CS-E4075 - Special Course in Machine Learning, Data Science and Artificial Intelligence D: Signed graphs: spectral theory and applications

Open problems in the spectral theory of signed graphs

Bruno Ordozgoiti

Aalto University 2021

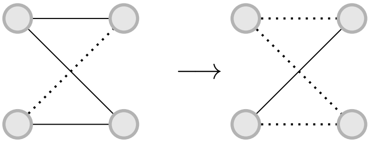
Sign-symmetric graphs

Sign-symmetric graphs

Theorem

A graph is bipartite if and only if its adjacency spectrum is symmetric with respect to the origin.

Bipartite signed graphs have an interesting property: they are switching equivalent to their negation. $(G, \sigma) \sim (G, -\sigma)$.

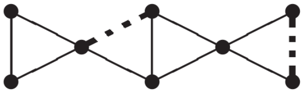


This property is known as sign-symmetry.

Sign-symmetric graphs

Definition

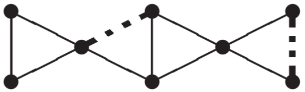
A signed graph $\Gamma = (G, \sigma)$ is said to be sign-symmetric if it is switching equivalent to its negation $-\Gamma = (G, -\sigma)$.

This property is not exclusive to bipartite graphs: 

Sign-symmetric graphs

Definition

A signed graph $\Gamma = (G, \sigma)$ is said to be sign-symmetric if it is switching equivalent to its negation $-\Gamma = (G, -\sigma)$.

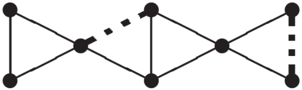
This property is not exclusive to bipartite graphs: 

Let Γ be sign-symmetric. Is its adjacency spectrum symmetric? Recall that the spectrum is invariant under switching...

Sign-symmetric graphs

Definition

A signed graph $\Gamma = (G, \sigma)$ is said to be sign-symmetric if it is switching equivalent to its negation $-\Gamma = (G, -\sigma)$.

This property is not exclusive to bipartite graphs: 

Let Γ be sign-symmetric. Is its adjacency spectrum symmetric? Recall that the spectrum is invariant under switching...

Theorem

Let Γ be a sign-symmetric graph. Then its adjacency spectrum is symmetric with respect to the origin.

Sign-symmetric graphs

Question

Are there signed graphs whose spectrum is symmetric with respect to the origin but are not sign-symmetric?

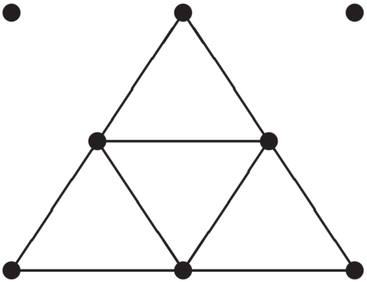
Sign-symmetric graphs

Question

Are there signed graphs whose spectrum is symmetric with respect to the origin but are not sign-symmetric?

Seidel matrix: $S(G) = J - I - 2A$.

Example from (Et-Taoui and Fruchard, 2018)



Sign-symmetric graphs

Problem

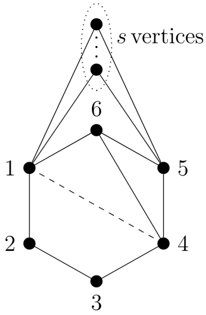
Are there ***non-complete*** *connected signed graphs whose spectrum is symmetric with respect to the origin but are not sign-symmetric?*

Sign-symmetric graphs

Problem

*Are there **non-complete** connected signed graphs whose spectrum is symmetric with respect to the origin but are not sign-symmetric?*

Solved: (Ghorbani et al., 2020)

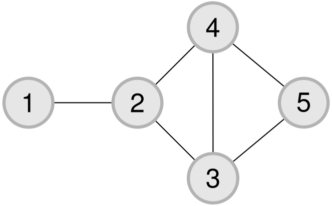


The number of distinct eigenvalues

The number of distinct eigenvalues

Counting cycles in unsigned graphs:

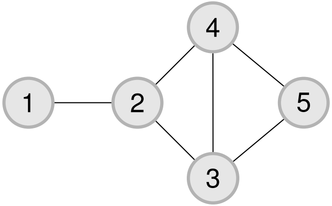
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$



The number of distinct eigenvalues

Counting cycles in unsigned graphs:

