle for  $n \gg 1$ .

For the small  $n$  in the figure, the right panel shows  $x$  for  $2^n$ -cycles. As  $n \rightarrow \infty$ , this set approaches a topological Cantor set that is not strictly self-similar.

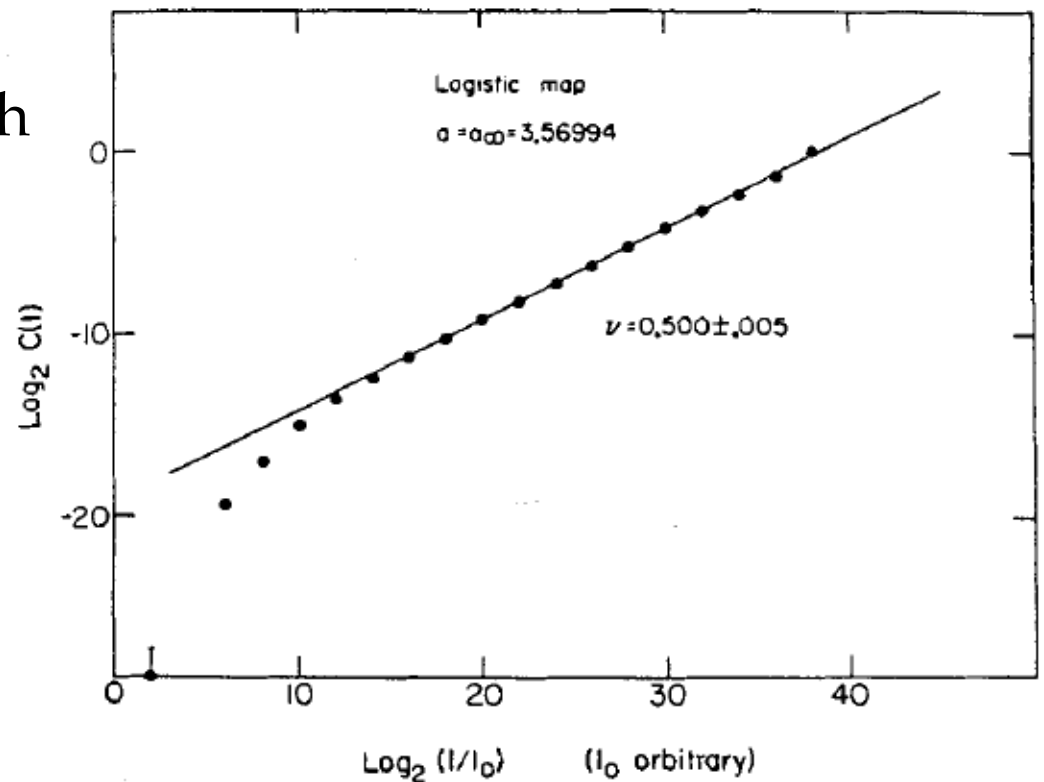


# Pointwise and Correlation Dimensions

The correlation dimension of the limiting set of the logistic map was estimated by Grassberger and Procaccia.

Trajectory of 30,000 points, starting from  $x_0 = 1/2$ :  
 $d_{\text{corr}} = 0.500 \pm 0.005$ , which is smaller than the box dimension  $d_{\text{box}} = 0.538$ .  
( $d_{\text{corr}} < d_{\text{box}}$ , because  $d_{\text{corr}}$  takes account of the density of points.)

Note: In an attractor there may be local variations in the scaling. They are called *multifractals*.  $d$  is quantified by a multifractal spectrum.



# Strange Attractors

Equipped with some understanding on maps and fractals we try to acquire some intuition on chaotic dynamics and start with

## Simple Examples

How can trajectories on the attractor remain confined to a bounded region of phase space and yet separate from their neighbours exponentially fast (initially...).

The basic mechanism involves repeated *stretching and folding*.



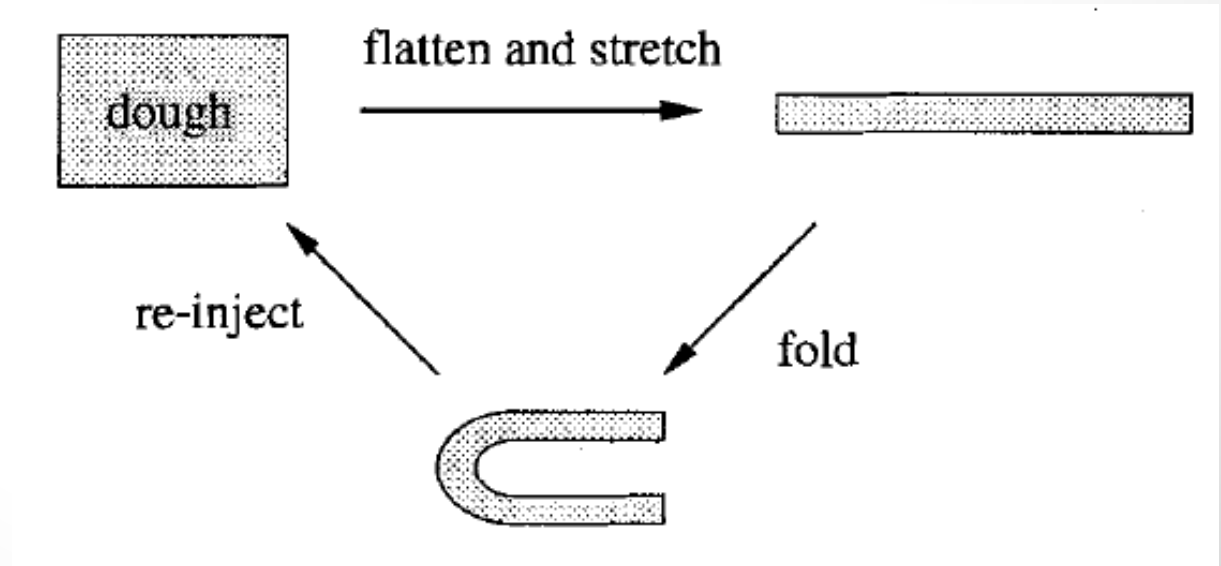
# Pastry map

Flow contracts in a blob in some directions (dissipation) and stretches in others → a strange attractor.

Stretching leads to sensitive dependence on initial conditions.

Stretching cannot go on forever → folding.

## Making pastry



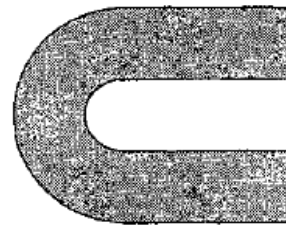
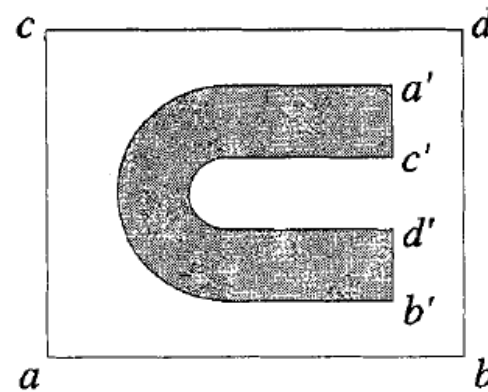
# Pastry map

“Pastry map”:

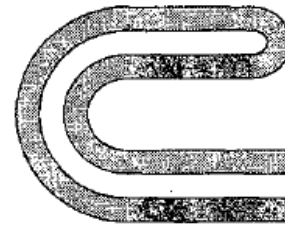
Flatten, stretch, and fold rectangle  $abcd$  into the horseshoe  $S_1$ .

Flatten, stretch, and fold  $S_1$  into  $S_2$  etc.

