CS-E4075 - Special Course in Machine Learning, Data Science and Artificial Intelligence D: Signed graphs: spectral theory and applications

Introduction to signed graphs

Bruno Ordozgoiti

Aalto University 2021

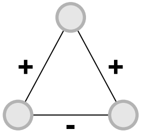
Signed graphs

Signed graphs

Signed graphs: each edge labeled $+$ or $-$.

Definitions:

- $G = (V, E^+, E^-)$,

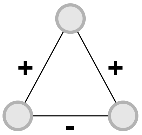


Signed graphs

Signed graphs: each edge labeled $+$ or $-$.

Definitions:

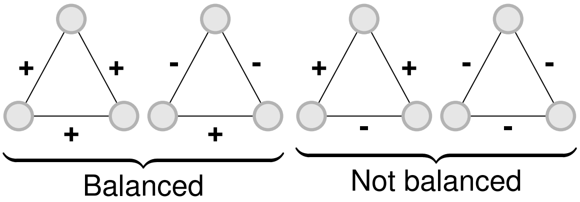
- $G = (V, E^+, E^-)$,
- $G = (V, E, \sigma), \sigma : E \to \{-, +\}$.



Signed graphs

Motivation: **balance** in social networks.

"*The friend of a friend is a friend*" (or "*the enemy of a friend is an enemy*").

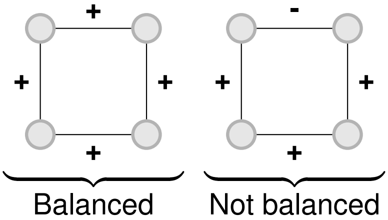


The four possible non-isomorphic signed triangles.

Signed graphs

Balance applies to cycles of any length.

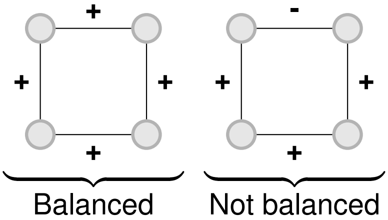
"*The friend of a friend of a friend is a friend*".

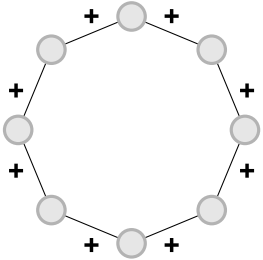


Signed graphs

Balance applies to cycles of any length.

"*The friend of a friend of a friend is a friend*".

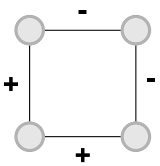




"*The friend of a friend of a friend of a friend of a friend of a friend of a friend is a friend*".

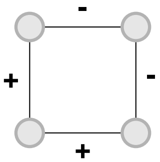
Signed graphs

Balanced or not balanced?

1. 

Signed graphs

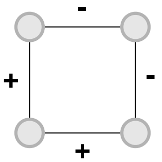
Balanced or not balanced?

1. 

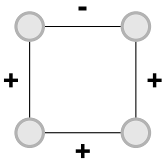
Balanced

Signed graphs

Balanced or not balanced?

1. 

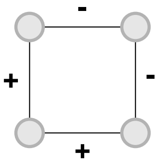
Balanced

2. 

Signed graphs

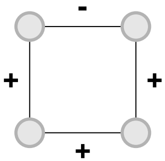
Balanced or not balanced?

1.



Balanced

2.

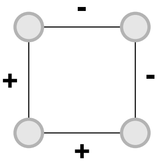


Not balanced

Signed graphs

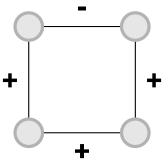
Balanced or not balanced?

1.



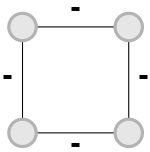
Balanced

2.



Not balanced

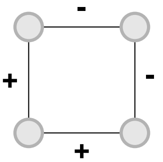
3.



Signed graphs

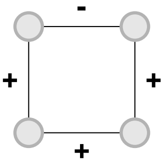
Balanced or not balanced?

1.



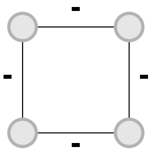
Balanced

2.



Not balanced

3.

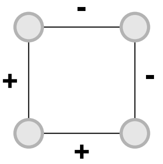


Balanced

Signed graphs

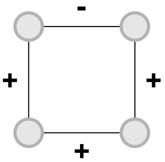
Balanced or not balanced?