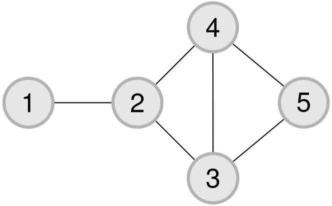
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \ A^2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 & 2 \\ 1 & 1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 1 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix}$$



The number of distinct eigenvalues

Counting cycles in unsigned graphs:

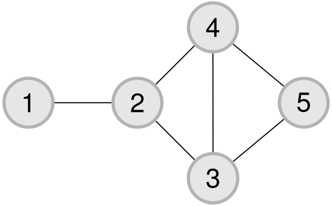
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \ A^2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 & 2 \\ 1 & 1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 1 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix}, \ A^3 = \begin{pmatrix} 0 & 3 & 1 & 1 & 2 \\ 3 & 2 & 6 & 6 & 2 \\ 1 & 6 & 4 & 5 & 5 \\ 1 & 6 & 5 & 4 & 5 \\ 2 & 2 & 5 & 5 & 2 \end{pmatrix}$$



The number of distinct eigenvalues

Counting cycles in unsigned graphs:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \; A^2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 & 2 \\ 1 & 1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 1 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix}, \; A^3 = \begin{pmatrix} 0 & 3 & 1 & 1 & 2 \\ 3 & 2 & 6 & 6 & 2 \\ 1 & 6 & 4 & 5 & 5 \\ 1 & 6 & 5 & 4 & 5 \\ 2 & 2 & 5 & 5 & 2 \end{pmatrix}$$

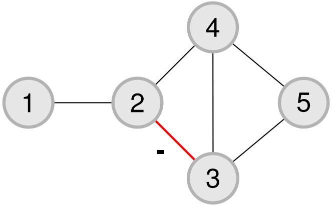


$A_{ii}^k = 2 \times \#(k\text{-cycles adjacent to vertex } i)$.

The number of distinct eigenvalues

Counting cycles in signed graphs:

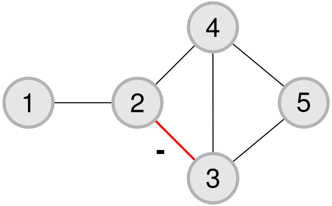
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$



The number of distinct eigenvalues

Counting cycles in signed graphs:

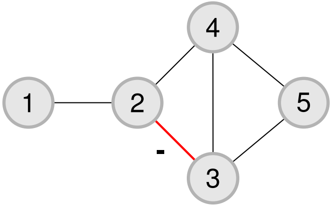
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \ A^2 = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 \\ -1 & 1 & 3 & 0 & 1 \\ 1 & -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$



The number of distinct eigenvalues

Counting cycles in signed graphs:

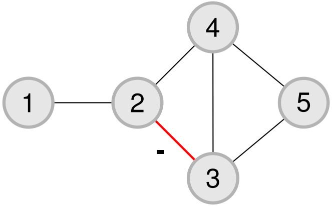
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, A^2 = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 \\ -1 & 1 & 3 & 0 & 1 \\ 1 & -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}, A^3 = \begin{pmatrix} 0 & 3 & 1 & -1 & 0 \\ 3 & -2 & -4 & 4 & 0 \\ 1 & -4 & 0 & 5 & 3 \\ -1 & 4 & 5 & 0 & 3 \\ 0 & 0 & 3 & 3 & 2 \end{pmatrix}$$



The number of distinct eigenvalues

Counting cycles in signed graphs:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \; A^2 = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 \\ -1 & 1 & 3 & 0 & 1 \\ 1 & -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}, \; A^3 = \begin{pmatrix} 0 & 3 & 1 & -1 & 0 \\ 3 & -2 & -4 & 4 & 0 \\ 1 & -4 & 0 & 5 & 3 \\ -1 & 4 & 5 & 0 & 3 \\ 0 & 0 & 3 & 3 & 2 \end{pmatrix}$$

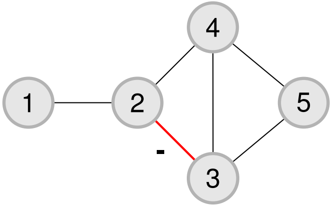


$$A^3_{ii} = 2 \times (\#\text{balanced 3-cycles} - \#\text{unbalanced 3-cyles}).$$

