The Determinant

Consider a $n \times n$ square matrix $\boldsymbol{A}$. If $n = 1$, define the determinant as $\det A = a_{11}$. For a general $n \times n$ matrix $\boldsymbol{A}$ and remove row i and and column j to get an $(n-1) \times (n-1)$ matrix $\boldsymbol{A}_{ij}$. Define the *minors* of $\boldsymbol{A}$ as:

$$M_{ij} := \det \boldsymbol{A}_{ij}.$$

The ij^{th} *cofactor* C_{ij} of $\boldsymbol{A}$ is defined as:

$$C_{ij} := (-1)^{i+j} M_{ij}.$$

The determinant of $\boldsymbol{A}$ is defined recursively as:

$$\det \boldsymbol{A} := \Sigma_{j=1}^{n} (-1)^{(i+j)} a_{ij} C_{ij}.$$

Notice that the recursive element in the definition comes from the fact that in order to compute M_{ij}, you need to evaluate the determinant of the smaller $(n-1) \times (n-1)$ matrix $\boldsymbol{A}_{ij}$. To evaluate that determinant, you need to evaluate the determinant of a yet smaller $(n-2) \times (n-2)$ matrix etc.

The determinant can also be computed by expanding similarly along a column:

$$\det \boldsymbol{A} = \Sigma_{i=1}^{n} (-1)^{(i+j)} a_{ij} C_{ij}.$$

Examples:

1.
$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

$$\det A = 2 \det \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} - 0 \det \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$$

$$+1 \det \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = 4 + 1 = 5.$$

2.
$$\det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn} \end{pmatrix} = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}.$$

3. The following observations regarding the determinant are clear from the definition of the determinant:

   i) If $\boldsymbol{A}_i^\lambda$ is obtained from $\boldsymbol{A}$ by a multiplying row $i \in \{1, ..., n\}$ by a real number λ, then
$$\det \boldsymbol{A}_i^\lambda = \lambda \det \boldsymbol{A}.$$

   ii) If two rows of matrix $\boldsymbol{A}$ are equal, then $\det \boldsymbol{A} = 0$.

   iii) Adding (scalar multiples) of one row to another row does not change the determinant.

   iv) Swapping two rows i and j in $\boldsymbol{A}$ reverses the sign of the determinant.

   Why are these facts true?

   i) Follows by expanding the determinant along row i.

   ii) Let rows i and j coincide. Claim ii) follows by expanding along $k \neq i, j$ and observing that 2×2 matrices with identical rows have a zero determinant.

   iii) When adding a multiple by λ of row j to row i, the result is a new matrix $\boldsymbol{A}'$, where $a'_{ik} = a_{ik} + \lambda a_{jk}$. Expanding along the i^{th} row, you get
$$\det \boldsymbol{A}' = \det \boldsymbol{A} + \lambda \det \boldsymbol{A}'',$$
   where $\boldsymbol{A}''$ has identical rows i and j so the claim follows by ii).

   iv) Follows from i) and iii) by going over the following steps: 1. Add row j to i This leaves the determinant unchanged by iii). 2. Subtract row i of the new matrix from row j. Again the determinant is unchanged by iii). 3. Multiply row j in this new matrix by -1. By i), this reverses the sign of the determinant. 4. Add the row j of the new matrix to row i. By iii), the determinant is unchanged and the resulting matrix is $\boldsymbol{A}$ with rows i and j swapped.

   Since Gaussian elimination performs the elementary row operations iii) and iv) repeatedly, we see that the determinant of a matrix is zero if and only if the determinant of its row echelon form is zero, i.e. if and only if the matrix does not have full rank.

4. Rules for computing the determinant:

$$\begin{aligned}
\det \boldsymbol{A}^\top &= \det \boldsymbol{A}. \\
\det \boldsymbol{AB} &= \det \boldsymbol{A} \det \boldsymbol{B}, \\
\det \boldsymbol{A}^{-1} &= \frac{1}{\det \boldsymbol{A}}, \\
\det(\boldsymbol{A}+\boldsymbol{B}) &\neq \det \boldsymbol{A} + \det \boldsymbol{B} \text{ in general.}
\end{aligned}$$

Cramer's rule

Assume that $\boldsymbol{A}$ has full rank and therefore $\det \boldsymbol{A} \neq 0$. The system of equations

$$\boldsymbol{Ax} = \boldsymbol{b}$$

has then a unique solution $\boldsymbol{x}$ with components:

$$x_i = \frac{\det \boldsymbol{B}_i}{\det \boldsymbol{A}},$$

where $\boldsymbol{B}_i$ is the matrix obtained by replacing the i^{th} column of $\boldsymbol{A}$ by the column vector $\boldsymbol{b}$.