1.



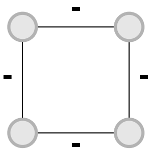
Balanced

2.



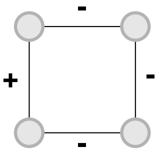
Not balanced

3.



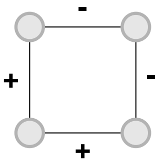
Balanced

4.

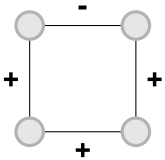


Signed graphs

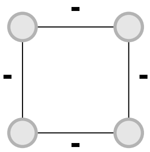
Balanced or not balanced?

1. 

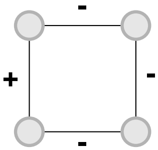
Balanced

2. 

Not balanced

3. 

Balanced

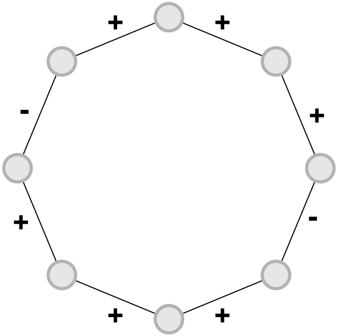
4. 

Not balanced

Signed graphs

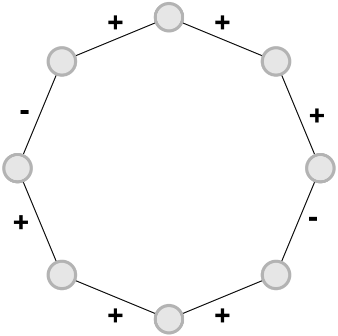
Balanced or not balanced?

5.



Signed graphs

Balanced or not balanced?

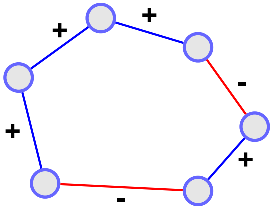
5. 

Balanced

Motivation

Definition of balanced cycle

A cycle is balanced if the product of its signs is positive.

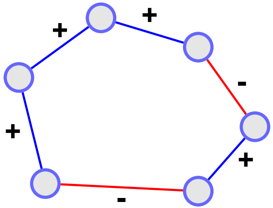


$+ \times - \times + \times - \times + \times + = +.$

Motivation

Definition of balanced cycle

A cycle is balanced if the product of its signs is positive.



$$+ \times - \times + \times - \times + \times + = +.$$

Definition of balanced graph

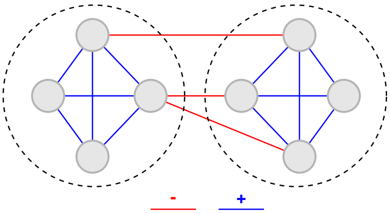
A graph is balanced if it contains no unbalanced (negative) cycles.

Signed graphs

Characterizations of balance

G is balanced iff

- There are no negative (unbalanced) cycles.

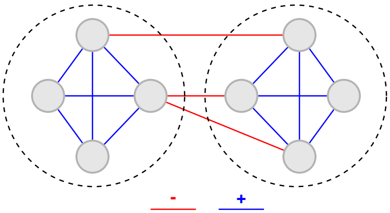


Signed graphs

Characterizations of balance

G is balanced iff

- There are no negative (unbalanced) cycles.
- There exists a sign-compliant partition $V = V_1 \cup V_2$.
  - **within** V_1, V_2, every edge is **+**;
  - **between** V_1, V_2, every edge is **-**.

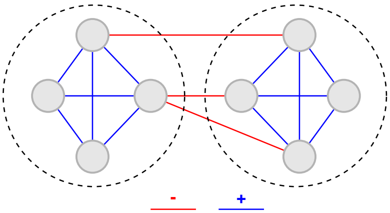


Signed graphs

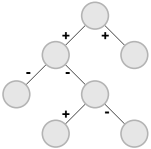
Characterizations of balance

G is balanced iff

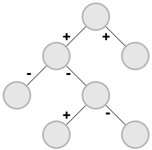
- There are no negative (unbalanced) cycles.
- There exists a sign-compliant partition $V = V_1 \cup V_2$.
  - **within** V_1, V_2, every edge is **+**;
  - **between** V_1, V_2, every edge is **-**.
- All paths between any pair u, v have the same sign.



