Mathematics for Economists: Lecture 1

Juuso Välimäki

Aalto University School of Business

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Welcome to the course

- Course logistics
  - Lectures Mon, Wed at 13:15 - 14:45
  - Review sessions with Amin Mohazab Thu 10:15 - 11:45
  - Weekly Problem Sets to be returned via MyCourses on specified due date
  - 20% of the grade based on problem sets, 80% on final exam
  - To succeed in the course, you should attempt all problem sets
  - Exam May 31, 9:00-12:00

Contents I

- Course contents: Part I
  - Lectures 1-2 Introduction and Applications of Linear Algebra Readings: Synopsis, Section 1, material for week 1, S&B: Chapter 6-11, 13.
  - Lectures 3-4: Multivariate Calculus Readings: Synopsis, Section 2, material for week 2, S&B: Chapters 14, 15.
  - Lecture 5: Unconstrained Optimization, Convexity and Concavity Readings: Synopsis, Section 3, material for week 3, S&B: Chapters 16, 17, 21.

Contents II

- Course contents: Part II
  - Lecture 7-8: Constrained Optimization Readings: Synopsis, Sections 4-5, material for week 4, S&B Chapters 18, 19
  - Lecture 9-10: Economic Applications of Constrained Optimization
    Readings: Synopsis, Sections 4-5: material for week 5, S&B:Chapters 20, 22
  - Lectures 11-12: Linear Dynamical Systems
    Readings: Synopsis, Sections 6: material for week 6, S&B: 23, 25.1, 25.2

Economic models

- Economics studies the allocation of scarce resources amongst competing ends
  - what are the ways to allocate?
  - how to evaluate the results?
  - what do we mean by scarcity?
  - how can we formalize such questions?
- Individualistic approach: economic agents are autonomous decision makers
- They act in pursuit of individual objectives or goals
  - agents do not make systematic mistakes in their choices, they choose according to their preferences
  - they act within constraints
  - they react to changes on their environment

Economic models

- Equilibrium analysis to guarantee the consistency of individual decisions
  - in competitive markets: equilibrium brought about by price mechanism
  - in games: equilibrium from consistency of expectations and realized behavior

Optimizing agents

- Economic agents (also called decision makers) have objectives summarized in their objective functions
  - utility function of a consumer
  - profit function of a firm
  - total surplus for an economic planner
- Autonomous decisions:
  - each agent chooses her own actions
- Decisions in line with the objectives
  - each agent maximizes her objective function
- Economic choices constrained by scarcity

Endogenous and exogenous variables

- Economic models are chosen by the modeler
- Idea is to pick the most important features of an economic situation and ignore the rest
- Every model has variables that are determined within the model
  - endogenous variables
- Interesting models have variables not determined within the model
  - exogenous variables
- Exogenous variables and parameters of the model are similar in nature

Mathematical formulation

- In order to formulate the problem, we need the following ingredients:
  - Choices x from the set of choice variables: X
  - Evaluation of choices: objective function $f : X \to \mathbb{R}$
  - Scarcity in the form of feasible set: $F = \{x | g(x) \leq 0\}$
  - Possible parameters and other (exogenous) variables, $\{\alpha, \beta, ...\}$ to include in f, g
  - Exogenous variables are not determined in the model
- For concreteness, let's consider a economic problems familiar from Principles of Economics I

Examples

- Consumer choice between food and leisure
  - x_1 food consumption, x_2 leisure, $X = \{(x_1, x_2) | x_i \in \mathbb{R}, \text{ for } i \in \{1, 2\}, x_1 \geq 0, 0 \leq x_2 \leq 24\}$.
  - Utility from x: $f(x_1, x_2)$, $f(x_1, x_2) \geq f(y_1, y_2)$ if (x_1, x_2) is at least as good for the consumer as $f(y_1, y_2)$.
  - Feasible set: $p_1 x_1 \leq w(24 - x_2)$.
  - Price of food p_1 and wages w are exogenous variables.
- Best responses of player 1 in two player games:
  - Own action $x_1 \in X_1$ (row in the matrix).
  - Payoff from own action: $f(x_1; x_2)$, where x_2 is the exogenous variable (for best responses of 1, pick the row with the largest payoff to 1).
  - In this context, no further feasibility constraint.
  - When solving the game, x_2 becomes also an endogenous variable.