Example:

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

$$x_1 = \frac{\det \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}}{\det \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}} = \frac{1}{5},$$

$$x_2 = \frac{\det \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}}{5} = \frac{3}{5},$$

$$x_3 = \frac{\det \begin{pmatrix} 2 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}}{5} = \frac{-1}{5},$$

Inverting a matrix

Cofactor matrix of $\boldsymbol{A}$ is given by:

$$\boldsymbol{C} = (C_{ij}),$$

where the C_{ij} are as above. The transpose of $\boldsymbol{C}$ is called the adjoint of A $adj\,(A)$:

$$\mathrm{adj}\,(\boldsymbol{A}) = \boldsymbol{C}^{\top}.$$

Then:

$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \cdot \mathrm{adj}\,(\boldsymbol{A}).$$

Example: compute adj($\boldsymbol{A}$), for

$$\boldsymbol{A} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} C_{11} &= 2,\ C_{12} = 1\ , C_{13} = -2, \\ C_{21} &= -3,\ C_{22} = 1,\ C_{23} = 3, \\ C_{31} &= 1,\ C_{32} = -1,\ C_{33} = 4. \end{aligned}$$

$$\mathrm{adj}\,(\boldsymbol{A}) = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -2 \\ -2 & 3 & 4 \end{pmatrix},$$

Therefore

$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \cdot \begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -2 \\ -2 & 3 & 4 \end{pmatrix},$$

which corresponds to what we computed with Gaussian elimination since $\det A = 5$.

Elementary row operations as matrix multiplication (extra material)

A permutation matrix is a matrix with zeros and ones as elements. Each row and each column has a single one. Permutation matrices are obtained from the unit matrix by interchanging (permuting) rows. For example with $n = 3$ we get

$$E_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

by permuting the last two rows of the identity matrix.

Elementary row operations can be represented as results of matrix multiplication as follows. Let E_{ij} be a permutation matrix where rows i and j have been permuted. Permuting the rows i and j of $\boldsymbol{A}$ can be written as matrix product:

$$E_{ij}\boldsymbol{A}.$$

Let $E_i\left(r\right)$ be the matrix obtained by multiplying row i of the unit matrix by scalar r.

$$E_2\left(r\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplying the i^{th} row of $\boldsymbol{A}$ by r corresponds to the product

$$E_i\left(r\right)\boldsymbol{A}.$$

Let $E_{ij}\left(r\right)$ be a matrix obtained by adding to the unit matrix a matrix whose element ji is r and all other elements are zeros.

$$E_{23}\left(r\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 1 \end{pmatrix}.$$

Adding row i multiplied by r to row j in matrix $\boldsymbol{A}$ is given by the matrix multiplication:

$$E_{ij}\left(r\right)A.$$

Hence we have shown that the elementary operations can be performed as matrix multiplications by elementary matrices $E_{ij}, E_i(r), E_{ij}(r)$.

Since the determinants of all elementary row operations are non-zero (for $r \neq 0$), we see by the product rule for computing the determinant that the determinant is non-zero if and only if the square matrix has full rank.

A somewhat challenging exercise is to prove the product rule for determinants using a representation of the Gaussian elimination via products of elementary matrices. Using only $E_{ij}, E_{ij}(r)$, any matrix $\boldsymbol{A}$ can be transformed to an upper triangular matrix. The determinant of an upper triangular matrix is the product of its diagonal elements. Premultiplying a matrix $\boldsymbol{A}$ E_{ij} changes the sign of the determinant, premultiplying by $E_{ij}(r)$ leaves the determinant unchanged. Also the determinant of the product of an upper triangular matrix and a lower triangular matrix is the product of their determinants.

Consider now premultiplying $\boldsymbol{AB}$ by the sequence of elementary matrices that transform $\boldsymbol{A}$ to an upper triangular matrix $\boldsymbol{A}_\Delta$. (Use $E(\boldsymbol{AB}) = (E\boldsymbol{A})\boldsymbol{B}$ repeatedly to see that the resulting product is indeed $\boldsymbol{A}_\Delta\boldsymbol{B}$

Use det($\boldsymbol{C}$) = det($\boldsymbol{C}^\top$), to consider the transpose $\boldsymbol{B}^\top\boldsymbol{A}_\Delta^\top$ of the resulting matrix and premultiply next by the elementary matrices transforming $\boldsymbol{B}^\top$ to a an upper triangular matrix $\boldsymbol{B}_\Delta^\top$. This gives you the result.