1.

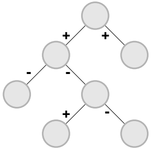


1.



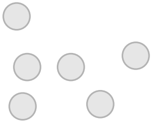
Balanced

1.

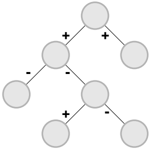


Balanced

2.

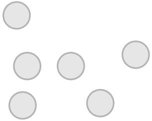


1.



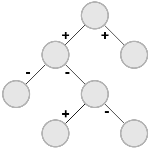
Balanced

2.



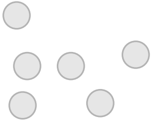
Balanced

1.



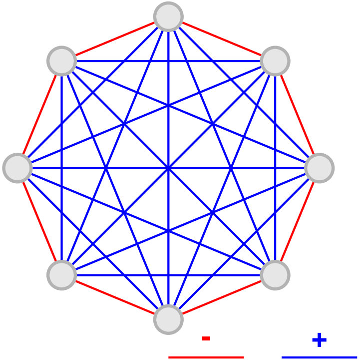
Balanced

2.

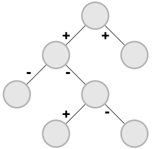


Balanced

3.

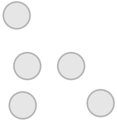


1.



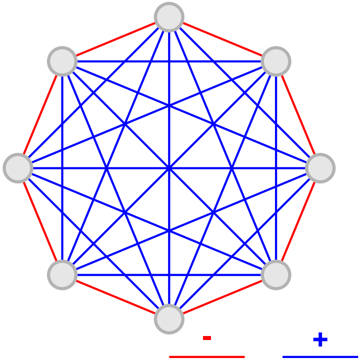
Balanced

2.



Balanced

3.

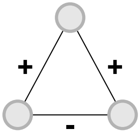


Not balanced

Spectral implications of negative weights

Signed graphs: spectral theory

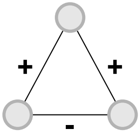
Signed adjacency matrix: $A = A_{E^+} - A_{E^-}$



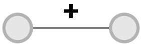
$$\mapsto A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Signed Laplacian: $L = D - A$.
Degree matrix:

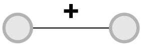
$$\bar{D}_{ij} = \begin{cases} d_i = \sum_k |A_{ik}| & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$



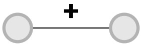
$$\mapsto L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

1. 

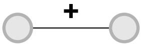
$$L = \begin{pmatrix} & & \\ & & \end{pmatrix}, V = \begin{pmatrix} & & \\ & & \end{pmatrix}, \lambda : (\quad)$$

1. 

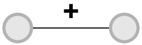
$$L = \left(\quad \right), V = \left(\quad \right), \lambda : (\quad)$$

1. 

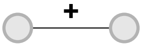
$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} & \\ & \end{pmatrix}, \lambda : (\quad)$$

1. 

$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (\quad\quad)$$

1. 

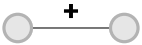
$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

1. 

$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

2. 

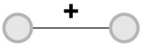
$$L = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}, V = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}, \lambda : (\quad)$$

1. 

$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

2. 

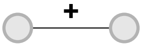
$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}, \lambda : (\qquad)$$

1. 

$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

2. 

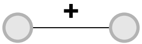
$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \lambda : (\qquad)$$

1. 

$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

2. 

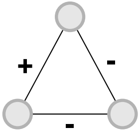
$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

1. 

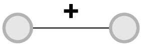
$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

2. 

$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

3. 

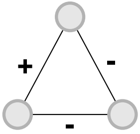
$$L = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}, V = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}, \lambda : (\qquad)$$

1. 

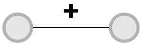
$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

2. 

$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

3. 

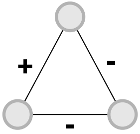
$$L = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, V = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}, \lambda : (\qquad)$$

1. 

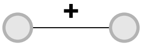
$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

2. 

$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

3. 

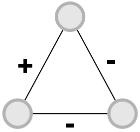
$$L = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, V = \begin{pmatrix} -1 & \dots & \dots \\ -1 & \dots & \dots \\ 1 & \dots & \dots \end{pmatrix}, \lambda : (\qquad\qquad)$$

1. 

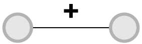
$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0,2)$$

2. 