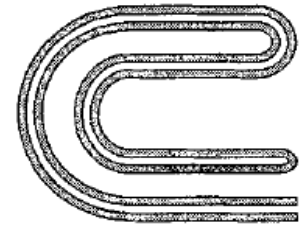
Proceeding this way, the layers become thinner and the number of them doubles at each stage.



$S_1$



$S_2$



$S_3$

The limiting set  $S_\infty$  consists of infinitely many smooth layers, separated by gaps of various sizes: A vertical cross section through the middle resembles a Cantor set.

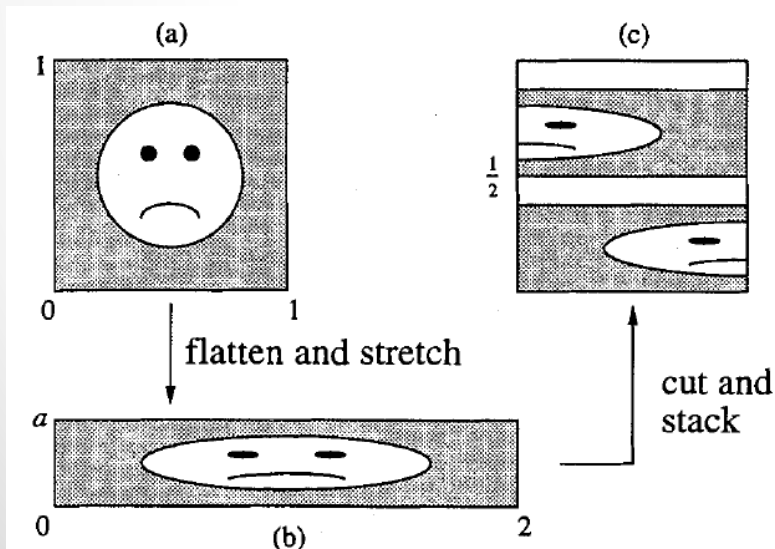
$S_\infty$  is an attractor with fractal structure.

# The baker's map

**Example 1.** The *baker's map*  $B$  of the square  $0 \leq x \leq 1, 0 \leq y \leq 1$  to itself is given by

$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & 0 \leq x_n \leq \frac{1}{2} \\ (2x_n - 1, ay_n + \frac{1}{2}) & \frac{1}{2} \leq x_n \leq 1 \end{cases}$$

where  $a$  is a parameter in the range  $0 \leq a \leq \frac{1}{2}$ .



The map is a product of two transformations: (a)  $\rightarrow$  (b) stretch & flatten the square into  $2 \times a$  rectangle, (b)  $\rightarrow$  (c) cut the rectangle in half and stack the one  $1 \times a$  rectangle on top of the other at  $y = 1/2$ .

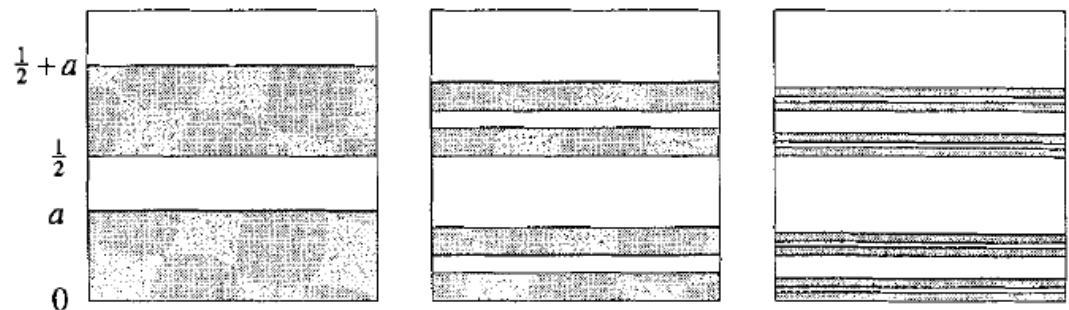
# The baker's map

Thanks to the stretching in the  $x$ -direction, the baker's map exhibits sensitive dependence on initial conditions. It has uncountably many chaotic orbits.

**Example 2.** Show that for  $a < \frac{1}{2}$ , the baker's map has a fractal attractor  $A$  that attracts all orbits. More precisely, show that there is a set  $A$  such that for any initial condition  $(x_0, y_0)$ , the distance from  $B^n(x_0, y_0)$  to  $A$  converges to zero as  $n \rightarrow \infty$ .

*Solution.* Construct the attractor: Let  $S_0$  denote the square  $0 \leq x \leq 1, 0 \leq y \leq 1$ , which includes all possible  $(x_0, y_0)$ .

$B^n(S)$  consists of  $2^n$  horizontal strips of height  $a^n$ .



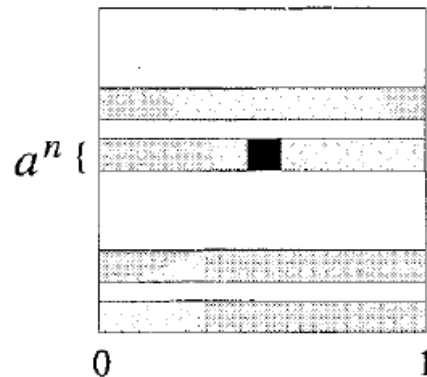
The limiting set  $A = B^\infty(S)$

is a fractal and topologically a Cantor set of line segments.

# The baker's map

**Example 3.** Find the box dimension of the attractor for the baker's map with  $a \leq \frac{1}{2}$ .

*Solution.* The attractor is approximated by  $B^n(S)$  consisting of  $2^n$  strips of height  $a^n$  and length 1. Cover  $A$  with square boxes of side  $\varepsilon = a^n$ .



It takes about  $a^{-n}$  boxes to cover each of the strips of length 1.

There are  $2^n$  strips altogether  $\rightarrow N \approx a^{-n} \times 2^n = (a/2)^{-n}$ .

$$d = \lim_{\varepsilon \rightarrow 0} \frac{\ln N}{\ln 1/\varepsilon} = \lim_{n \rightarrow 0} \frac{\ln[(a/2)^{-n}]}{\ln(a^{-n})} = 1 + \frac{\ln(1/2)}{\ln a} \quad (d \rightarrow 2 \text{ as } a \rightarrow 1/2)$$

# The baker's map

## The importance of dissipation

The baker's map shrinks areas in phase space for  $a < 1/2$ .

For any region  $R$  in the square,  $\text{area}(B(R)) = 2a \times \text{area}(R) < \text{area}(R)$  (stretching by 2 + flattening by  $a$ ).

This area contraction is analogous to the volume contraction found for Lorenz equations.

- (1) The attractor  $A$  for the baker's map must have a zero area.
- (2) The baker's map cannot have repelling fixed points.

As we've seen, area and volume contraction in phase space is due to dissipation. **Without dissipation there are no attractors.**



# Hénon Map

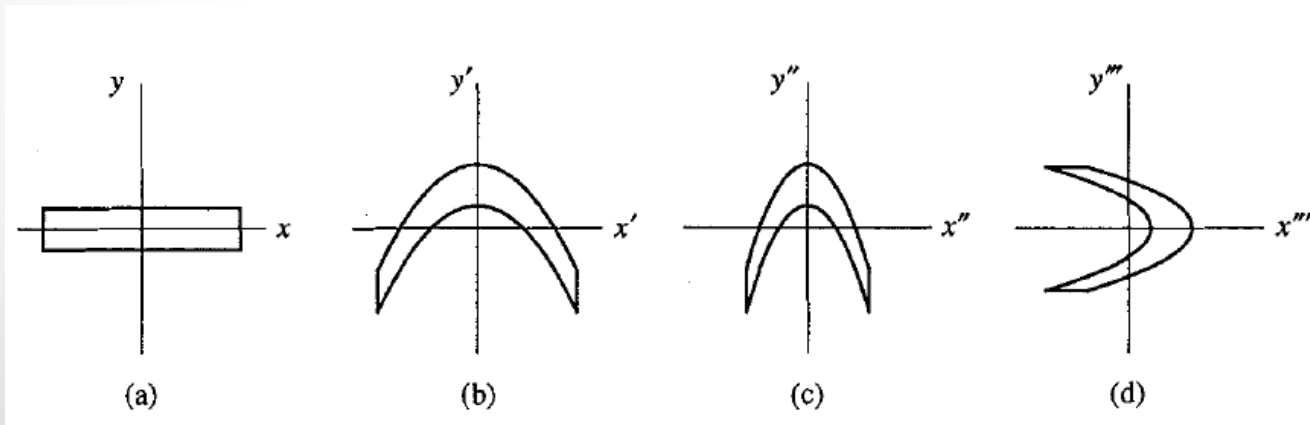
Michel Hénon introduced this map to study Lorenz system.