General form:

- In general, we have the problem

$$\max_{x \in X} f(x;\alpha)$$
$$\text{subject to } g(x;\beta) \leq 0.$$

- What is a solution to the problem? An x^* such that
  - i) The solution must be feasible:
  $$x^* \in F,$$
  - ii) No other feasible alternative achieves a higher value of the objective function:
  $$f(x^*;\alpha) \geq f(y;\alpha) \text{ for all } y \in F.$$
  - Can you formulate the monopolists production problem from Principles of Economics in this framework?

Mathematical structure:

- What kinds of variables are x, α, β?
  - most often real numbers, real vectors or sometimes discrete choices (such as choosing the row in a matrix or choosing between a red and a blue car)
- When does a solution to an economic model exist?
  - Solutions to equilibrium systems: for linear systems, conditions on the rank (in the material for week 1 and Lecture 2). Other existence theorems (these are hard and only hinted at in this course).
  - Optimal choices: Weierstrass theorem for maximization problems (you will see this in part II of the course).
  - For dynamic systems, we construct the solutions.
- How to find a solution?
  - one of the main questions for this course
  - usually with the help of calculus
  - calculus is not of much help for discrete problems, in more advanced courses tools for handling this to some extent

Mathematical structure:

- Is the solution unique?
  - concavity and convexity of the objective function key for this
- How do endogenous variables react to changes in exogenous variables?
  - comparative statics
  - implicit function theorem is the key tool for this and one of our first goals in this course

Linear economic models

- The key object of study in linear algebra are *linear equations*
- A linear equation has the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

  where

  - $a_1, a_2, \ldots, a_n, b$ are fixed real numbers (parameters)
  - $x_1, x_2, \ldots, x_n$ are real valued variables
- A system of linear equations is a collection of such equations that hold simultaneously

Examples of systems

A system of two linear equations in two unknowns:

$$2x_1 + 3x_2 = 7 \tag{1}$$
$$2x_1 + x_2 = 4 \tag{2}$$

A system of two non-linear equations in two unknowns:

$$2x_1x_2 + 3x_2 = 7 \tag{3}$$
$$2x_1^2 + x_2 = 4 \tag{4}$$

Market Equilibrium for Two Goods

- Goal: Determine competitive equilibrium prices and quantities for two goods $i \in \{1, 2\}$.
- Demand Q_i^d depends on the prices of the two goods P_1 and P_2, on disposable income Y and on other factors K_i as follows:

$$Q_1^d = K_1 P_1^{\alpha_{11}} P_2^{\alpha_{12}} Y^{\beta_1},$$

$$Q_2^d = K_1 P_1^{\alpha_{21}} P_2^{\alpha_{22}} Y^{\beta_2}.$$

- How would you interpret the parameters α_{ij} ja β_i?
- Think back to Principles 1 and elasticities

Market Equilibrium for Two Goods

- Y and K_i are the exogenous variables (i.e. ones not determined in the model).
- Supply functions Q_i^s for the two products are assumed to take the form:

$$Q_1^s = M_1 P_1^{\gamma_1},$$

$$Q_2^s = M_2 P_2^{\gamma_2}.$$

- Again, we take the variables M_i to be exogenous to the model.
- In equilibrium, supply equals demand so that

$$Q_1^d = Q_1^s,$$

and

$$Q_i^d = Q_i^s.$$

- Six equations for six endogenous variables $(Q_i^s, Q_i^d, P_i)_{i=1,2}$, but not linear

Market Equilibrium for Two Goods

- But a change of variables helps: define the following new variables for $i \in \{1, 2\}$:

$$q_i^d = \ln Q_i^d, \; q_i^s = \ln Q_i^s \; p_i = \ln P_i, y = \ln Y, \; m_i = \ln M_i, \; k_i = \ln K_i.$$