$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \lambda : (0,2)$$

3. 

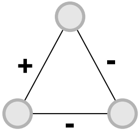
$$L = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, V = \begin{pmatrix} -1 & \ldots & \ldots \\ -1 & \ldots & \ldots \\ 1 & \ldots & \ldots \end{pmatrix}, \lambda : (0,3,3)$$

1. 

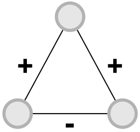
$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

2. 

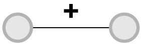
$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

3. 

$$L = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, V = \begin{pmatrix} -1 & \dots & \dots \\ -1 & \dots & \dots \\ 1 & \dots & \dots \end{pmatrix}, \lambda : (0, 3, 3)$$

4. 

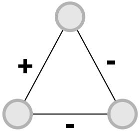
$$L = \begin{pmatrix} \quad \end{pmatrix}, V = \begin{pmatrix} \quad \end{pmatrix}, \lambda : (\quad)$$

1. 

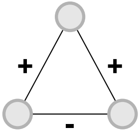
$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

2. 

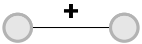
$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

3. 

$$L = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, V = \begin{pmatrix} -1 & \dots & \dots \\ -1 & \dots & \dots \\ 1 & \dots & \dots \end{pmatrix}, \lambda : (0, 3, 3)$$

4. 

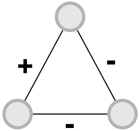
$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, V = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}, \lambda : (\qquad)$$

1. 

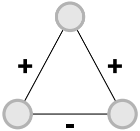
$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

2. 

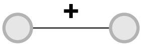
$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

3. 

$$L = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, V = \begin{pmatrix} -1 & \dots & \dots \\ -1 & \dots & \dots \\ 1 & \dots & \dots \end{pmatrix}, \lambda : (0, 3, 3)$$

4. 

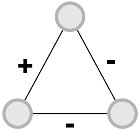
$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, V = \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}, \lambda : (\qquad\qquad)$$

1. 

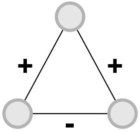
$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

2. 

$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \lambda : (0, 2)$$

3. 

$$L = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, V = \begin{pmatrix} -1 & \dots & \dots \\ -1 & \dots & \dots \\ 1 & \dots & \dots \end{pmatrix}, \lambda : (0, 3, 3)$$

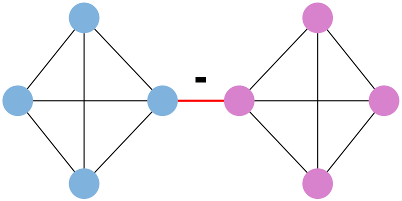
4. 

$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, V = \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}, \lambda : (1, 1, 4)$$

Signed graphs: spectral theory

Spectral characterization of balance

1. A connected signed graph is balanced if and only if $\lambda_1(L_G) = 0$.

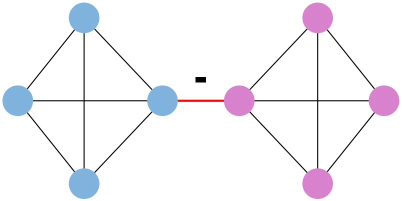


$$v_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \lambda_1(L) = 0.$$

Signed graphs: spectral theory

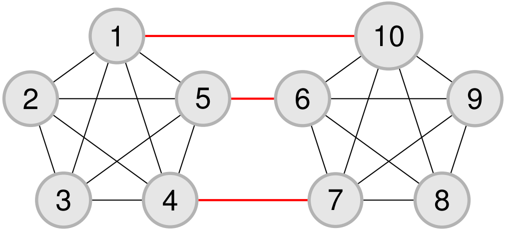
Spectral characterization of balance

1. A connected signed graph is balanced if and only if $\lambda_1(L_G) = 0$.
2. A signed graph is balanced if and only if its Laplacian has 0 as an eigenvalue with multiplicity equal to the number of connected components.