The *Hénon map*

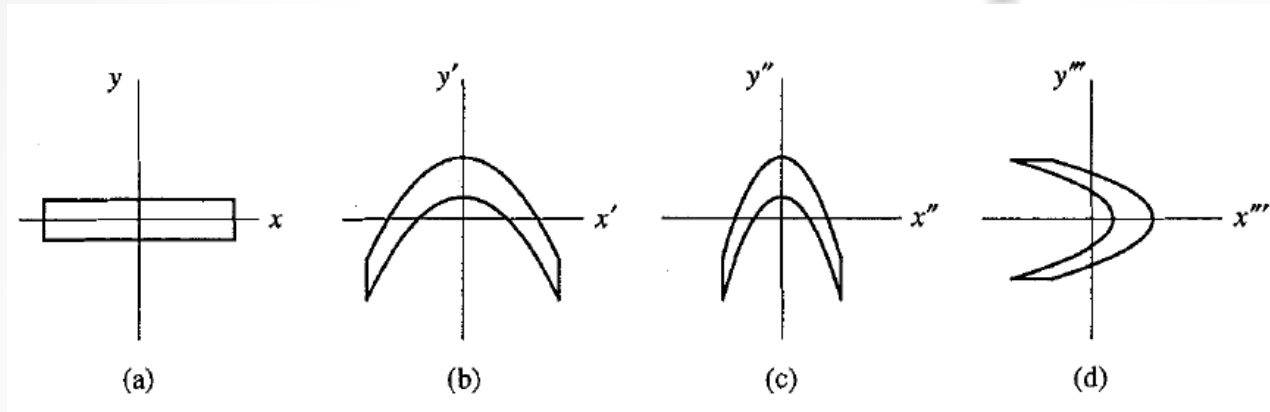
$$x_{n+1} = y_n + 1 - ax_n^2$$
$$y_{n+1} = bx_n,$$

where  $a$  and  $b$  are adjustable parameters. The amount of dissipation can be adjusted via  $b$ .

The equivalent chain of transformations:



# Hénon Map



(a)-(b) Stretch & fold  $T': x' = x, y' = 1 + y - ax^2$ .  
(Parameter  $a$  controls the folding.)

(b)-(c) Fold further by contracting along the  $x$ -axis  
 $T'': x'' = bx', y'' = y'$ , where  $-1 < b < 1$ .

(c)-(d) Rotate to orient the area along the  $x$ -axis  
 $T''': x''' = y'', y''' = x''$ .

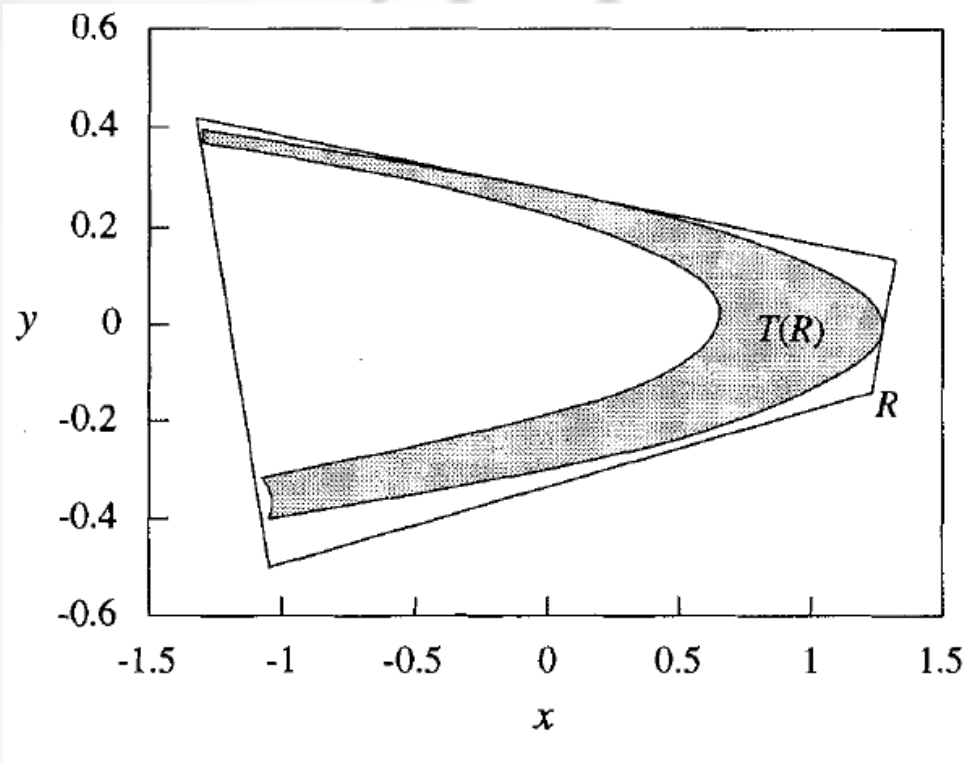
→ The composite transformation  $T = T'''T''T'$  yields the Hénon mapping (notation:  $(x_n, y_n) = (x, y)$  and  $(x_{n+1}, y_{n+1}) = (x''', y''')$ ).

# Elementary properties of the Hénon map

The Hénon map captures several essential properties of the Lorenz system.

1. The Hénon map is invertible. Consequently, trajectories are unique – no crossings.
2. The Hénon map is dissipative. It contracts areas at the same rate everywhere in phase space. (Analogous to constant negative divergence in the Lorenz system.)
3. For certain parameter values, the Hénon map has a trapping region, that is, there is a region  $R$  that gets mapped inside itself. (As in the Lorenz system, the strange attractor is enclosed in the trapping region.)

# Elementary properties of the Hénon map



4. Some trajectories of the Hénon map escape to infinity. (In contrast, all trajectories in the Lorenz system are bounded; they all eventually enter and stay inside a certain large ellipsoid.)

# Elementary properties of the Hénon map

**Example 1.** Show that the Hénon map  $T$  is invertible if  $b \neq 0$  and find the inverse  $T^{-1}$ .

*Solution.* We solve  $x_{n+1} = y_n + 1 - ax_n^2$  for  $x_n$  and  $y_n$ .

$$y_{n+1} = bx_n,$$

$\Rightarrow x_n = b^{-1}y_{n+1}, y_n = x_{n+1} - 1 + ab^{-2}(y_{n+1})^2$ . Thus  $\exists T^{-1} \forall b \neq 0$ . ■

# Elementary properties of the Hénon map

**Example 2.** Show that Hénon map contracts areas if  $-1 < b < 1$ .

*Solution:* Compute the Jacobian

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

If  $|\det \mathbf{J}| < 1 \forall (x, y)$ , the map is area-contracting.