The number of distinct eigenvalues

Counting cycles in signed graphs:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 \\ -1 & 1 & 3 & 0 & 1 \\ 1 & -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 3 & 1 & -1 & 0 \\ 3 & -2 & -4 & 4 & 0 \\ 1 & -4 & 0 & 5 & 3 \\ -1 & 4 & 5 & 0 & 3 \\ 0 & 0 & 3 & 3 & 2 \end{pmatrix}$$



$A^3_{ii} = 2 \times (\#\text{balanced 3-cycles} - \#\text{unbalanced 3-cyles})$. Thus,

$$\frac{Tr(A^3) + Tr(|A|^3)}{2\,Tr(|A|^3)} = \text{fraction of balanced triangles.}$$

Note: $|A|$ is the adj. matrix of the *underlying* (unsigned) graph.

The number of distinct eigenvalues

Definition

The diameter of a graph is the maximum distance between two vertices.

Theorem

Let G have a diameter of d. The number of distinct adjacency eigenvalues of G is at least $d + 1$.

[1] www.math.caltech.edu/~2014-15/2term/ma006b/22%20spectral%202.pdf

The number of distinct eigenvalues

Definition

The diameter of a graph is the maximum distance between two vertices.

Theorem

Let G have a diameter of d. The number of distinct adjacency eigenvalues of G is at least $d + 1$.

Proof sketch[1]: $A^0, A^1, A^2, \ldots, A^d$ are linearly independent, and the degree of the minimal polynomial is the number of distinct eigenvalues.

[1] www.math.caltech.edu/~2014-15/2term/ma006b/22%20spectral%202.pdf

The number of distinct eigenvalues

Definition

The diameter of a graph is the maximum distance between two vertices.

Theorem

Let G have a diameter of d. The number of distinct adjacency eigenvalues of G is at least $d + 1$.

Proof sketch[1]: $A^0, A^1, A^2, \ldots, A^d$ are linearly independent, and the degree of the minimal polynomial is the number of distinct eigenvalues.

Question

What is the only unsigned graph with exactly two distinct eigenvalues?

[1] www.math.caltech.edu/~2014-15/2term/ma006b/22%20spectral%202.pdf

The number of distinct eigenvalues

Definition

The diameter of a graph is the maximum distance between two vertices.

Theorem

Let G have a diameter of d. The number of distinct adjacency eigenvalues of G is at least $d + 1$.

Proof sketch[1]: $A^0, A^1, A^2, \ldots, A^d$ are linearly independent, and the degree of the minimal polynomial is the number of distinct eigenvalues.

Question

What is the only unsigned graph with exactly two distinct eigenvalues?

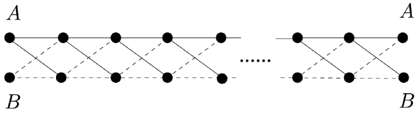
K_n

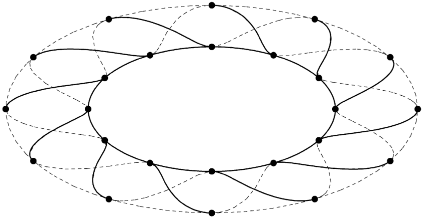
[1] www.math.caltech.edu/~2014-15/2term/ma006b/22%20spectral%202.pdf

The number of distinct eigenvalues

In signed graphs, it is not true in general that # of distinct eigenvalues $>$ diameter.
A counterexample, with diameter $\lfloor \frac{k}{2} \rfloor$, two distinct eigenvalues:

(McKee and Smyth, 2007)





The number of distinct eigenvalues

Therefore, the answer to the next question is not easy for signed graphs:

Problem

Characterize all connected signed graphs whose spectrum consists of two distinct eigenvalues.

Definition

Two vertices are at signed distance k if they are at distance k and the difference between the numbers of positive and negative walks of length k among them is nonzero. Otherwise the signed distance is set to 0. The maximum signed distance is the signed diameter $diam^{\pm}(\Gamma)$.

The number of distinct eigenvalues

Theorem

Let Γ be a connected signed graph with m distinct eigenvalues. Then $m \geq diam^{\pm}(\Gamma) + 1$.

The number of distinct eigenvalues

Theorem

Let Γ be a connected signed graph with m distinct eigenvalues. Then $m \geq diam^{\pm}(\Gamma) + 1$.

Finally...

Question for you

Can you think of other signed graphs with exactly two distinct eigenvalues?

The number of distinct eigenvalues

Theorem

Let Γ be a connected signed graph with m distinct eigenvalues. Then $m \geq diam^{\pm}(\Gamma) + 1$.