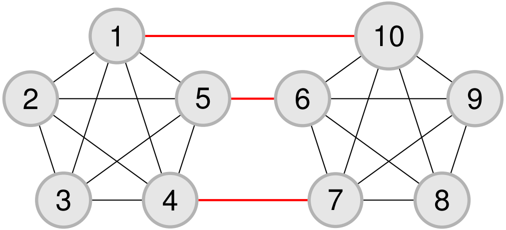
$$v_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \lambda_1(L) = 0.$$

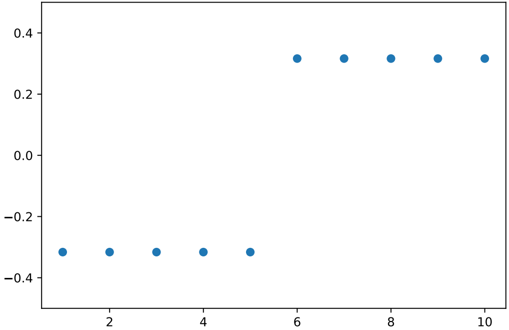
Fiedler vector, v_1, corresponding to the smallest eigenvalue, λ_1, of the signed Laplacian.



$v_1 =$

Fiedler vector, v_1, corresponding to the smallest eigenvalue, λ_1, of the signed Laplacian.

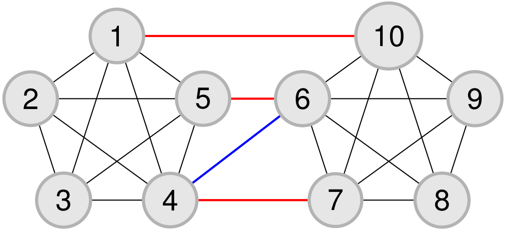


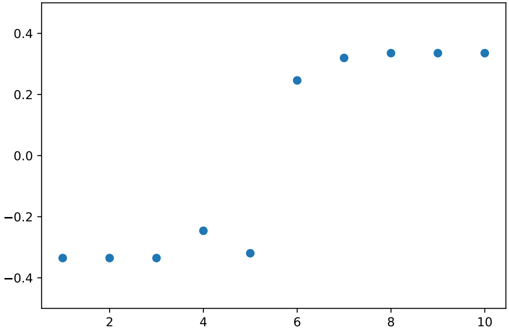


$v_1 =$
$(-0.316, -0.316, -0.316, -0.316, -0.316, 0.316, 0.316, 0.316, 0.316, 0.316)$

$\lambda_1 = 0$

Fiedler vector, v_1, corresponding to the smallest eigenvalue, λ_1, of the signed Laplacian.

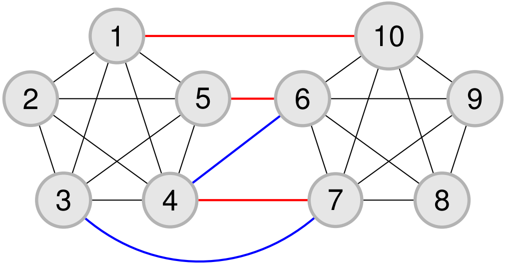


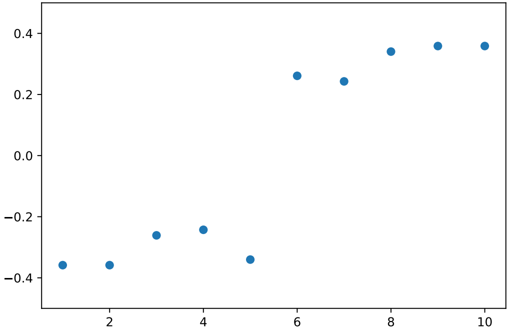


$$v_1 = (-0.335, -0.335, -0.335, -0.246, -0.32, 0.246, 0.32, 0.335, 0.335, 0.335)$$

$$\lambda_1 \approx 0.313$$

Fiedler vector, v_1, corresponding to the smallest eigenvalue, λ_1, of the signed Laplacian.





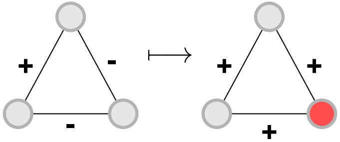
$$v_1 = (-0.359, -0.359, -0.261, -0.243, -0.34, 0.261, 0.243, 0.34, 0.359, 0.359)$$

$\lambda_1 \approx 0.645$

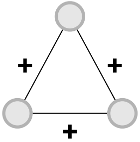
Switching

Switching

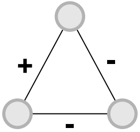
Switch $S \subset V$: flip signs of edges between S and $V \backslash S$.



The spectrum of both A and L is invariant w.r.t. switching.