For the Hénon map,  $f(x, y) = 1 - ax^2 + y$  and  $g(x, y) = bx$ . Hence,

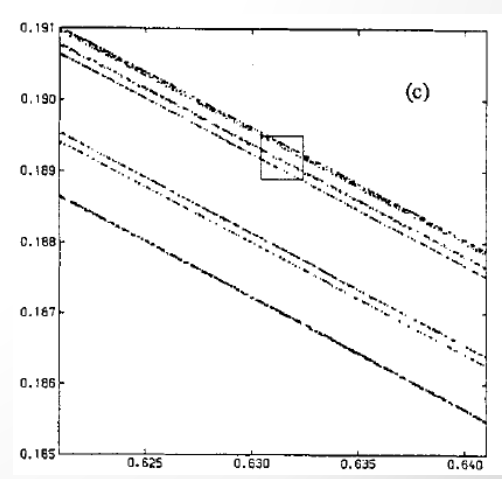
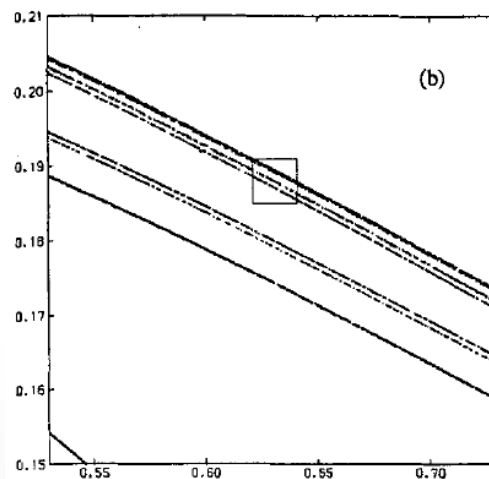
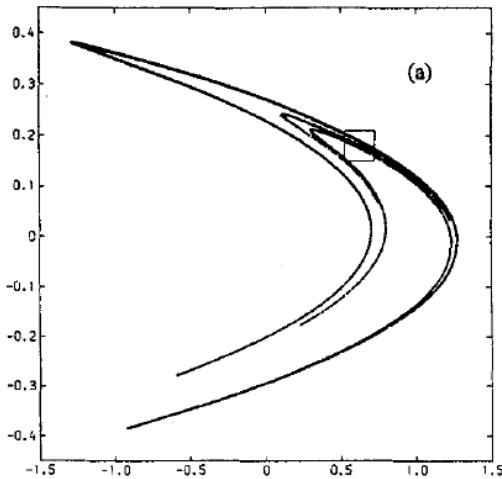
$$\mathbf{J} = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

$|\det \mathbf{J}| = -b \forall (x, y)$ . So, the map is area-contracting for  $-1 < b < 1$ . ■

# Zooming in on a Strange Attractor

Hénon chose parameter values  $a = 1.4$  and  $b = 0.3$  to have the system well into the chaotic region. He generated the attractor by computing ten thousand successive iterates starting from the origin. The points  $(x_n, y_n)$  hop around erratically, soon the attractor begins to take form. → This was **the first visualisation of the fractal structure of a strange attractor**.

Self-similarity: six parallel curves, a lone curve, two closely spaced curves above, and three more curves above them. The pattern repeats at an arbitrarily small scale.



# Rössler System

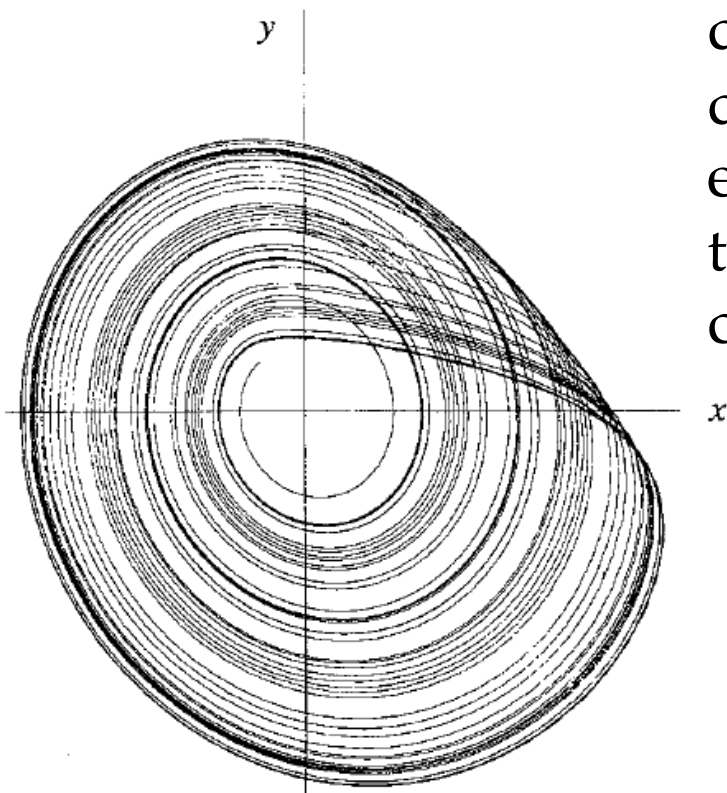
$$\dot{x} = -y - z$$

$$\dot{y} = x + ay$$

$$\dot{z} = b + z(x - c).$$

Numerical integration on the system with  $a = b = 0.2, c = 5.7$ : the system has a strange attractor.

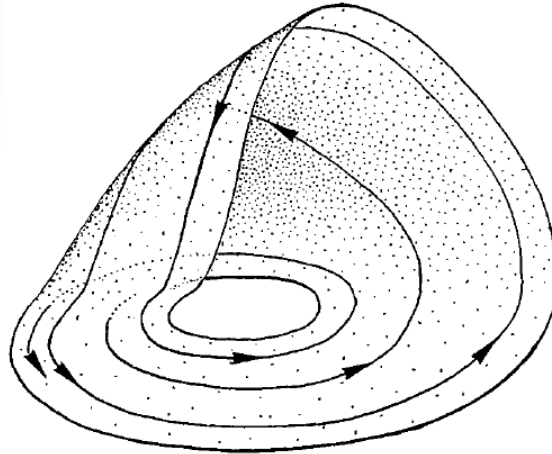
**Note:** In the previous mappings chaotic dynamic could be seen for discrete dynamics in 2D. Flow (differential equations) can be chaotic only in 3D: the wildly moving trajectories do not cross.



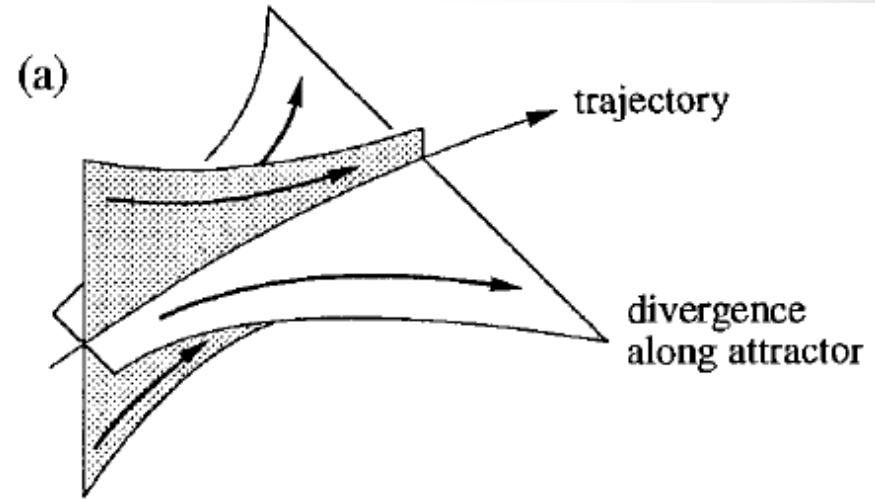


# Rössler System

(In fact trajectories “fold” onto another surface close to the previous one.)