- By taking logarithms on both sides of each equation, we can write the six equations for $i \in \{1, 2\}$:

$$q_i^d = k_i + \alpha_{ii} p_i + \alpha_{ij} p_j + \beta_i y,$$

$$q_i^s = m_i + \gamma_i p_i,$$

$$q_i^s = q_i^d.$$

Market Equilibrium for Two Goods

- By the third equation, $q_i^d = q_i^s$ for $i \in \{1, 2\}$, and therefore the right hand sides in the first and the second equations are equalized:

$$k_i + \alpha_{ii} p_i + \alpha_{ij} p_j + \beta_i y = m_i + \gamma_i p_i, \;\; i \in \{1, 2\}.$$

- The only remaining endogenous variables are: p_1 ja p_2.
- Let's write the exogenous variables on the right-hand side and the endogenous variables on the left-hand side:

$$\begin{array}{ccl} (\alpha_{11} - \gamma_1)\, p_1 & +\alpha_{12} p_2 & = m_1 - k_1 - \beta_1 y, \\ \alpha_{21} p_1 & (\alpha_{22} - \gamma_2)\, p_2 & = m_2 - k_2 - \beta_2 y. \end{array}$$

Market Equilibrium for Two Goods

- Let's solve for p_1 from the top equation:

$$p_1 = \frac{m_1 - k_1 - \beta_1 y - \alpha_{12} p_2}{(\alpha_{11} - \gamma_1)}.$$

- Substituting into the second equation gives:

$$\begin{aligned} p_2 &= \frac{m_2 - k_2 - \beta_2 y - \alpha_{21} p_1}{(\alpha_{22} - \gamma_2)} \\ &= \frac{m_2 - k_2 - \beta_2 y - \alpha_{21} \frac{m_1 - k_1 - \beta_1 y - \alpha_{12} p_2}{(\alpha_{11} - \gamma_1)}}{(\alpha_{22} - \gamma_2)}. \end{aligned}$$

- Multiplying both sides by $(\alpha_{22} - \gamma_2)(\alpha_{11} - \gamma_1)$ gives:

$$(\alpha_{22} - \gamma_2)(\alpha_{11} - \gamma_1) p_2 = (\alpha_{11} - \gamma_1)(m_2 - k_2 - \beta_2 y)$$
$$-\alpha_{21}(m_1 - k_1 - \beta_1 y) + \alpha_{12}\alpha_{21} p_2,$$

Market Equilibrium for Two Goods

- Solving for p_2,

$$p_2 = \frac{(\alpha_{11} - \gamma_1)(m_2 - k_2 - \beta_2 y) - \alpha_{21}(m_1 - k_1 - \beta_1 y)}{(\alpha_{22} - \gamma_2)(\alpha_{11} - \gamma_1) - \alpha_{12}\alpha_{21}}.$$

- And substituting back gives:

$$\begin{aligned} p_1 &= \frac{m_1 - k_1 - \beta_1 y - \alpha_{12}\frac{(\alpha_{11}-\gamma_1)(m_2-k_2-\beta_2 y)-\alpha_{21}(m_1-k_1-\beta_1 y)}{(\alpha_{22}-\gamma_2)(\alpha_{11}-\gamma_1)-\alpha_{12}\alpha_{21}}}{(\alpha_{11} - \gamma_1)} \\ &= \frac{(\alpha_{22} - \gamma_2)(m_1 - k_1 - \beta_1 y) - \alpha_{12}(m_2 - k_2 - \beta_2 y)}{((\alpha_{22} - \gamma_2)(\alpha_{11} - \gamma_1) - \alpha_{12}\alpha_{21})}. \end{aligned}$$

Market Equilibrium for Two Goods

- The (logarithmic) equilibrium quantities are solved most easily from the supply curves.
- Finally P_i, Q_i are solved by exponentiating p_i, q_i.
- Exercise: Can you see how an improvement in the production technology for good 2 changes the equilibrium?
- **Lessons from the example**
- A nonlinear model can sometimes be transformed to a linear model (logarithmic transforms are particularly useful)
- Solving by substitution is clumsy and prone to errors.
- Gaussian elimination (familiar from Matrix Algebra) is a systematic representation of this process .

Next Lecture

- A brief review of some of the main concepts of Matrix Algebra or Linear Algebra
- Input-output models of economic production
- Linear models of exchange