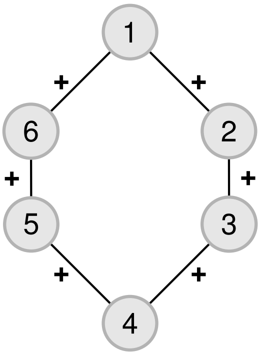


$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, V : \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \lambda : (0, 3, 3)$$



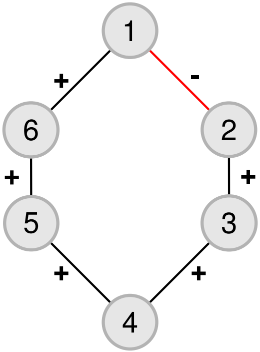
$$L = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, V : \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \lambda : (0, 3, 3)$$

Example: cycle



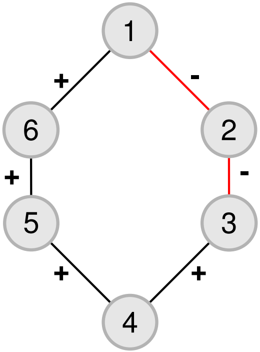
- Spectrum: $(0, 1, 1, 3, 3, 4)$
-
-

Example: cycle



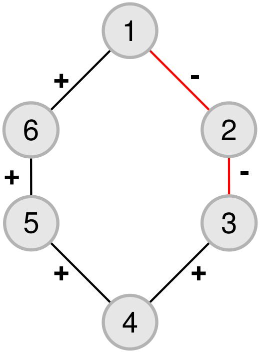
- Spectrum: $(0, 1, 1, 3, 3, 4)$
- Spectrum: $(0.27, 0.27, 2, 2, 3.73, 3.73)$
-

Example: cycle



- Spectrum: $(0, 1, 1, 3, 3, 4)$
- Spectrum: $(0.27, 0.27, 2, 2, 3.73, 3.73)$
- Spectrum:

Example: cycle

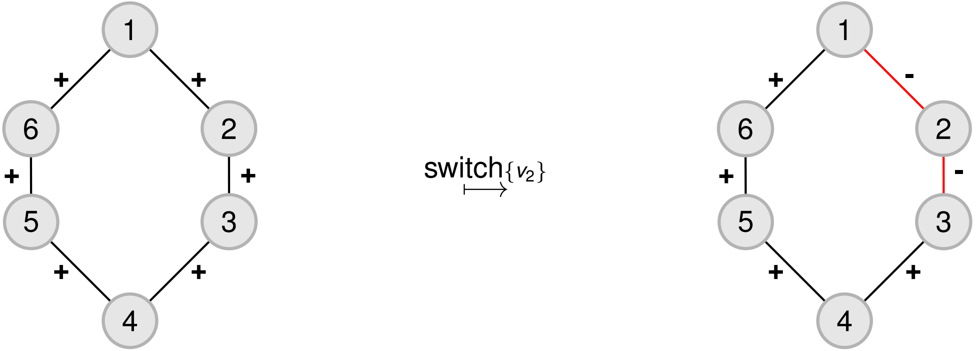


- Spectrum: $(0, 1, 1, 3, 3, 4)$
- Spectrum: $(0.27, 0.27, 2, 2, 3.73, 3.73)$
- Spectrum: $(0, 1, 1, 3, 3, 4)$

Switching

Other spectral characterizations of balance

1. Spectrum (A or L) of G = spectrum of $|G|$ (**underlying** graph).
2. Switching equivalent to all-positive.



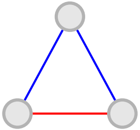
Frustration

Frustration

Vertex frustration f_v: N. of vertices that need to be removed to achieve balance.

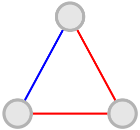
Edge frustration f_e: N. of edges that need to be removed to achieve balance.

1.



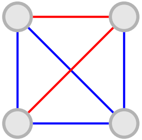
f_e : f_v :

2.



f_e : f_v :

3.



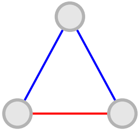
f_e : f_v :

Frustration

Vertex frustration f_v: N. of vertices that need to be removed to achieve balance.

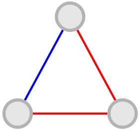
Edge frustration f_e: N. of edges that need to be removed to achieve balance.

1.



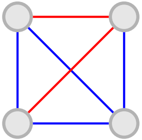
f_e : f_v :

2.



f_e : f_v :

3.