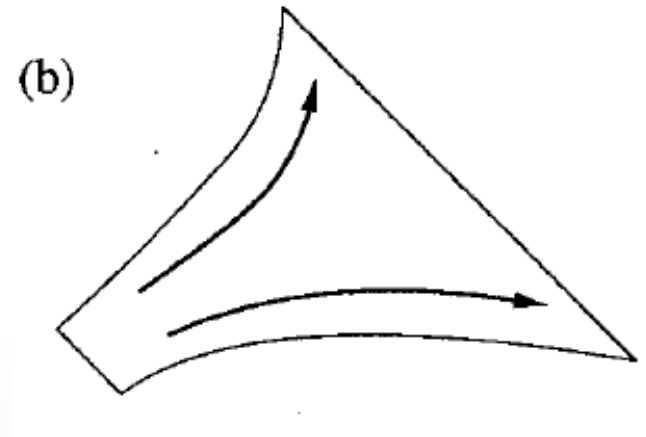


Flow near a typical trajectory



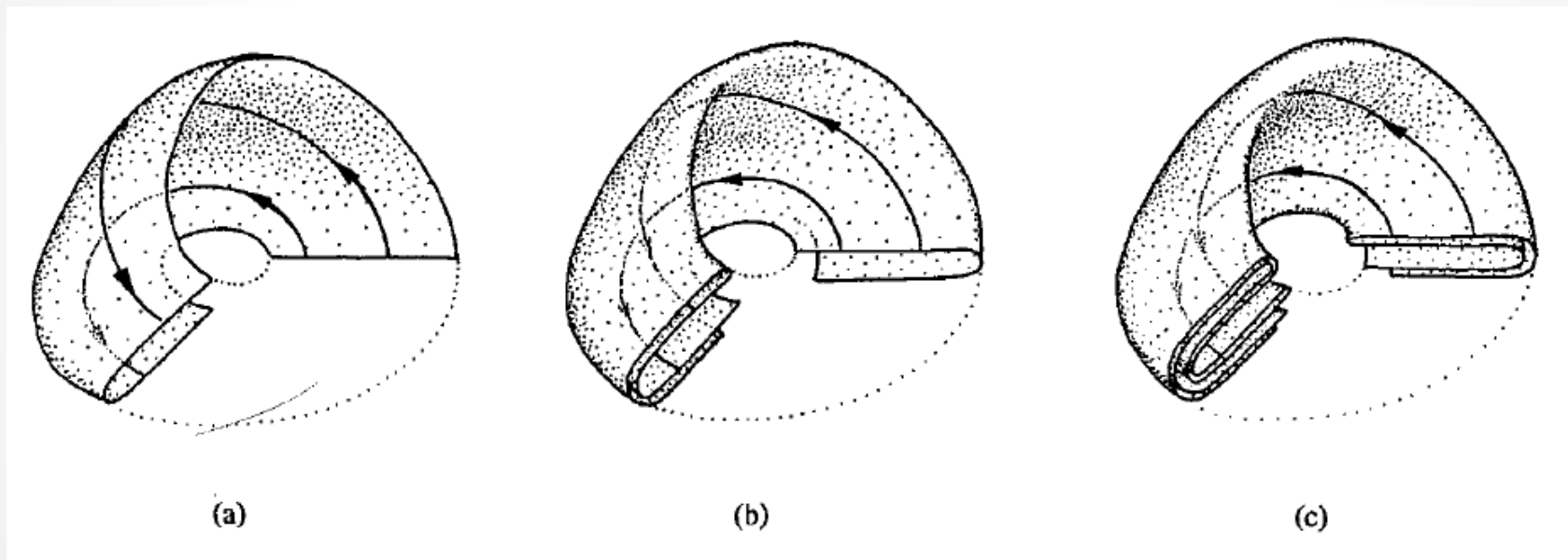
There's *compression* toward the attractor in one direction  
*divergence* along the attractor in the other direction.

The sheet on which there's sensitive dependence on initial conditions →



# Rössler System

(a) The flow takes a single sheet and produces two sheets after one circuit. (b) The process repeats and the two sheets produce four sheets. (c) These four sheets produce eight sheets.



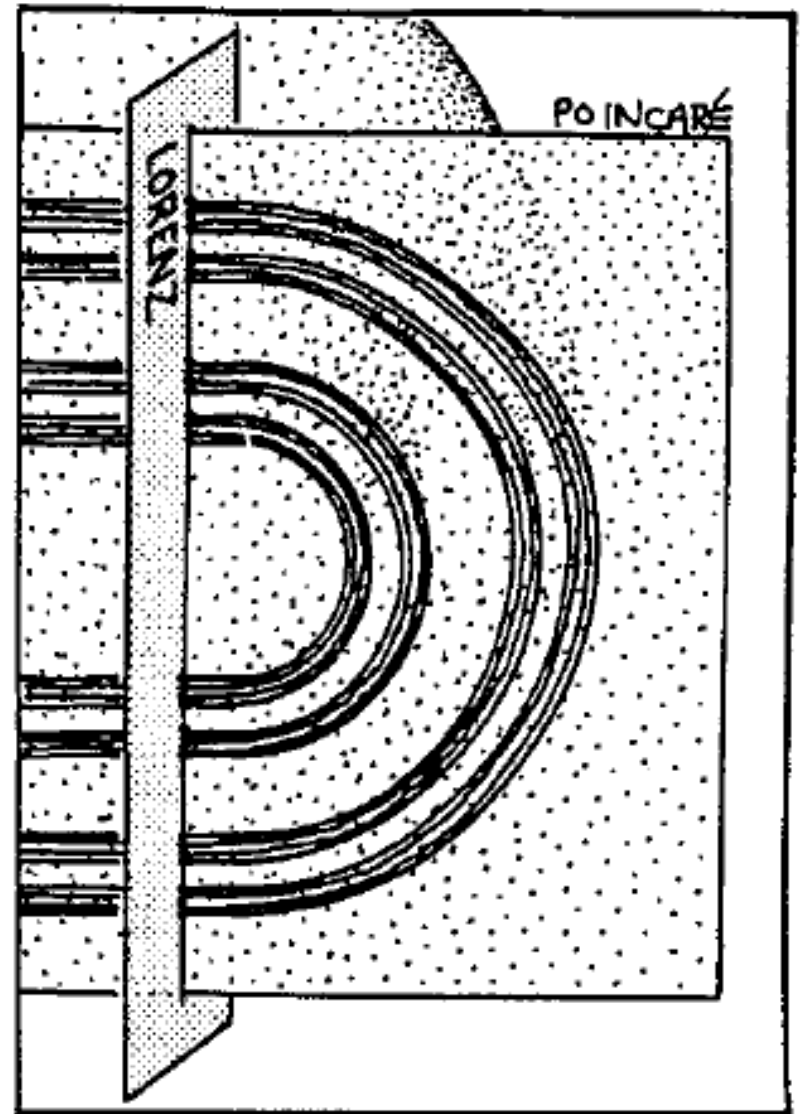
Ultimately the flow generates an infinite complex of tightly packed surfaces: the strange attractor.

# Rössler System

1. Slice the attractor with a plane to expose its cross section. → Poincaré section.
2. Take a one-dimensional slice of the Poincaré section → Lorenz section: An infinite set of points separated by gaps of various sizes.

The resulting pattern of dots and gaps is a topological Cantor set.