Finally...

Question for you

Can you think of other signed graphs with exactly two distinct eigenvalues?

- $\pm K_n$

The number of distinct eigenvalues

Theorem

Let Γ be a connected signed graph with m distinct eigenvalues. Then $m \geq diam^{\pm}(\Gamma) + 1$.

Finally...

Question for you

Can you think of other signed graphs with exactly two distinct eigenvalues?

- $\pm K_n$
- Huang's hypercube for the Sensitivity Conjecture! (Huang, 2019)

Signature minimizing the spectral radius

Signature minimizing the spectral radius

Theorem

Perron-Frobenius: Let G be a graph with adjacency matrix A. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *be the eigenvalues of A. Then* $\lambda_1 \geq |\lambda_i|$*, for all* i*.*

This does not hold for signed graphs.

Signature minimizing the spectral radius

Let $\rho(\Gamma) = \max_i\{|\lambda_i(\Gamma)|\}$.

Problem

Let Γ be a simple and connected unsigned graph. Determine the signature $\bar{\sigma}$ such that for any signature σ of Γ, we have $\rho(\Gamma, \bar{\sigma}) \leq \rho(\Gamma, \sigma)$.

This problem is very important! Let's see why.

Definition

A d-regular graph G is a Ramanujan graph if $\max\{|\lambda_2|, |\lambda_n|\} \leq 2\sqrt{d-1}$.

Signature minimizing the spectral radius

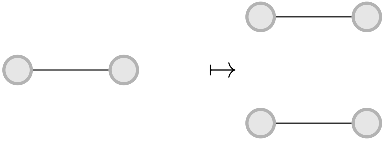
Definition

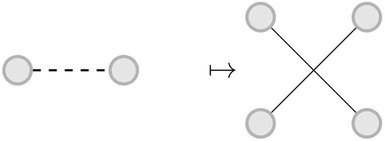
Consider a signed graph Γ. The 2-lift of Γ is an unsigned graph $\Gamma' = (V \times \{+1, -1\}, E)$ where (x, s) is adjacent to $(y, s\sigma(xy))$, for $s = \pm 1$.

Signature minimizing the spectral radius

Definition

Consider a signed graph Γ. The 2-lift of Γ is an unsigned graph $\Gamma' = (V \times \{+1, -1\}, E)$ where (x, s) is adjacent to $(y, s\sigma(xy))$, for $s = \pm 1$.





Theorem

Let G be the underlying graph of Γ. The spectrum of Γ' is the union of the spectra of G and Γ.

Proof: The adjacency matrix of Γ' is $A_{\Gamma'} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$, where A_1 (resp. A_2) is the adjacency matrix of $(V, s) \times (V, s)$ (resp. $(V, s) \times (V, -s)$), where $s = \pm 1$.

Signature minimizing the spectral radius

Recall. Ramanujan: $\lambda_1 = d, \max\{|\lambda_2|, |\lambda_n|\} \leq 2\sqrt{d-1}$.

Signature minimizing the spectral radius

Recall. Ramanujan: $\lambda_1 = d, \max\{|\lambda_2|, |\lambda_n|\} \leq 2\sqrt{d-1}$.

Theorem

Every connected d-regular graph has a signing with spectral radius at most $c\sqrt{d \log^3 d}$, where $c > 0$ is some absolute constant. (Bilu and Linial, 2006)

Signature minimizing the spectral radius

Recall. Ramanujan: $\lambda_1 = d, \max\{|\lambda_2|, |\lambda_n|\} \leq 2\sqrt{d-1}$.

Theorem

Every connected d-regular graph has a signing with spectral radius at most $c\sqrt{d \log^3 d}$, where $c > 0$ is some absolute constant. (Bilu and Linial, 2006)

Conjecture

Every connected d-regular graph has a signing with spectral radius at most $2\sqrt{d-1}$. (Bilu and Linial, 2006)

Signature minimizing the spectral radius

Recall. Ramanujan: $\lambda_1 = d, \max\{|\lambda_2|, |\lambda_n|\} \leq 2\sqrt{d-1}$.