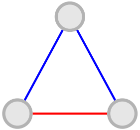


f_e : f_v :

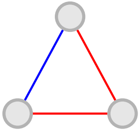
Frustration

Vertex frustration f_v: N. of vertices that need to be removed to achieve balance.

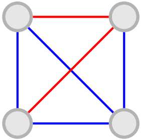
Edge frustration f_e: N. of edges that need to be removed to achieve balance.

1. 

f_e : 1 f_v :

2. 

f_e : f_v :

3. 

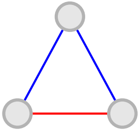
f_e : f_v :

Frustration

Vertex frustration f_v: N. of vertices that need to be removed to achieve balance.

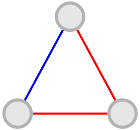
Edge frustration f_e: N. of edges that need to be removed to achieve balance.

1.



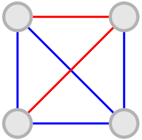
f_e : 1 f_v : 1

2.



f_e : f_v :

3.



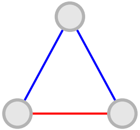
f_e : f_v :

Frustration

Vertex frustration f_v: N. of vertices that need to be removed to achieve balance.

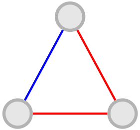
Edge frustration f_e: N. of edges that need to be removed to achieve balance.

1.



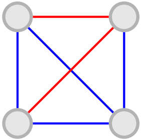
f_e : 1 f_v : 1

2.



f_e : 0 f_v :

3.



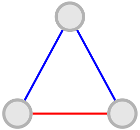
f_e : f_v :

Frustration

Vertex frustration f_v: N. of vertices that need to be removed to achieve balance.

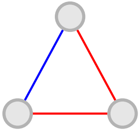
Edge frustration f_e: N. of edges that need to be removed to achieve balance.

1.



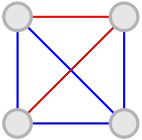
$f_e : 1$ $f_v : 1$

2.



$f_e : 0$ $f_v : 0$

3.



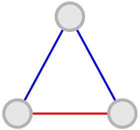
$f_e :$ $f_v :$

Frustration

Vertex frustration f_v: N. of vertices that need to be removed to achieve balance.

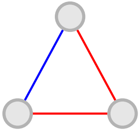
Edge frustration f_e: N. of edges that need to be removed to achieve balance.

1.



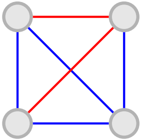
$f_e : 1 \ f_v : 1$

2.



$f_e : 0 \ f_v : 0$

3.



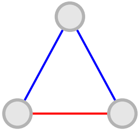
$f_e : 1 \ f_v :$

Frustration

Vertex frustration f_v: N. of vertices that need to be removed to achieve balance.

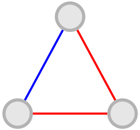
Edge frustration f_e: N. of edges that need to be removed to achieve balance.

1.



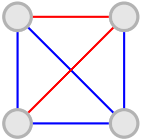
$f_e : 1 \; f_v : 1$

2.



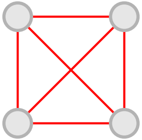
$f_e : 0 \; f_v : 0$

3.



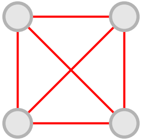
$f_e : 1 \; f_v : 1$

Frustration



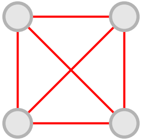
f_e : f_v :

Frustration



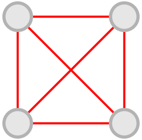
$f_e : 2 \; f_v :$

Frustration



$f_e : 2 \; f_v : 2$

Frustration



$f_e : 2 \; f_v : 2$

Finding the edge frustration is NP-hard (MAXCUT).

Spectral frustration inequalities

The spectral connection

$$\lambda_{min}(L) \leq f_e.$$

Spectral frustration inequalities

The spectral connection

$$\lambda_{min}(L) \leq f_e.$$

Try $\mathbf{1}_{\pm}^T L \mathbf{1}_{\pm}$, where $\mathbf{1}_{\pm} =$ partition indicator.

Spectral graph theory

Take-aways from this lecture:

- Balance theory.
- Characterizations of balance.
- Spectral characterizations of balance.
- Fiedler vector in signed graphs.
- Switching.
- Frustration.