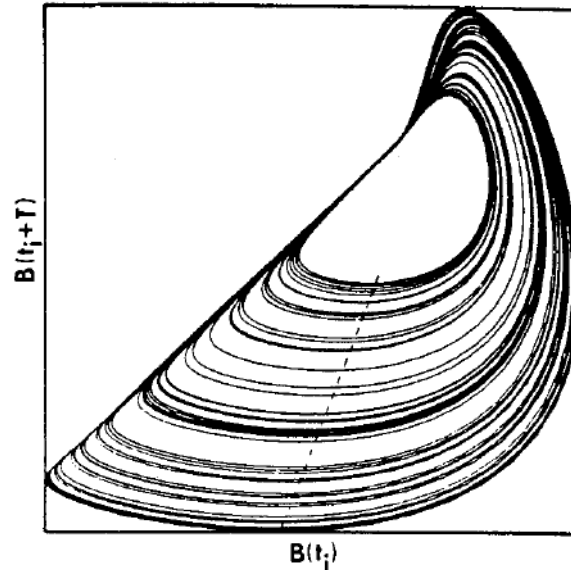
Each dot corresponds to one layer of the complex → **Rössler attractor is a Cantor set of surfaces.**



# Chaos and Strange Attractors in the Real World

Existence of strange attractors has been verified for some real-world systems. The Belousov-Zhabotinsky chemical reaction system that exhibits limit-cycle behaviour in autonomous form can be driven to chaos. This requires feeding in fresh reactants. This driven system is **non-autonomous**. Such a manoeuvre is often required to introduce chaos: we drive the system out of equilibrium.

The way to map out the attractor is to plot dynamical variable at  $t$  and delayed at  $t + T$ .



## Read Strogatz: 12.4 and 12.5

These sections in the book elucidate some general concepts and practical methods in analysing nonlinear/chaotic systems. There may be a question in the exam based on these sections.

# The End



# What we have covered...

Transformation of **ordinary 2-dimensional** differential equations...

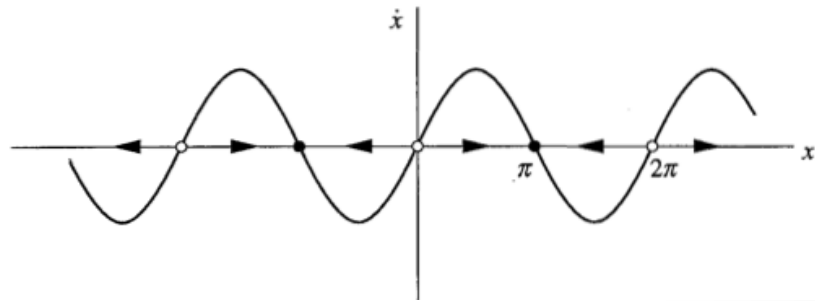
$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \dots, x_n) \end{aligned} \quad \left( \dot{x}_i = \frac{dx_i}{dt} \right) \quad \text{- trick: } x_1 = x; \quad x_2 = \dot{x}_1$$

$$\rightarrow \frac{d^2x}{dt^2} = \dot{x}_2$$

... to **one-dimensional flows**:  $\dot{x} = f(x)$ .  $\rightarrow$  Perform linear stability analysis via  $f'(x)$ .

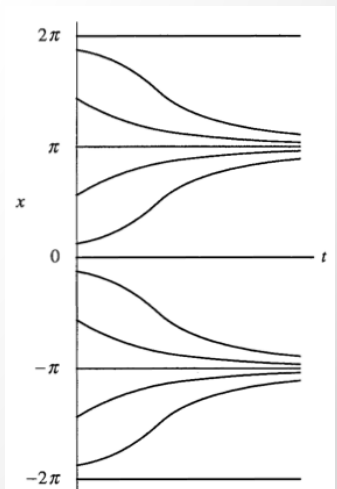
- **vector field**: how the velocity of the particle depends on its position

- **fixed points**; stable and unstable



- time dependence  $x(t)$

- potential  $V$ :  $f(x) = -\frac{dV}{dx}$



# Bifurcations in 1D

Three kinds of bifurcations:

1. Saddle-node:

2. Transcritical:

3. Pitchfork:

a) Supercritical

b) Subcritical

4. Imperfect bifurcations  
for example

Normal form

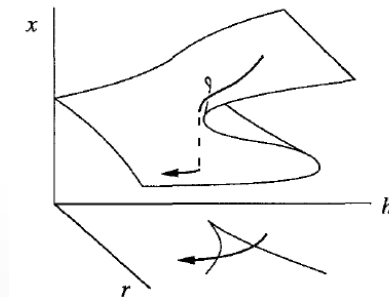
$$\dot{x} = r + x^2$$

$$\dot{x} = rx - x^2$$

$$\dot{x} = rx - x^3$$

$$\dot{x} = rx + x^3$$

$$\dot{x} = h + rx - x^3$$

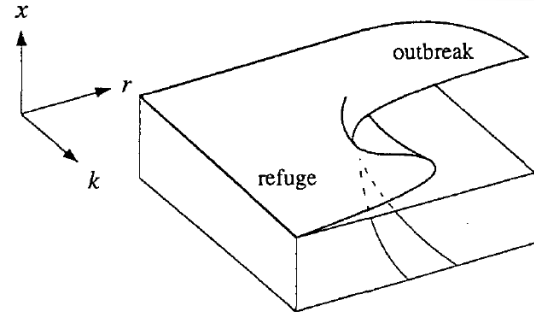


# Growth models and flows on the line

Analysis of growth model having **logistic growth** part and predation. Insect outbreak.

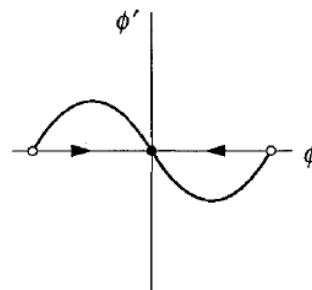
$$\dot{N} = RN \left( 1 - \frac{N}{K} \right) - p(N)$$

Emergence of a **cusp catastrophe**:

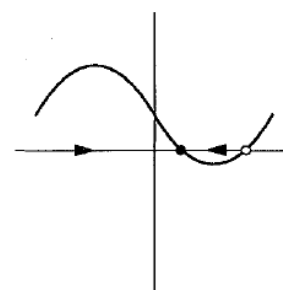


**Flows on the circle** to describe one-dimensional periodic systems.

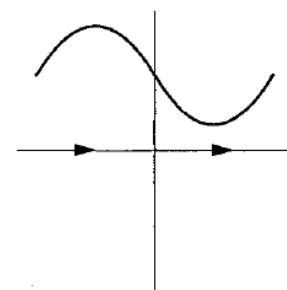
Attempted **phase-locking** of fireflies:



(a)  $\mu = 0$



(b)  $0 < \mu < 1$



(c)  $\mu > 1$



# 2D Linear systems

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}$$

Matrix form

$$\dot{\mathbf{x}} = A\mathbf{x}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