Theorem

Every connected d-regular graph has a signing with spectral radius at most $c\sqrt{d \log^3 d}$, where $c > 0$ is some absolute constant. (Bilu and Linial, 2006)

Conjecture

Every connected d-regular graph has a signing with spectral radius at most $2\sqrt{d-1}$. (Bilu and Linial, 2006)

Theorem

Let G be a connected d-regular graph. Then there exists a signature σ of G such that the largest eigenvalue of A_σ is at most $2\sqrt{d-1}$. (Marcus et al., 2013)

Signature minimizing the spectral radius

Problem

Let Γ be a simple and connected unsigned graph. Determine the signature $\bar{\sigma}$ such that for any signature σ of Γ, we have $\rho(\Gamma, \bar{\sigma}) \leq \rho(\Gamma, \sigma)$.

"As an amusing exercise, we challenge the readers to solve this problem by finding a signature of the Petersen graph or of their favorite graph that minimizes the spectral radius." (Belardo et al., 2019)

Spectral determination

Spectral determination

Question

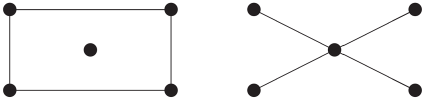
Are unsigned graphs determined by their spectrum (up to isomorphism)?

Spectral determination

Question

Are unsigned graphs determined by their spectrum (up to isomorphism)?

No Van Dam and Haemers (2003)



Spectral determination

Some are, such as the path P_n and the cycle C_n.

But of course, not the case for signed graphs (Akbari et al., 2018a,b).

Spectral determination

Some are, such as the path P_n and the cycle C_n.

But of course, not the case for signed graphs (Akbari et al., 2018a,b).

Theorem

The signed path P_n is determined by its spectrum if and only if $n \equiv 0, 1, 2 \pmod 4$ unless $n \in \{8, 13, 14, 17, 29\}$, or $n = 3$.

Theorem

- *Odd signed cycles C^+_{2n+1}, C^-_{2n+1} and C^-_4 are determined by their spectrum.*
- *Even signed cycles C^+_{2n}, C^-_{2n} except C^-_4 are not determined by their spectrum.*

Spectral determination

Proposition

From the eigenvalues of a signed graph Γ we obtain the following invariants:

- *number of vertices and edges*
- *the difference between the number of positive and negative triangles:* $\frac{1}{6}\sum_i \lambda_i^3$;
- *the difference between the number of positive and negative closed walks of length p:* $\sum_i \lambda_i^p$.

References I

Akbari, S., Belardo, F., Dodongeh, E., and Nematollahi, M. A. (2018a). Spectral characterizations of signed cycles. *Linear Algebra and its Applications*, 553:307–327.

Akbari, S., Haemers, W. H., Maimani, H. R., and Majd, L. P. (2018b). Signed graphs cospectral with the path. *Linear Algebra and its Applications*, 553:104–116.

Belardo, F., Cioabă, S. M., Koolen, J. H., and Wang, J. (2019). Open problems in the spectral theory of signed graphs. *arXiv preprint arXiv:1907.04349*.

Bilu, Y. and Linial, N. (2006). Lifts, discrepancy and nearly optimal spectral gap. *Combinatorica*, 26(5):495–519.

Et-Taoui, B. and Fruchard, A. (2018). On switching classes of graphs. *Linear Algebra and its Applications*, 549:246–255.

References II

Ghorbani, E., Haemers, W. H., Maimani, H. R., and Majd, L. P. (2020). On sign-symmetric signed graphs. *arXiv preprint arXiv:2003.09981*.

Huang, H. (2019). Induced subgraphs of hypercubes and a proof of the sensitivity conjecture. *Annals of Mathematics*, 190(3):949–955.

Marcus, A., Spielman, D. A., and Srivastava, N. (2013). Interlacing families i: Bipartite ramanujan graphs of all degrees. In *2013 IEEE 54th Annual Symposium on Foundations of computer science*, pages 529–537. IEEE.

McKee, J. and Smyth, C. (2007). Integer symmetric matrices having all their eigenvalues in the interval [- 2, 2]. *Journal of algebra*, 317(1):260–290.

van Dam, E. R. (1998). Nonregular graphs with three eigenvalues. *Journal of Combinatorial Theory, Series B*, 73(2):101–118.

Van Dam, E. R. and Haemers, W. H. (2003). Which graphs are determined by their spectrum? *Linear Algebra and its applications*, 373:241–272.

References III

Van Dam, E. R., Koolen, J. H., and Xia, Z.-J. (2014). Graphs with many valencies and few eigenvalues. *arXiv preprint arXiv:1405.3383*.