# Classification of linear systems

Eigenvalues and eigenvectors

$$A\mathbf{v} = \lambda\mathbf{v}$$

Characteristic equation

$$\det(A - \lambda I) = 0$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

$$\lambda^2 - \tau\lambda + \Delta = 0$$

$$\tau = \text{trace}(A) = a + d$$

$$\Delta = \det(A) = ad - bc$$

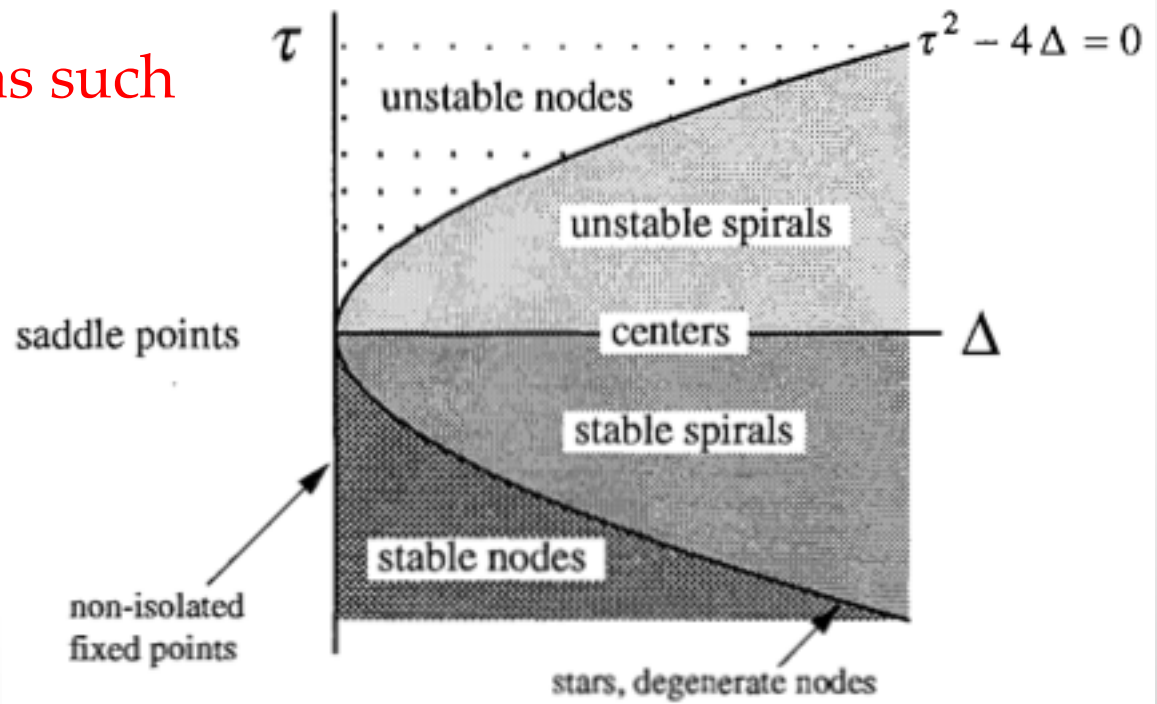
# Classification of fixed points

$$\lambda_{1,2} = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\Delta} \right), \quad \Delta = \lambda_1 \lambda_2, \quad \tau = \lambda_1 + \lambda_2$$

$\Delta$  and  $\tau$  are solved from

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2 = \lambda^2 - \tau\lambda + \Delta = 0$$

Note that the analysis as such is only valid in 2D!



# Phase portraits

The general form of a vector field on the phase plane:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

In vector notation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

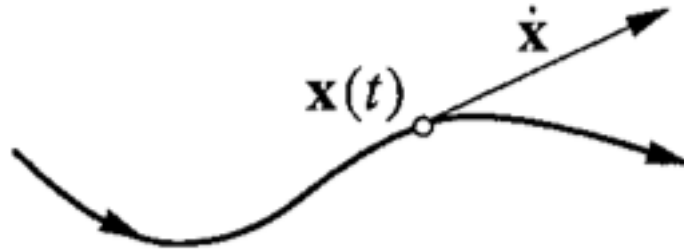
$$[\mathbf{x} = (x_1, x_2), \quad \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))]$$

$\mathbf{x}$  = point in phase plane

$\dot{\mathbf{x}}$  = velocity at that point

# Phase portraits

Solution  $\mathbf{x}(t)$  describes a **trajectory** on the phase plane



The whole plane is filled with (non-intersecting) trajectories starting from different phase points.

For nonlinear systems there is no hope to find trajectories analytically + the analytical solutions would not provide much insight.

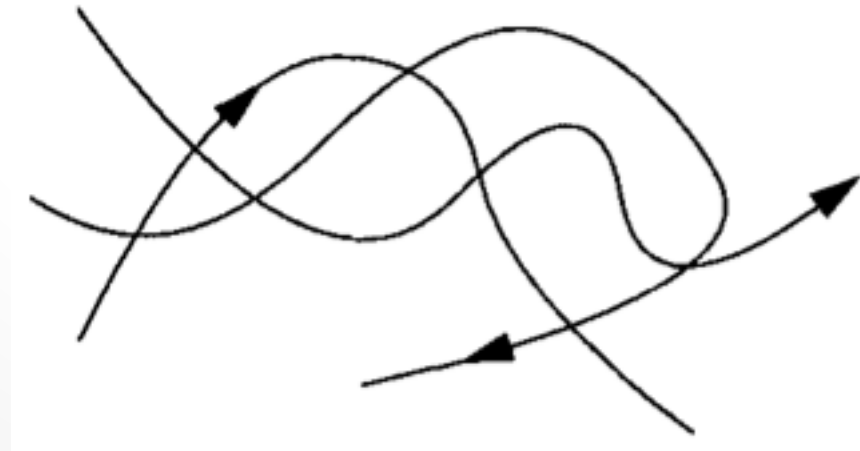
**Our approach:** determine the **qualitative behavior** of the solutions via phase portraits. **Nullclines!**

# Existence, uniqueness and topological consequences

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

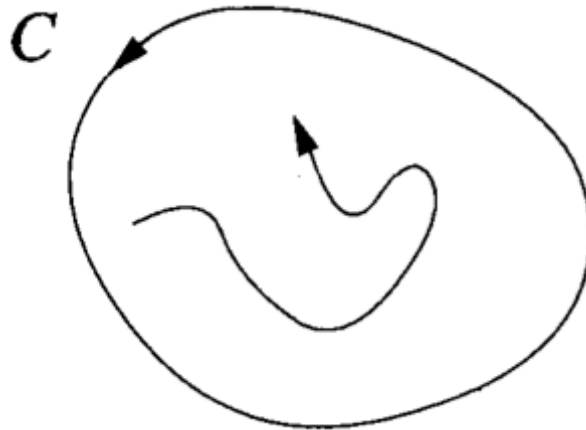
**Corollary:** different trajectories never intersect!

If two trajectories did intersect there would be **two solutions starting from the same point** (the crossing point).



# Existence, uniqueness and topological consequences

**Consequence in two dimensions:** any trajectory starting from inside a closed orbit will be trapped inside it forever!



# 2D nonlinear systems

How to identify:

1. Conservative systems
2. Reversible systems

And what they implicate:

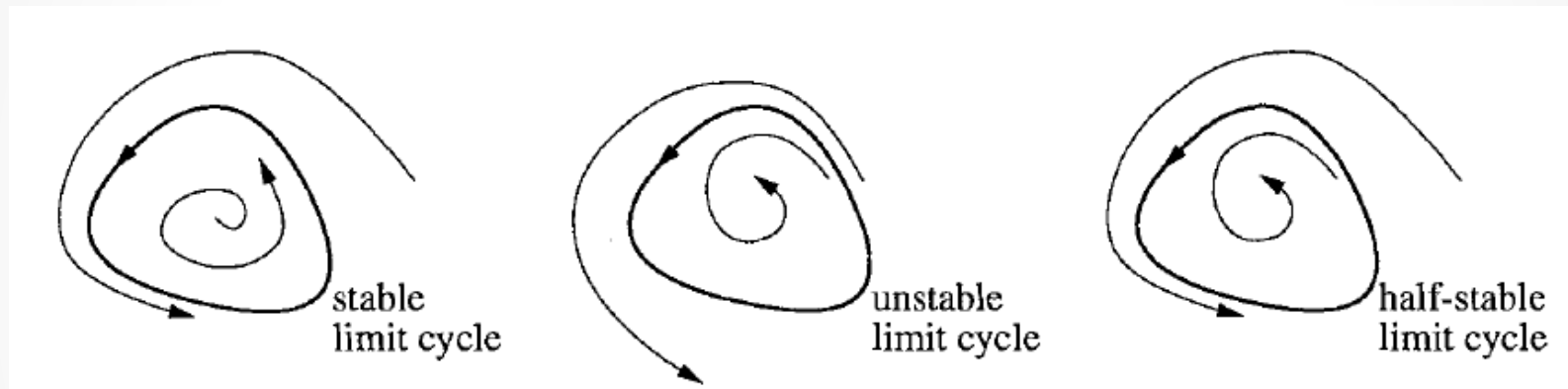
For example, possibility/impossibility of closed orbits

Memorise the tricks pointed out in the lecture notes!



# Limit cycles in 2D

Stable and unstable limit cycles.



Ways to rule out closed orbits.

Ways to prove existence of closed orbits.

Nonlinear oscillations.

Limit cycles are the basis for a new kind of bifurcation in 2D.

# Bifurcations in 2D

Ones that have corresponding bifurcations in 1D:

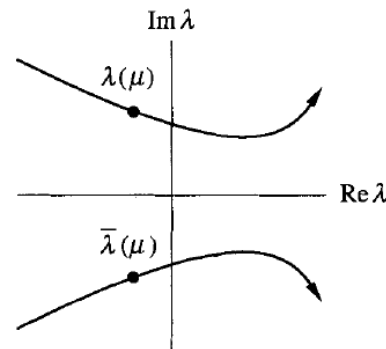
- saddle-node
- transcritical
- supercritical and subcritical pitchfork

These all are zero-eigenvalue bifurcations: They occur at  $\Delta = 0$ , which means that one of the eigenvalues must be zero ( $\Delta = \lambda_1 \lambda_2$ ).

# Hopf bifurcation in $D \geq 2$

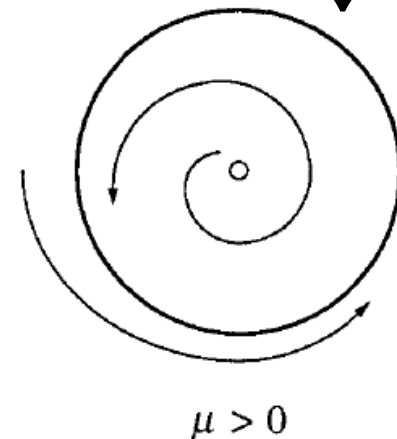
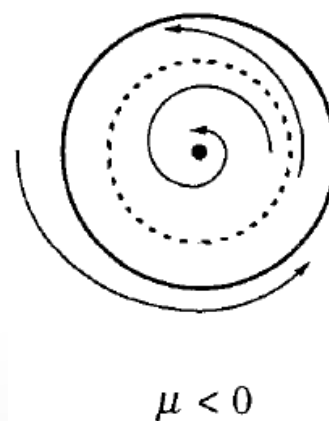
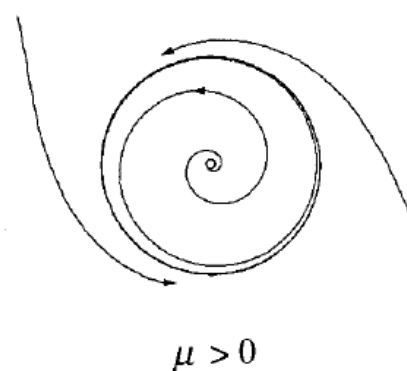
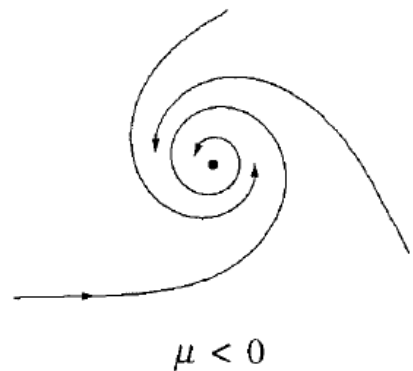
**Hopf bifurcation.** Complex conjugate eigenvalue pair passes through  $Re(\lambda) = 0$ .

In 3D there's something weird lurking here.



Supercritical

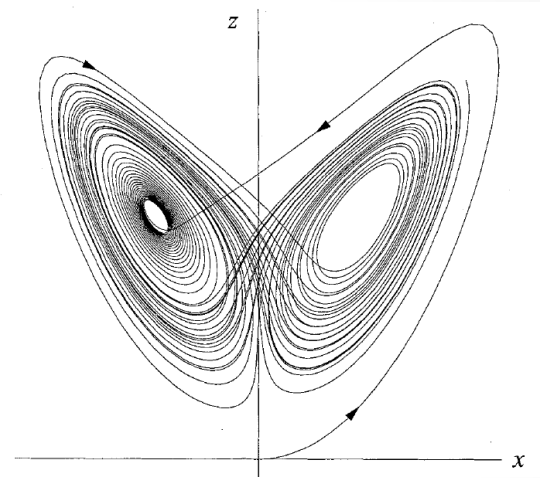
Subcritical (hysteresis)



# Chaotic dynamics in 3D

**Continuous time:** Lorenz' equations and the butterfly – a strange attractor.

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned}$$



Learn the Lorenz system **qualitatively** – different regimes etc. – so that you have a general understanding of its behaviour.  
Make sure you know what chaos is and what it is not.

**Discrete time:**

**One-directional maps.**

**Logistic map**       $x_{n+1} = rx_n(1 - x_n)$        $x_n \geq 0, r \geq 0.$

**Routes to chaos:**

intermittency, periodic doubling.

**Universality**

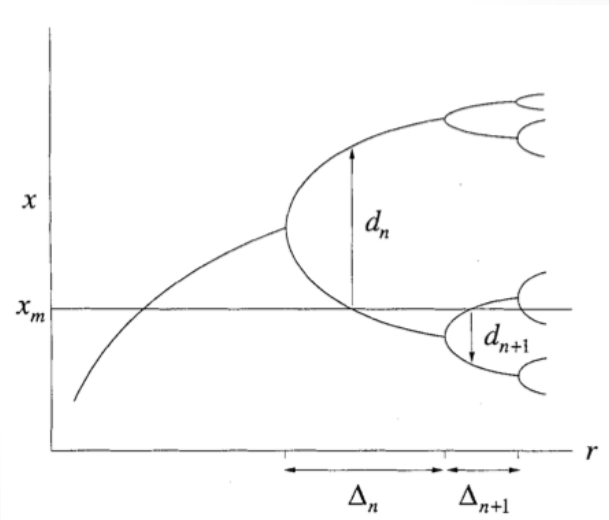
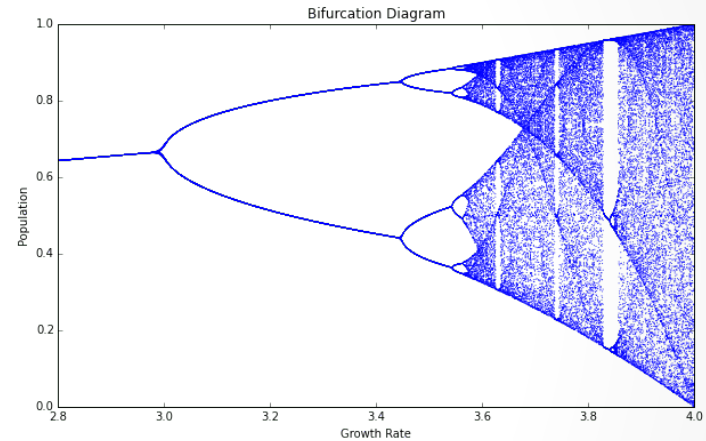
For example,

$$\frac{d_n}{d_{n+1}} \rightarrow \alpha \approx -2.5029$$

as  $n \rightarrow \infty,$

independent of the form of  $f$  in

$$x_{n+1} = f(x_n)$$



# Sets and fractals

Fractal properties of the Cantor set.

Definitions of dimensions: similarity, box, pointwise, and correlation dimensions.

Strange attractors and the maps constructed to understand their properties.

The relation of the Cantor set to the strange attractor of the Lorenz and the Rössler systems.

The exam will mostly consist of calculations. There will probably be either one larger essay question or a couple of smaller ones.

# The End

Although chaotic dynamics has been around in science for quite some time, its application to complex systems is still in its infancy. Application of machine learning techniques is changing the game.

**Hopefully**, after this course **you won't be intimidated** by nonlinear phenomena. **Thanks for the company!**

